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An equivariant, inclination flip, heteroclinic bifurcation

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Abstract: We examine a heteroclinic bifurcation occurring in families of equivariant vector fields. Within these families, the flows contain structurally stable heteroclinic cycles. The flow can twist around the cycle to produce what is the equivalent of an "inclination flip" homoclinic orbit. An unfolding of this bifurcation in a generic one-parameter equivariant family shows that Smale horseshoes are embedded in the flow for an open set of parameter values.

1 Introduction

In this paper, we study an equivariant heteroclinic bifurcation which produces chaotic dynamics. When the quotient of the phase space with respect to the group action is taken, the heteroclinic cycles we consider are reduced to homoclinic orbits, so we may conduct our analysis within the mathematical framework of homoclinic bifurcation. This heteroclinic bifurcation has two special features. The first is that the heteroclinic orbits are not broken because they are artifacts of the symmetry. The second is that the flow in a neighborhood of the heteroclinic orbits respects a rotational symmetry. Consequently, we are not studying the equivalent of a generic codimension two homoclinic bifurcation, but are looking at a generic, codimension one, equivariant heteroclinic bifurcation. Equivariant homoclinic bifurcations have been studied before and, for example, one type occurs in the Lorenz equations [10] (where the symmetry is a simple rotation through π). However, this particular bifurcation is new and predicts chaotic behavior.

We will be studying bifurcations in a one-parameter family of vector fields $\dot{x} = f_\lambda(x)$ which are generic within a classs of vector fields with symmetry. Each element of the family has the same symmetry as defined by the following:

Definition 1.1 *Let G be a finite subgroup of $O(n)$ acting on R^n. A vector field $\dot{x} = f(x)$, $x \in R^n$, is G-equivariant or symmetric if*

$$f(gx) = gf(x) \text{ for all } g \in G \ .$$

For a fixed group G, we wish to study generic bifurcation, both local and global, within the class of G-equivariant vector fields. In other words, we are interested in the bifurcations which persist under sufficiently small G-equivariant perturbations.

Consider the action of a group $\tilde{G}$ on R^3 generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and studied by Guckenheimer and Holmes [9]. A few short calculations show that the degree three $\tilde{G}$-equivariant normal form may be written as:

$$\begin{aligned} \dot{x}_1 &= (\lambda + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2) x_1 \\ \dot{x}_2 &= (\lambda + a_1 x_2^2 + a_2 x_3^2 + a_3 x_1^2) x_2 \\ \dot{x}_3 &= (\lambda + a_1 x_3^2 + a_2 x_1^2 + a_3 x_2^2) x_3 \ . \end{aligned} \tag{1}$$

Guckenheimer and Holmes observed that this system of equations has structurally stable heteroclinic cycles for open regions of the parameter space.

Theorem 1.2 *[9] If either $a_3 < a_1 < a_2 < 0$ or $a_2 < a_1 < a_3 < 0$ and $2a_1 > a_2 + a_3$ then Equations 1 have attracting heteroclinic cycles for $\lambda > 0$ and a globally attracting equilibrium at the origin for $\lambda \leq 0$.*

This theorem essentially follows from realizing that the coordinate planes are invariant and examining the flow in them. The consequence is that within families of $\tilde{G}$-equivariant vector fields there are generic one-parameter bifurcations from an attracting equilibrium point to structurally stable attracting limit cycles.

Now consider the action of $\bar{G}$ on R^4 generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and studied by Field and Swift [7]. The cubic $\bar{G}$-equivariant normal form is given by

$$\begin{aligned} \dot{x}_1 &= (\lambda + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2) x_1 \\ \dot{x}_2 &= (\lambda + a_1 x_2^2 + a_2 x_3^2 + a_3 x_4^2 + a_4 x_1^2) x_2 \\ \dot{x}_3 &= (\lambda + a_1 x_3^2 + a_2 x_4^2 + a_3 x_1^2 + a_4 x_2^2) x_3 \\ \dot{x}_4 &= (\lambda + a_1 x_4^2 + a_2 x_1^2 + a_3 x_2^2 + a_4 x_3^2) x_4 \, . \end{aligned} \tag{2}$$

If all the $a_i = -1$ and if $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, then $\dot{r} = \lambda r - r^3$. So for $\lambda \leq 0$ the origin is a globally attracting equilibrium point and for $\lambda > 0$ there is an invariant 3-sphere of radius $\sqrt{\lambda}$. The invariant sphere is normally hyperbolic and all trajectories which are not identically zero are asymptotic to it. If the a_i are close to -1, then the sphere will persist as an invariant topological 3-sphere in the post bifurcation flow. This is also true for Equations 1 and a stronger result is given in the invariant sphere theorem by Field and Swift.

Theorem 1.3 *(Invariant Sphere Theorem) [8] Let $\dot{x} = \lambda x + Q(x)$ be a G-equivariant vector field where $Q(x)$ is a homogeneous cubic polynomial equivariant. If $(Q(x), x) < 0$ for all $x \neq 0$ ($Q(x)$ is contracting), then for every $\lambda > 0$ there exists a unique $(n-1)$-dimensional sphere $S(\lambda) \subset R^n - \{0\}$ which is invariant under the flow. This topological sphere is globally attracting in the sense that every trajectory which is not identically zero is forward asymptotic to $S(\lambda)$. Furthermore, the flow on $S(\lambda)$ is topologically conjugate to the flow of $u' = Q(u) - (Q(u), u)u$, where $u \in S^{n-1} \equiv \{u \in R^n : |u| = 1\}$.*

This theorem allows us to study the dynamics on the invariant sphere. Those features which are robust will be generic properties of the equivariant bifurcation. In other words, these features will not be removed by small perturbations or the addition of higher order terms to the normal form.

Varying the parameters a_i in Equation 2, reveals a wide range of behavior on the sphere including limit cycles and heteroclinic cycles with a variety of different stability properties. This analysis by Field and Swift proved the existence of a stationary bifurcation to limit cycles.

This brings us to the symmetry group which is the focus of this paper. Consider the group $G \subset \bar{G}$, the elements of the group G which are orientation preserving. A new term must be added to the cubic equivariant normal form given in Equation 2 which results in

$$\begin{aligned} \dot{x}_1 &= (\lambda + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2) x_1 + e x_2 x_3 x_4 \\ \dot{x}_2 &= (\lambda + a_1 x_2^2 + a_2 x_3^2 + a_3 x_4^2 + a_4 x_1^2) x_2 - e x_1 x_3 x_4 \\ \dot{x}_3 &= (\lambda + a_1 x_3^2 + a_2 x_4^2 + a_3 x_1^2 + a_4 x_2^2) x_3 + e x_1 x_2 x_4 \\ \dot{x}_4 &= (\lambda + a_1 x_4^2 + a_2 x_1^2 + a_3 x_2^2 + a_4 x_3^2) x_4 - e x_1 x_2 x_3 \ . \end{aligned} \tag{3}$$

The geometry associated with this group action appears to be very rich and was first examined by Guckenheimer and Worfolk [11]. Once again, for $a_i \approx -1$, $\lambda > 0$, and all e, there is an invariant, attracting, topological sphere in the flow on which we study the dynamics. As in the earlier examples, there are structurally stable heteroclinic cycles and limit cycles. The work [11] includes a numerical study which reveals unmistakable signs of chaotic behavior: period doubling cascades and Šilnikov homoclinic orbits. This leads to the conclusion that the flow on the invariant sphere can contain hyperbolic chaotic sets (embedded Smale horseshoes) and nonhyperbolic chaotic attractors. Due to the persistence of hyperbolic sets, we can expect to bifurcate directly from a trivial equilibrium to a flow containing chaotic invariant sets of small amplitude.

In this paper we prove that there exist generic codimension one heteroclinic bifurcations in G-equivariant vector fields which create hyperbolic chaotic invariant sets (embedded Smale horseshoes) in the flow. We examine G-equivariant vector fields which contain structurally stable heteroclinic cycles between equilibria in the same G-orbit. We will call these cycles *homoclinic cycles* since they are homoclinic orbits in the quotient space R^4/G. Unlike standard homoclinic bifurcation, the cycles do not break under small parameter variations. Instead, we are interested in small perturbations of a vector field with a degenerate twist in the flow along a homoclinic cycle. We will give conditions for a one-parameter family of vector fields to contain this bifurcation. The second section of this paper presents the geometric aspects of the bifurcation which is a discussion of how the horseshoes are created. The third section presents the analytic problems associated with proving that a specific family of vector fields. In particular, we can prove that our cubic normal form satisfies all but one of the geometric conditions presented in the second section. We believe that it satisifes all the conditions, but have been unable to prove one of the nondegeneracy conditions.

The creation of horseshoes in a codimension two homoclinic bifurcation with a twist degeneracy has been studied by Deng [4] and Homburg *et. al.* [15]. The

strategy we use to study homoclinic cycle bifurcations was introduced by Šilnikov. We introduce a codimension one cross section to the homoclinic cycle and study a return map on the section. This style of analysis leads to basic results on the asymptotic stability of the heteroclinic cycles. The stability is determined by the flow near the equilibrium points and is therefore a local analysis. The existence of more complicated behavior generally depends on the flow along the homoclinic cycle and is a global and harder analysis.

This is a standard method for constructing a geometric model, but analytically proving the existence of such a global model in a particular family of equations is still a problem fraught with technical difficulties. Our presentation is similar to the work of Rychlik [18] and Robinson [16, 17]. In their work they prove that families of equations closely related to the Lorenz equations contain a homoclinic bifurcation to a flow containing geometric Lorenz attractors.

2 The Geometric Model

In this section we present the geometric model for an equivariant homoclinic bifurcation. We give a set of conditions which if a family of G-equivariant vector fields satisfies will guarantee the existence of horseshoes in the flow for an open set of parameters. Let $C^r_G(R^4, R^4)$ denote the set of all G-equivariant vector fields of class C^r. For our geometric analysis we need only that $r \geq 3$, but we are specifically interested in analytic vector fields since we wish to apply this analysis to our family of cubic normal forms. To avoid the overuse of subscripts, we will write the cubic G-equivariant vector field in the following way:

$$\dot{x} = \lambda x + Q(x, a, b, c, d, e) \, ,$$

where

$$Q(x, a, b, c, d, e) = \begin{pmatrix} ax_1r^2 + bx_1x_2^2 + cx_1x_3^2 + dx_1x_4^2 + ex_2x_3x_4 \\ ax_2r^2 + bx_2x_3^2 + cx_2x_4^2 + dx_2x_1^2 - ex_1x_3x_4 \\ ax_3r^2 + bx_3x_4^2 + cx_3x_1^2 + dx_3x_2^2 + ex_1x_2x_4 \\ ax_4r^2 + bx_4x_1^2 + cx_4x_2^2 + dx_4x_3^2 - ex_1x_2x_3 \end{pmatrix}$$

$$r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \, .$$

We will want to apply the invariant sphere theorem later so we present the conditions under which $Q(x)$ is contracting.

Lemma 2.1 *([7], Lemma 3.1) $Q(x)$ is contracting (i.e., $(Q(x), x) < 0$ for all $x \neq 0$) if and only if*

$$a < 0, \; 2a + c < 0, \; 4a + b + d < 0, \; 4a + b + c + d < 0 \, .$$

Let $\mathcal{M}_0$ be an open subset of $C^r_G(R^4, R^4)$ consisting of those vector fields X which satisfy the following four conditions:

(A1) X is a perturbation of the vector field $\dot{x} = \lambda x + Q(x, a, 0, 0, 0, e)$ so that the flow contains an invariant topological 3-sphere, S. The perturbation should be small enough so that the sphere is attracting and sufficiently smooth.

(A2) There exists $p = (p_1, 0, 0, 0)$, $p_1 > 0$, such that $X(p) = 0$. The eigenvalues of $DX(p)$ denoted by $\lambda_a, \lambda_b, \lambda_c$, and λ_d are real, satisfy the inequalities $\lambda_a < 0, \lambda_b > 0, \lambda_c < 0$, and $\lambda_d < 0$ and have eigenvectors $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, and $(0, 1, 0, 0)$, respectively;

(A3) Let $g \in G$ map p to $q = (0, p_1, 0, 0)$, then $W^u(q) \subset W^s(p) \cup W^s(-p)$;

(A4) $\lambda_b + 3\lambda_d > 0, \lambda_b + \lambda_c + \lambda_d < 0$.

Here are a few comments on (A1)-(A4):

1. The vector field we are perturbing from in (A1) contains a normally hyperbolic invariant 3-sphere. Therefore, for small enough perturbations the invariant 3-sphere will persist as a topological 3-sphere. The smoothness of the sphere is determined by the eigenvalues at the equilibrium points and we will in general require only that it is C^3.

2. Due to the sign constraints on the eigenvalues in (A2), the unstable manifold of p, $W^u(p)$, is one-dimensional, and the stable manifold, $W^s(p)$, is three-dimensional. This is also true for each point in the G-orbit of p, in particular q. From a simple count of dimensions, $W^u(q)$ and $W^s(p)$ cannot intersect transversally, but they do intersect G-transversally so (A3) is an open condition. The symmetry and (A3) together imply that p is the equilibrium point in a structurally stable homoclinic cycle. For a discussion of G-transversality see [6].

3. The conditions placed on the eigenvalues in (A4) will ensure the appropriate behavior in the neighborhood of the equilibrium point in the cycle and will be discussed later. We note that they imply $\lambda_c < \lambda_d < 0$.

4. The normal form $\dot{x} = \lambda x + Q(x)$ satisfies (A1)-(A4) when $\lambda > 0$, $Q(x)$ is contracting, $b > 0$, $d < 0$, $b + 3d > 0$, and $b + c + d < 0$ and b, c, d are scaled small enough to ensure sufficient smoothness of S. See Section 3.

Before continuing we give two definitions which are useful when working with equivariant vector fields.

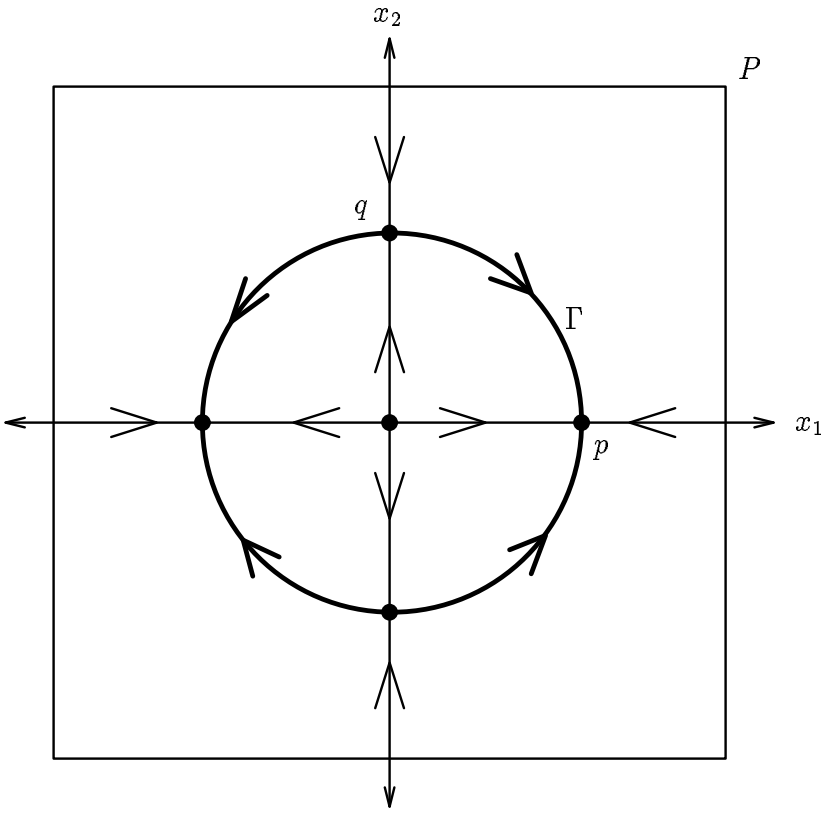

Figure 1: The geometry of the flow in P.

Definition 2.2 *The isotropy subgroup of a point $x \in R^4$ is*

$$\Sigma_x = \{g \in G \ : \ gx = x\}.$$

An isotropy subgroup for G is a subgroup which is the symmetry group for some point. We next define the set of points which have the same symmetry.

Definition 2.3 *The fixed point subspace of a subgroup $\Sigma \subset G$ is*

$$Fix(\Sigma) = \{x \in R^4 \ : \ gx = x \ for \ all \ g \in \Sigma\}.$$

We note that the set of points with a particular symmetry forms a subspace of R^4 and is invariant under the flow of a G-equivariant vector field.

Let Γ be the trajectory containing p and q as limit points. Denote by P the fixed point subspace with isotropy subgroup $(Z_2^2)'$ which contains p, q, and Γ. The geometry of the flow in the invariant plane P is sketched in Figure 1. Because of the equivariance, DX has a block structure along Γ. One 2×2 block corresponds to vectors in the coordinate plane P and a second 2×2 block for the orthogonal complement to P. The subbundle of TR^4 restricted to Γ and orthogonal to P is therefore invariant under the flow. We define two other invariant C^{r-1} subbundles as follows:

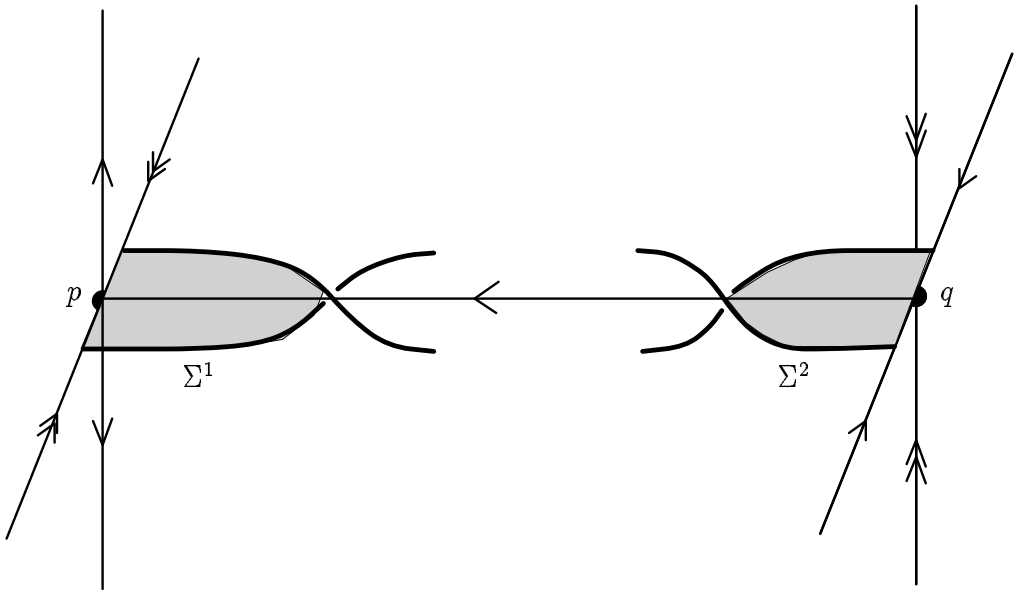

Figure 2: The geometry of Σ^1 and Σ^2.

Σ^1 is the subbundle of TR^4 invariant under the flow and restricted to Γ such that for all $x \in \Gamma$, $X_x \in \Sigma^1$ and whose closure at p is given by $\Sigma^1(p) = \text{span}\{(0,1,0,0),(0,0,1,0)\}$. In other words, $\Sigma^1(p)$ is the span of the eigenvectors of the stable eigenvalues λ_c and λ_d at p.

Σ^2 is the subbundle of TR^4 invariant under the flow and restricted to Γ such that for all $x \in \Gamma$, $X_x \in \Sigma^2$ and whose closure at q is given by $\Sigma^2(q) = \text{span}\{(1,0,0,0),(0,0,1,0)\}$. So $\Sigma^2(q)$ is the span of the eigenvectors of the eigenvalues λ_b and λ_d at q.

The existence and uniqueness of these subbundles follow from standard analyses, e.g. [14]. The condition that $\lambda_c < \lambda_d$ is necessary for the uniqueness of Σ^2. In Figure 2 we indicate the local geometry of Σ^1 and Σ^2.

We now consider a subset $\mathcal{M}_1 \subset \mathcal{M}_0$ of vector fields X which satisfy $\Sigma^1 = \Sigma^2$. This condition is analogous to the twist degeneracy condition for homoclinic orbits [4, 12, 15]. We claim, but do not show, that $\mathcal{M}_1$ is a C^{r-1} submanifold of $\mathcal{M}_0$ of codimension one.

We will now consider families of $C^r_G(R^4, R^4)$ vector fields X_μ on R^4 where μ is a parameter from some open set $U \subset R^m$ with m a positive integer. We need to place a transversality condition on the families to be considered. We do this in conditions (A5) and (A6):

(A5) $X_\mu \in \mathcal{M}_0$ for all $\mu \in U$;

(A6) There exists $\mu_0 \in U$ such that $X_{\mu_0} \in \mathcal{M}_1$ and the mapping $\mu \mapsto X_\mu$ is transversal to $\mathcal{M}_1$ at μ_0.

We will need to linearize in a neighborhood of p in order to analyze the dynamics and state this condition now:

(A7) There is a neighborhood of $p \in R^4$ and $\mu_0 \in U$ which is uniformly C^3 equivariantly linearizable.

This means that there is a family of C^3 diffeomorphisms that linearize X_μ in a neighborhood of p while preserving the equivariance. This family of diffeomorphisms varies smoothly with μ. Sufficient conditions for (A7) to hold are given by Rychlik [18], Theorem 1.1. The linearizability question is a perennial problem in the theory of dynamical systems [1]. For our purposes we will state simply that (A7) does not hold when there are low order resonances between the eigenvalues at the equilibrium point. For any set of eigenvalues for which (A7) does not hold, there is another set close by for which (A7) will hold.

Our final condition is most easily stated in terms of the vector field once it has been linearized in a neighborhood of the equilibrium points. We assume that X_μ is a family satisfying (A1)-(A7) and let $\tilde{X}_\mu$ be the family for which the flow is linear in a neighborhood V of the equilibrium point p for some neighborhood of μ_0. The subsets of V and gV given by $W^1 \equiv V \cap \{(p_1, x_2, x_3, 0) : x_2 \geq 0\}$ and $W^2 \equiv gV \cap \{(x_1, p_1, x_3, 0) : x_1 \geq 0\}$ (where $gp = q$) are invariant under the linear flow. See Figure 3.

We define the two-dimensional manifolds $\mathcal{W}^1$ and $\mathcal{W}^2$ by the flow under the vector field $\tilde{X}_{\mu_0}$ of W^1 and W^2, respectively. By their construction, these two manifolds intersect along $\tilde{\Gamma}$ (the homoclinic cycle of $\tilde{X}_{\mu_0}$) and condition (A6) implies that they are tangent along $\tilde{\Gamma}$. The next condition states that the tangency transverse to $\tilde{\Gamma}$ is third order. The symmetry of the problem requires the tangency to be flat to an even order.

(A8) Let $r \in \tilde{\Gamma} \cap V$ and let $w = (w_1(x_2, x_3), x_2, x_3, w_4(x_2, x_3))$ be a local parametrization of $\mathcal{W}^2$ in a neighborhood of r. Then $\frac{\partial^3 w_4}{\partial {x_3}^3}|_r \neq 0$.

This is an open condition due to the inequality. Also, if $\frac{\partial^3 w_4}{\partial {x_3}^3}|_r = 0$, then an arbitrarily small perturbation to the vector field within $\mathcal{M}_1$ can make it non-zero. This is analogous to the idea behind the argument that vector fields with homoclinic connections can be perturbed an arbitrarily small amount to break the connection. This means that vector fields within $\mathcal{M}_1$ satisfy (A8) generically.

The main result of this section is stated in the following theorem and the remainder of this section is used to prove it.

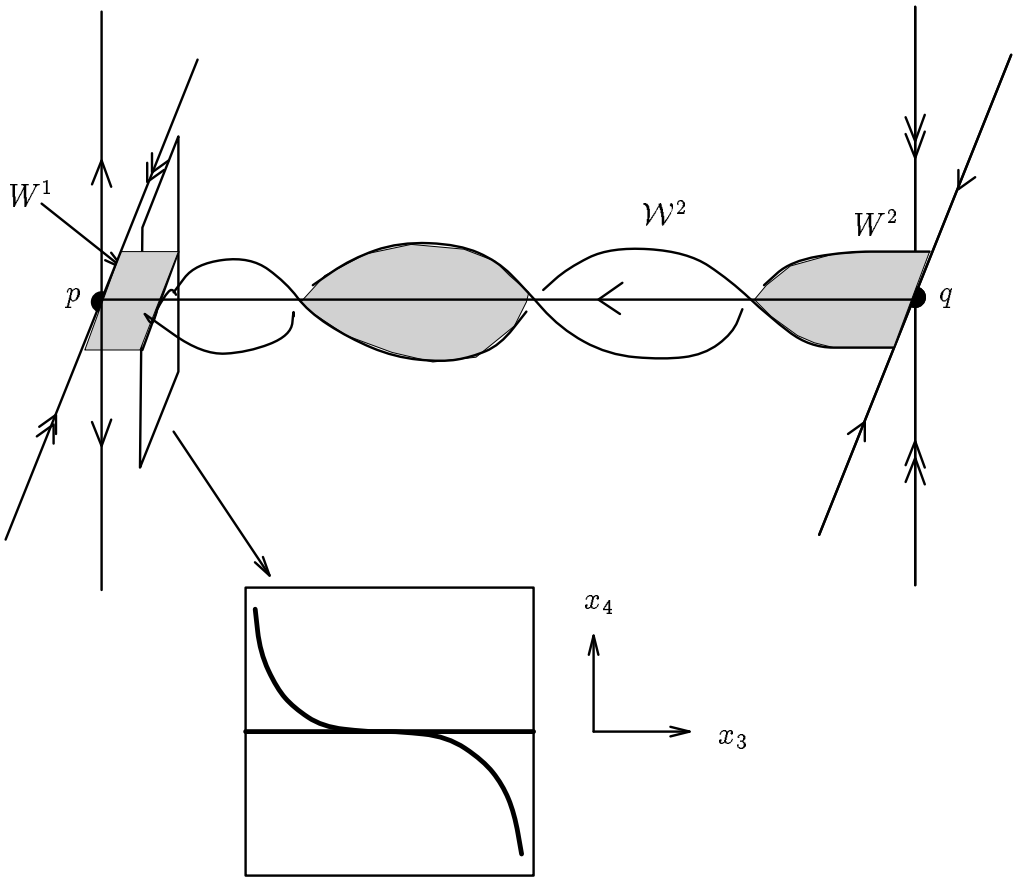

Figure 3: Model of locally linearized flow.

Theorem 2.4 *Let X_μ be a family of vector fields satisfying (A1)-(A8). Then there are values of μ arbitrarily close to μ_0 such that the flow associated with X_μ contains hyperbolic invariant chaotic sets (horseshoes).*

We prove this theorem by constructing a Poincaré map and demonstrating that this map contains horseshoes. Suppose that X_μ is a family satisfying (A1)-(A8). We work with the family $\tilde{X}_\mu$ constructed according to (A7) since the dynamics are the same, *i.e.*, the flows generated are C^3 equivalent. We shall assume without loss of generality that the flow at p is linear in a neighborhood of the unit cube at p and then construct maps between boundaries of the unit cube. Furthermore we restrict ourselves to the flow within S, the invariant topological 3-sphere.

In the linear region near p, lying in S is equivalent to setting $x_1 = p_1$. We define the cross sections, $\Sigma_0 \equiv \{(p_1, 1, x_3, x_4) : |x_3|, |x_4| < 1\}$ and $\Sigma_1 \equiv \{(p_1, x_2, x_3, 1) : |x_2|, |x_3| < 1\}$. See Figure 4.

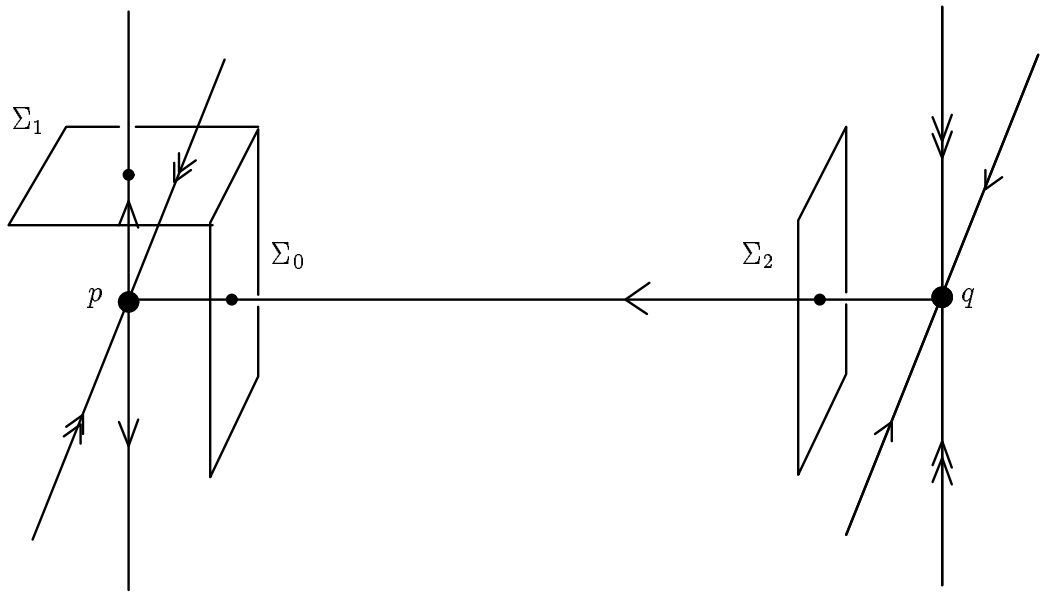

Figure 4: The sections Σ_0, Σ_1, and Σ_2.

Let L_μ be the map from Σ_0 to Σ_1 induced by the flow. This map is singular and is computed by explicitly integrating the linear flow. So $L_\mu : \Sigma_0 \to \Sigma_1$ is given by

$$(p_1, 1, x_3, x_4) \mapsto (p_1, x_4^\alpha, x_3 x_4^\beta, 1) \tag{4}$$

where

$$\alpha = -\lambda_d/\lambda_b > 0,\ \beta = -\lambda_c/\lambda_b > 0.$$

For a point on Σ_0 with $x_4 < 0$, the linear flow brings it to the surface $\hat{\Sigma}_1 = \{(p_1, x_2, x_3, -1) : |x_2|, |x_3| < 1\}$. However, $\hat{\Sigma}_1$ may be mapped to Σ_1 by an element of G. The resulting map may be extended to be both continuous and invertible across $x_4 = 0$ and is the map specified by Equation 4.

Define $\Sigma_2 = g\Sigma_1$ to be a cross section in the neighborhood of q where we are assuming the flow is linear. We reduce by the symmetry by considering the composition $g \circ L_\mu$ which maps Σ_0 to Σ_2 with g defined by

$$(p_1, x_2, x_3, 1) \mapsto (1, p_1, x_2, -x_3)\ .$$

We note that this map is only unique up to a rotation by π in the final two coordinates. In other words we have a choice as to which coordinate needs a minus sign. This is eliminated if we consider the full quotient space R^4/G.

Let Ψ_μ denote the map from Σ_2 to Σ_0 induced by the flow $\tilde{X}_\mu$. We assume that Ψ_μ is a family of C^3 diffeomorphisms and it will be approximated by a Taylor series. The parameter μ will be varied to produce the homoclinic bifurcation. The heteroclinic cycle has a symmetry since points on the cycle have a nontrivial isotropy subgroup.

This symmetry persists in the maps Ψ_μ with the requirement that $\Psi_\mu(-x_2,-x_3) = -\Psi_\mu(x_2,x_3)$. The map $\Psi_\mu : \Sigma_2 \to \Sigma_0$ may be defined by

$$(1, p_1, x_3, x_4) \mapsto (p_1, 1, \psi_3(x_3,x_4), \psi_4(x_3,x_4))$$

or, by neglecting the first two terms, as

$$(x_3, x_4) \mapsto (\psi_3(x_3,x_4), \psi_4(x_3,x_4))$$

with $\psi_i(-x_3,-x_4) = -\psi_i(x_3,x_4)$. Note that we did not express the explicit dependence on the parameter μ for notational convenience.

We write the derivative of the map Ψ_μ at $(0,0)$ as

$$D\Psi_\mu|_{(0,0)} = \begin{pmatrix} \gamma_{31}(\mu) & \gamma_{32}(\mu) \\ \gamma_{41}(\mu) & \gamma_{42}(\mu) \end{pmatrix} .$$

Condition (A6) says that

$$D\Psi_{\mu_0}|_{(0,0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \; c_1 \neq 0 ,$$

implying that $\gamma_{31}(\mu_0) \neq 0$ and $\gamma_{41}(\mu_0) = 0$. The invertibility of Ψ_μ further requires that $\gamma_{42}(\mu_0) \neq 0$. We may smoothly reparametrize the family $\tilde{X}_\mu$ and write using a Taylor series expansion

$$\Psi_\mu = \begin{pmatrix} \gamma_{31}x_3 + \gamma_{32}x_4 \\ (\mu-\mu_0)x_3 + Ax_3^3 + \gamma_{42}x_4 \end{pmatrix} + \text{ h.o.t. } .$$

The term Ax_3^3 has been introduced since we wish to study this map near $\mu = \mu_0$ and require that the higher order terms all be dominated by the given terms when x_3, x_4 are sufficiently small. The coefficient A is derived from $\frac{\partial^3 \Psi_4}{\partial {x_3}^3}|_{\mu_0,(0,0)}$. Condition (A8) is exactly the condition that $A \neq 0$. We see this Taylor series term is nonzero because the manifold $\mathcal{W}^2$ intersects Σ_2 along the parameterized curve $(1, p_1, s, 0)$ for $s \in (-1,1)$. Consequently $\mathcal{W}^2 \cap \Sigma_0$ is given by the curve $(p_1, 1, \Psi_3(s,0), \Psi_4(s,0))$. A straightforward calculation shows that the tangency condition in (A8) is given by $\frac{d\Psi_4}{d\Psi_3}|_{s=0} = \frac{{\Psi_4}'''}{{\Psi_3}'}|_{s=0} \neq 0$ where the prime denotes differentiation with respect to s. So ${\Psi_4}'''|_{s=0} \neq 0$ and ${\Psi_4}'''|_{s=0} = \frac{\partial^3 \Psi_4}{\partial {x_3}^3}|_{\mu_0,(0,0)}$. The geometrical significance of the nonzero cubic term is that the intersection $\mathcal{W}^2 \cap \Sigma_0$ is qualitatively like the graph from Figure 3 with a cubic tangency.

The composition of the linear map $g \circ L_\mu$ and the return map Ψ_μ gives the family $P_\mu : \Sigma_0 \to \Sigma_0$ defined by $(x_3, x_4) \mapsto \Psi_\mu \circ g \circ L_\mu(x_3, x_4)$ and given by

$$P_\mu \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma_{31}x_4^\alpha - \gamma_{32}x_3x_4^\beta \\ (\mu-\mu_0)x_4^\alpha + Ax_4^{3\alpha} - \gamma_{42}x_3x_4^\beta \end{pmatrix} + \text{ h.o.t. } .$$

This family of maps models the flow in a neighborhood of the homoclinic cycles. We shall proceed with an analysis of a family of maps of this type.

By rescaling the map P_μ, changing variables and dropping higher order terms, one gets the nonlinear map:

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu x^\alpha - x^{3\alpha} + x^\beta y \\ b_1 x^\alpha + b_2 x^\beta y \end{pmatrix} . \tag{5}$$

The only assumption necessary to perform these calculations was that $A \neq 0$. Conditions (A2) and (A4) restrict the values of α and β to the following ranges:

$$0 < \alpha = -\lambda_d/\lambda_b < 1/3,\ 1 - \alpha < \beta = -\lambda_c/\lambda_b .$$

In the following theorem, we show that this mapping has a complicated attracting set for values of α and β in this range. This gives us the necessary behavior for the family $\tilde{X}_\mu$ near $\mu = \mu_0$ and finishes the proof of Theorem 2.4.

Theorem 2.5 *Assume that* $0 < 3\alpha < 1$ *and* $1 - \alpha < \beta$ *then the mapping* Φ *defined by Equation 5 contains a hyperbolic chaotic invariant set for* $\mu > 0$*, but sufficiently small.*

Proof We study this mapping on the rectangle $R_\mu = [-2\mu^{1/2\alpha}, 2\mu^{1/2\alpha}] \times [-\mu^\gamma, \mu^\gamma]$ (where $\gamma > 0$ is still to be determined) and are interested in the limit as $\mu \to 0$. We will demonstrate that the map may be thought of as stretching and folding as pictured in Figure 5.

Rescaling under the singular transformation given by $z = \mu^{-1/2\alpha} x, w = \mu^{-\gamma} y$, the mapping becomes

$$\hat{\Phi} \begin{pmatrix} z \\ w \end{pmatrix} = \mu^{(3\alpha-1)/(2\alpha)} \begin{pmatrix} z^\alpha - z^{3\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu^{\gamma+(\beta-1)/2\alpha} z^\beta w \\ \mu^{\beta/2\alpha} b_1 z^\beta w + \mu^{-\gamma+1/2} b_2 z^\alpha \end{pmatrix} \tag{6}$$

and $\hat{R}_\mu = [-2, 2] \times [-1, 1]$. Observe that since $(3\alpha - 1)/2\alpha < 0$ that the first term in the right hand side of Equation 6 is singular with respect to μ. Requiring that $\gamma \in ((1 - \beta)/2\alpha, 1/2)$ makes all powers of μ in the second term of the right had side positive. So this mapping converges in a singular limit to a rescaling of the rank one map given by

$$\Psi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu x^\alpha - x^{3\alpha} \\ 0 \end{pmatrix} .$$

We have reduced the problem to the study of a one-dimensional map which comes from the first component of Ψ,

$$\psi(x) = \mu x^\alpha - x^{3\alpha} .$$

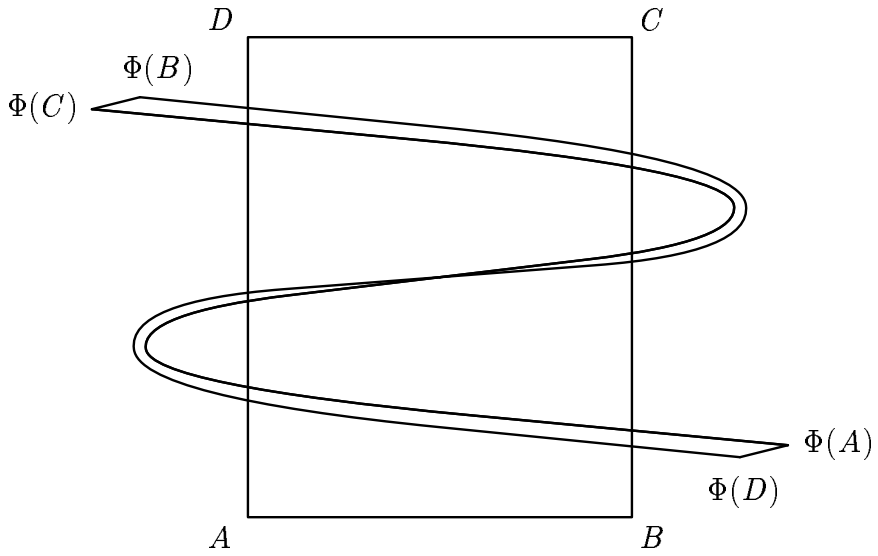

Figure 5: The image of the rectangle $R_\mu = ABCD$ under Φ.

A graph of $\psi(x)$ is given in Figure 6.

Consider the points

$$a = 0,\ b = (\mu/4)^{1/2\alpha},\ c = (\mu/2)^{1/2\alpha} \text{ and } d = \mu^{1/2\alpha}.$$

Assuming that μ is small, we have $\psi(a) = \psi(d) = a$ and $\psi(b) = 3(\mu/4)^{3/2} \gg d$ and $\psi(c) = (\mu/2)^{3/2} \gg d$. Further, $\psi' > 1$ on $[a, b]$ and $\psi' < 1$ on $[c, d]$. So ψ has a hyperbolic invariant set in the interval $[a, d]$. The dynamics on this set are chaotic and are well known from the study of the logistic map. See, for example, [5]. Due to the symmetry, there will be identical behavior for $x < 0$. The interaction between the negative and positive humps of the map produce additional invariant sets and the resulting dynamics are topologically conjugate to the full shift on three symbols.

We conclude that the hyperbolic invariant set of the singular limit will persist as horseshoes in Φ when we leave the singular limit. This is analogous to the behavior of the Hénon map in the limit as the map tends to rank one [13]. □

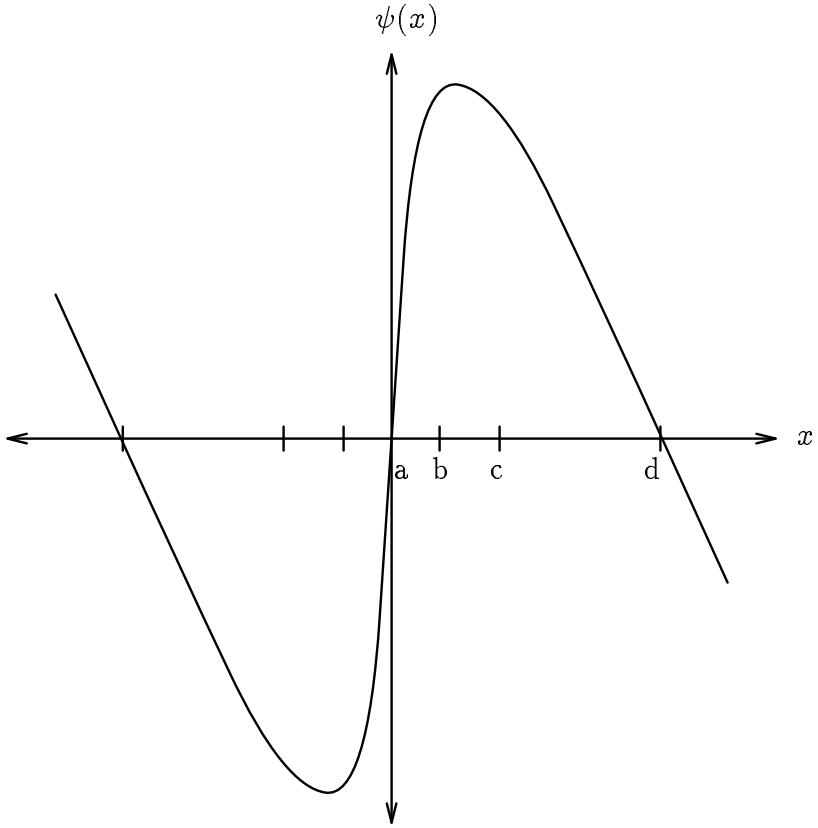

Figure 6: The one-dimensional map $\psi(x)$.

3 The Analytic Problem

In the previous section we constructed a geometric model for a G-equivariant homoclinic bifurcation. We would like to be able to prove that this bifurcation occurs in the cubic normal for G-equivariant vector fields given in Equation 3. This would then give a proof for the existence of a direct bifurcation from an attracting fixed point to a chaotic invariant set. The primary difficulty, and one which we have not resolved, is in proving that within the four parameter family (rescaled with $\lambda = 1$) there is a one parameter family which satisfies the nondegeneracy condition expressed in (A8). However, we do prove the following theorem about the family of normal forms:

Theorem 3.1 *There are open sets of values for b, c, d such that there exists a countably infinite number of functions $\tilde{e}_i(b,c,d)$, $i \in Z$, such that the flow of $\dot{x} = x + Q(x, -1, b, c, d, e)$ contains an inclination flip heteroclinic orbit if and only if $e \in \{\tilde{e}_i(b,c,d)\}_{i \in Z}$.*

We conjecture that a countably infinite number of these bifurcation points will be nondegenerate. The bifurcation at $e = \tilde{e}_0(b,c,d) \equiv 0$ will always be degenerate. We proceed to prove this theorem by examining the heteroclinic cycles in the normal form and finding conditions on the parameter values for which the normal form satisfies assumptions (A1)-(A7) of the previous section. Because there exists a rescaling with respect to a that is valid whenever $a < 0$, we have assumed $a = -1$.

Structurally stable heteroclinic cycles can occur in the flow of $\dot{x} = x + Q(x)$. We are interested in heteroclinic cycles whose components lie in the invariant $x_i - x_{i+1}$ coordinate planes joining zeros on the coordinate axes.

Proposition 3.2 *Heteroclinic cycles exist in the flow of $\dot{x} = x + Q(x)$ lying in invariant coordinate planes and joining zeros on the coordinate axes whenever*

$$\begin{aligned} a &< 0 \ , \\ bd &< 0 \ . \end{aligned}$$

Proof The conditions are sufficient to show (using the Poincaré-Bendixson Theorem) that in each invariant $x_i - x_{i+1}$ coordinate plane, the one-dimensional stable manifolds of the saddle equilibrium points on one of the coordinate axes lie in the two-dimensional stable manifolds of the sink equilibrium points on the other coordinate axis. The images of these connecting orbits under the group action form heteroclinic cycles. □

From now on we will assume that Q is contracting, so that the flow of the normal form contains an invariant sphere S. The heteroclinic cycles of Proposition 3.2 are given by the intersection of S with the invariant $x_1 - x_{i+1}$ coordinate planes. The

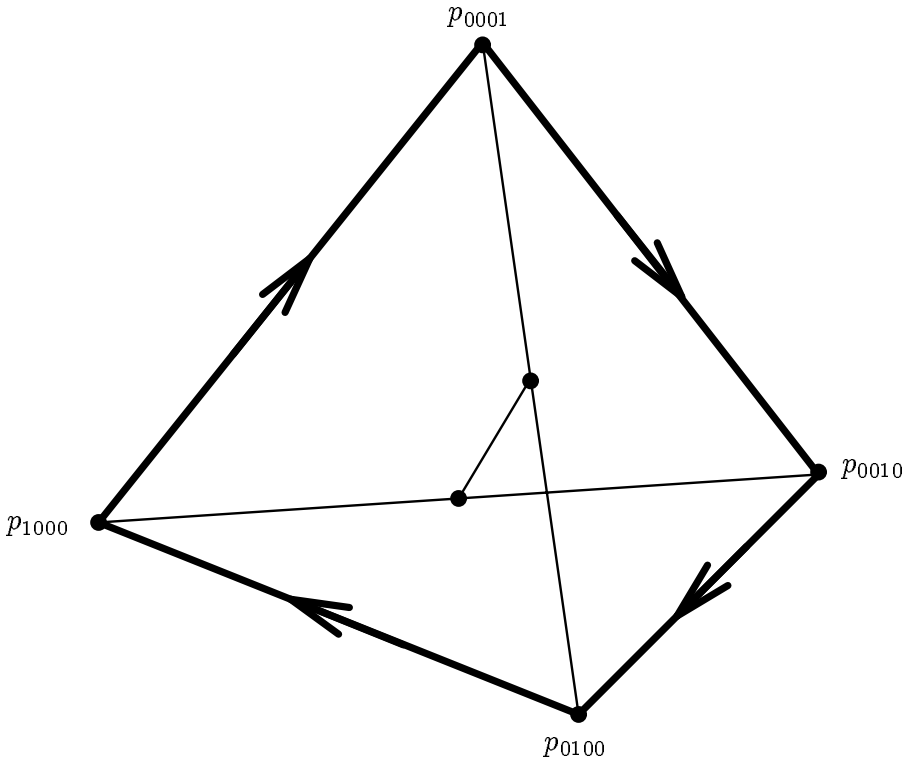

Figure 7: A heteroclinic cycle on Δ.

flow is most easily visualized by considering a "fundamental domain" in the phase space defined as

$$\Delta \equiv S \cap \{x \ : \ x_i \geq 0\}.$$

The region Δ may be thought of as a tetrahedron where the corners correspond to points in S on coordinate axes, the edges correspond to points in S in coordinate planes, and the faces correspond to points in S in coordinate hyperplanes. The fundamental domain Δ containing a heteroclinic cycle is pictured in Figure 7. The equilibrium points at the corners are labeled according to the convention that p_{1000} corresponds to the equilibrium point on the x_1-axis, and similarly for p_{0100}, p_{0010}, and p_{0001}. If the flow on Δ is known then the flow on the entire attracting invariant sphere S is known by application of elements of the symmetry group G.

The normal form $\dot{x} = x + Q(x, -1, b, c, d, e)$ has eigenvalues of $\lambda_a = -2, \lambda_b = b, \lambda_c = c, \lambda_d = d$ at p_{1000} with the correct eigenvectors as given in (A2). Therefore, it satisfies (A1)-(A4) when $Q(x)$ is contracting, $b > 0$, $b + 3d > 0$, and $b + c + d < 0$ and b, c, d are scaled small enough to give sufficient smoothness of S in a neighborhood of the heteroclinic cycle. The twist conditions (A5) and (A6) can be satisfied and we examine this now.

Twist Degeneracy in the Normal Form

In order to study the twist around the homoclinic cycle, we first develop an approximation to the cycle. It lies in the coordinate planes and as b, c, d approach zero, must approach one quarter of a circle. We therefore scale the parameters b, c, d by $(b, c, d) \mapsto (\epsilon b, \epsilon c, \epsilon d)$ and perform a perturbation analysis assuming that $\epsilon \ll 1$. The equations in the $x_1 - x_2$ plane reduce to

$$\begin{aligned} \dot{x_1} &= x_1(1 - x_1^2 - x_2^2 + \epsilon b x_2^2) \\ \dot{x_2} &= x_2(1 - x_1^2 - x_2^2 + \epsilon d x_1^2) \ . \end{aligned}$$

Define $\alpha = (b + d)/4$ and $\beta = (b - d)/4$, then writing the above vector field in polar coordinates gives

$$\begin{aligned} \dot{r} &= r(1 - r^2) + \epsilon \alpha r^3 \sin^2 2\theta \\ \dot{\theta} &= \epsilon r^2 \sin 2\theta (\alpha \cos 2\theta - \beta) \ . \end{aligned}$$

Since $\beta > \alpha > 0$, we see that $\dot{\theta} < 0$ for $\theta \in (0, \pi/2)$. For small $\epsilon > 0$ there exists a connecting orbit from $(r, \theta) = [1, \pi/2]$ to $[1, 0]$ close to the curve defined by $r = 1$. We perform a perturbation analysis to study this curve. We wish to parameterize the curve by $(g(\theta), \theta)$ and write

$$g(\theta) = 1 + \epsilon g_1(\theta) + \epsilon^2 g_2(\theta) + O(\epsilon^3).$$

Setting $\dot{r} = \frac{dg(\theta)}{d\theta}\dot{\theta}$, we can equate powers of epsilon and solve for g_1 and g_2. This gives

$$g(\theta) = 1 + \epsilon \frac{\alpha}{2} sin^2 2\theta + \epsilon^2 \frac{\alpha}{2} sin^2 2\theta (5\alpha \sin^2 2\theta - (3 + 4\alpha)/4 + 2\beta \cos 2\theta) + O(\epsilon^3).$$

To examine the twist around the cycle, we transform x_3 and x_4 into polar coordinates (s, ϕ). The differential equation for $\phi \in S^1 \equiv R \bmod 2\pi$ becomes

$$\begin{aligned} \dot{\phi} &= -\tfrac{e}{2} \sin 2\theta + s^2 \tfrac{e}{2} \sin 2\theta + \epsilon \sin 2\phi (\tfrac{b}{2} \cos^2 \theta - \tfrac{c}{2} \cos 2\theta - \tfrac{d}{2} \sin^2 \theta) + \\ &\quad \epsilon s^2 \sin 2\phi (-\tfrac{b}{2} (\cos^2 \theta - \sin^2 \phi) + \tfrac{c}{2} \cos 2\theta + \tfrac{d}{2} (\sin^2 \theta + \cos^2 \phi)) \ . \end{aligned}$$

The twist about the cycle is given by integrating the above equation with $s = 0$ along the homoclinic cycle. Although we cannot do this explicitly, the results we need follow from studying the flow of θ and ϕ on the cylindrical section $(\theta, \phi) \in [0, \pi/2] \times S^1$. The flow is given by the equations

$$\begin{aligned} \dot{\theta} &= \epsilon g^2(\theta) \sin 2\theta (\alpha \cos 2\theta - \beta) \\ \dot{\phi} &= -\tfrac{e}{2} \sin 2\theta + \epsilon \sin 2\phi (\tfrac{b}{2} \cos^2 \theta - \tfrac{c}{2} \cos 2\theta - \tfrac{d}{2} \sin^2 \theta) \ . \end{aligned}$$

This flow has equilibrium points which are saddles at $(0, 0)$, $(0, \pi)$, $(\pi/2, 0)$, $(\pi/2, \pi)$, sinks at $(0, \pi/2)$, $(0, 3\pi/2)$, and sources at $(\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$. The twist condition (A6) corresponds to a connecting orbit between the saddles $(\pi/2, 0)$ and $(0, 0)$. This is most easily understood by considering the geometry as pictured in Figure 8.

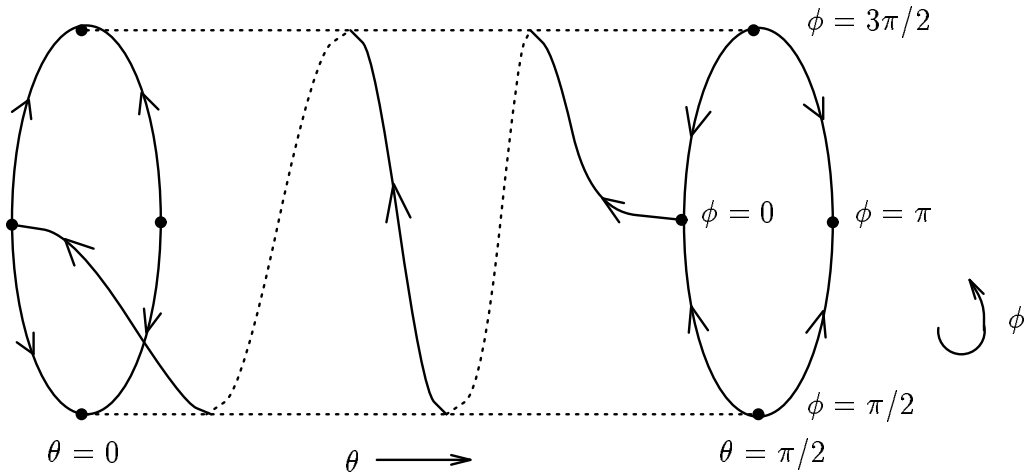

Figure 8: A connecting orbit in the flow representing twisting about the homoclinic cycle.

We know that θ decreases monotonically from $\pi/2$ to 0 and note that $\dot{\phi}$ decreases pointwise as e is increased. This means that as we increase e, the number of times that the unstable manifold of $(\pi/2, 0)$ twists around the cylinder increases and there will be a countably infinite number of values of $e = e_0(\epsilon b, \epsilon c, \epsilon d)$ for which the desired connection occurs.

We now turn to condition (A5) and the remainder of (A6). Taking a one-parameter family of the normal forms which is transversal to $\mathcal{M}_1$ is accomplished by considering e to be the distinguished parameter. This follows directly from consideration of the vector field on the cylindrical section again. Perturbing e from a value at which a connecting orbit exists will break the orbit, so our family is transversal to $\mathcal{M}_1$.

The eigenvalues at p are given by $-2, b, c, d$ and do not vary with e. It is therefore easy to choose them with no low order resonances so that the normal form will be linearizable in a neighborhood of p, satisfying condition (A7). Further, we may insist that for all integer vectors Q,

$$|\,(Q, (-2, b, c, d))\,| > \epsilon |Q|^{-\nu}$$

where $|Q| = \sum_{i=1}^{4} |q_i|$ for some $\nu > 0, \epsilon > 0$. This condition holds for almost all values of b, c, d with respect to the Lebesgue measure and gives the existence of a linearizing change of coordinates which is analytic at p [3, 19]. Consequently, we may choose linearizing coordinates as smooth as necessary so that the manifolds $\mathcal{W}^1$ and $\mathcal{W}^2$ are well defined and smooth enough for our calculations. We do not have a good way to approximate these manifolds to determine whether the normal form contains degenerate inclination flip heteroclinic cycles. We cannot prove that the normal form

family with critical twist does not lie in the submanifold of $\mathcal{M}_1$ that contains vector fields with higher order tangencies, thus satisfying (A8).

We have performed some numerical calculations designed to examine the degeneracy of the twist tangency in the normal form. We observe that the twisting about the heteroclinic cycle appears to be nonlinear and has the same properties as the global flow of the geometrical model. These calculations indicate that the tangency is indeed nondegenerate, but of course due to the sensitivity of the situation consist of far less than a proof.

4 Conclusions

Symmetries arise naturally in many simple mathematical models. The variety of behavior which typically occurs is frequently model independent – it depends largely on the symmetry. The specific behavior depends on the actual parameters in the model, but the possibilities can be independently studied. Equivariant bifurcation theory is aimed at classifying the different generic bifurcations with symmetry and uses techniques from group theory and singularity theory. We have exhibited a specific symmetry group whose equivariant bifurcations can have extraordinary complexity.

As well as altering the generic local bifurcations, the presence of symmetry affects many global structures. Direct global analysis of the normal form seems difficult, instead we have examined a geometrical model which shares the same symmetry group. This equivariant family possesses structurally stable heteroclinic cycles for large parameter sets. In a one-parameter family, these heteroclinic cycles can undergo a global bifurcation which does not break them, but the flow around them twists degenerately. This bifurcation point corresponds to an inclination flip codimension two point for an asymmetric homoclinic orbit. For an open set of parameter values in a neighborhood of the bifurcation point there are Smale horseshoes embedded in the flow and the heteroclinic cycles still exist. We believe that this bifurcation occurs in the cubic normal form of the instant chaos bifurcation.

This is one example of a homoclinic/heteroclinic bifurcation as a mechanism for the generation of chaos. It is a common phenomenon for homoclinic and heteroclinic orbits to organize the dynamics of a nonlinear system. This is especially true in many symmetric systems where the heteroclinic cycles can be structurally stable. In the instant chaos equivariant normal form, not only do the the heteroclinic cycles play an important role in the dynamics, but as the numerical evidence presented indicates, they may be involved in a one-parameter bifurcation which creates chaotic hyperbolic invariant sets in the flow.

5 Acknowledgements

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References

[1] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer-Verlag, New York, 1988.

[2] A. Back, J. Guckenheimer, M. Myers, F. Wicklin, and P. Worfolk. dstool: Computer assisted exploration of dynamical systems. *Notices Amer. Math. Soc.*, 39(4):303–309, 1992.

[3] A. D. Brjuno. Analytical form of differential equations. *Trans. Moscow Math. Soc.*, 25:131–288, 1971.

[4] B. Deng. Homoclinic twisting bifurcation and cusp horseshoe maps. Preprint, Dec. 1991.

[5] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison Wesley, New York, 2nd edition, 1989.

[6] M. Field. Transversality in G-manifolds. *Trans. Amer. Math. Soc.*, 231:429–450, 1977.

[7] M. Field and J. Swift. Stationary bifurcation to limit cycles and heteroclinic cycles. *Nonlinearity*, 4:1001–1043, 1991.

[8] M. J. Field. Equivariant bifurcation theory and symmetry breaking. *J. Dynamics Differential Equations*, 1(4):369–421, 1989.

[9] J. Guckenheimer and P. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Cambridge Philos. Soc.*, 103:189–192, 1988.

[10] J. Guckenheimer and P. J. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, New York, 1983.

[11] J. Guckenheimer and P. Worfolk. Instant chaos. *Nonlinearity*, 5:1211–1222, 1992.

[12] J. Guckenheimer and P. Worfolk. Dynamical systems: Some computational problems. In D. Schlomiuk, editor, *Bifurcations and Periodic Orbits of Vector Fields*. NATO ASI Series. Series C: Mathematical and Physical Sciences, D. Reidel Publishing Company, 1993. In Press.

[13] M. Hénon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50:69–77, 1976.

[14] M. Hirsch, C. Pugh, and M. Shub. *Invariant Manifolds*. Lecture Notes in Mathematics. Springer-Verlag, New York, 1977.

[15] A. J. Homburg, H. Kokubu, and M. Krupa. The cusp horseshoe and its bifurcations in the unfolding of am incliniation-flip homoclinic orbit. Technical Report W-9307, University of Groningen, Department of Mathematics, 1993.

[16] C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. *Nonlinearity*, 2:495–518, 1989.

[17] C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type, II. *SIAM J. Math. Anal.*, 23(5):1255–1268, 1992.

[18] M. Rychlik. Lorenz attractors through šilnikov-type bifurcation - Part I. *Ergodic Theory & Dynamical Systems*, 10:793–821, 1989.

[19] C. L. Siegel. Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Nachr. Akad. Wiss. Göttingen*, pages 21–30, 1952.

[20] J. W. Swift and E. Barany. Chaos in the Hopf bifurcation with tetrahedral symmetry. *Eur. J. Mech. B / Fluids*, 10-Suppl. 2:99–104, 1991.