Performance Limits for Cellular Systems with Handoffs

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Abstract.

In cellular systems with handoffs, a call can be dropped when it is handed off from one cell to another. If we assume that a customer is to be reimbursed for dropped calls, not all calls in progress contribute to the revenue generated by the system. To capture the notion of "useful carried traffic" we define the *delivered traffic* as the expected number of calls in progress that will complete successfully (without being dropped). We derive an upper bound on the delivered traffic achievable by any channel assignment algorithm for a given cellular system. We also show that there exist channel assignment algorithms whose delivered traffic approaches this upper bound arbitrarily closely when the number of available channels in the system is sufficiently large.

1. Introduction.

Consider a *static* cellular system, which we denote by S, where calls arrive at random times and have a random duration but originate and terminate in the same cell, i.e., there are no handoffs. A *channel assignment algorithm* must either assign a channel to a call that arrives or block it. In assigning a channel to a call, the channel assignment algorithm must obey certain restrictions on simultaneous uses of the same channel that are imposed by the system in order to control the interference experienced by a user. Let n denote the number of available channels in the system and let the offered traffic to the ith cell in the system be $r_i n$ Erlangs. Then, it was shown in [1,2] that there exists a function $T_S(r)$ for the system, called the *carried traffic function* which is an upper bound on the carried traffic per channel for *any* channel assignment algorithm for that system. Moreover, for a wide class of traffic models (including Poisson arrivals and arbitrary holding times) there exist channel assignment algorithms whose carried traffic per channel approaches $T_S(r)$ arbitrarily closely for large n [1,2].

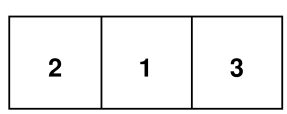

Figure 1. A 3-cell system.

Example. Consider the 3-cell system of Figure 1 and assume that there are *no handoffs*. In this system, the constraints on channel reuse are that the same channel may not be simultaneously used in two adjacent cells. Let the offered traffic be $\rho = nr/3$ Erlangs per cell. Using the results of [1,2] we can show the following. The carried traffic function for this system is $T_S(r) = \min\{r, r/3 + 1, 2\}$. A *fixed channel assignment algorithm* is one that allocates a fixed number of channels to each cell and is a greedy channel assignment algorithm in that it blocks a call in a cell if and only if all channels allocated to that cell are occupied. For $r \leq 3/2$, the carried traffic per channel of a fixed channel assignment algorithm that allocates $n/2$ channels to each of the cells approaches $T_S(r) = r$ arbitrarily closely for large n, i.e., this fixed channel assignment algorithm is asymptotically optimal for $r \leq 3/2$. Similarly, for $r \geq 3$, a fixed channel assignment algorithm that allocates all n available channels to each of the end-cells 2 and 3 is asymptotically optimal since its carried traffic approaches $T_S(R) = 2$. For every $r \in (3/2, 3)$ there exists a (different) fixed channel assignment algorithm that is similarly asymptotically optimal.

In this paper, we generalize the results of [1,2] to cellular systems with handoffs.

2. Traffic Model.

We assume there are T types of calls in the system each with a distinct *handoff sequence*. The handoff sequence of a call is the ordered list of cells that the call passes through, starting with the cell in which the call originates and ending with the cell in which the call terminates.

Example. Consider the 3-cell system and assume calls get handed off at most once (and to one of the neighbouring cells). Then, we have 7 types of calls in this system with handoff sequences: (1), (2), (3), (1, 2), (1, 3), (2, 1), and (3, 1).

If the handoff sequence of calls of type i, $i = 1, \ldots, T$, has length l_i, and $L = \max_i l_i$, the handoff sequences can be represented by a $T \times L$ matrix $H = (h_{ij})$ where h_{ij} is the jth element in the handoff sequence of calls of type i. ($h_{ij} = 0$, if $j > l_i$.) *

For the 3-cell example,

$$H = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 1 & 2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}.$$

For static cellular systems, all handoff sequences have length 1.

The (normalized) *offered traffic* for the system is represented by a $T \times L$ matrix $R = (r_{ij})$ where nr_{ij}, $j \leq l_i$, is the expected number of type i calls that would be in progress in cell h_{ij} if *all* call requests were honored in *all* cells, and $r_{ij} = 0$, if $j > l_i$.

The offered traffic is related to the arrival rates of calls and their expected durations as follows. If λ_i denotes the rate of origination of type i calls in the system and T_{ij} their expected duration in cell h_{ij}, then $nr_{ij} = \lambda_i T_{ij}$.

Example. Consider the 3-cell example again. Let the rate of arrival of *new* calls be λ in each cell. Let 1/3 of the calls originating in cell 1 terminate there without being handed off, 1/3 get handed off to cell 2 and 1/3 to cell 3, i.e., $\lambda_i = \lambda/3$, $i = 1, 4, 5$. Let 1/2 the calls originating in each of cells 2 and 3 terminate there without being handed off and let the other 1/2 get handed off to cell 1, i.e., $\lambda_i = \lambda/2$, $i = 2, 3, 6, 7$. Further let the expected total duration of all calls be T and let the expected durations of a handoff call in the originating and terminating cells be equal to $T/2$, i.e., $T_{11} = T_{22} = T_{33} = T$

* If we assume that the number of handoffs experienced by a call is bounded by a constant D and the number of neighboring cells to which a call may be handed off is bounded by a constant Δ, then the number of handoff sequences is bounded by $N\Delta^D$. Therefore, the number of handoff sequences only grows linearly with the number of cells, N.

and $T_{ij} = T/2$, $i = 4, 5, 6, 7$, $j = 1, 2$. If $3\lambda T = nr$, the offered traffic (per channel) matrix for this example is

$$R = \begin{pmatrix} 1/9 & 0 \\ 1/6 & 0 \\ 1/6 & 0 \\ 1/18 & 1/18 \\ 1/18 & 1/18 \\ 1/12 & 1/12 \\ 1/12 & 1/12 \end{pmatrix} r.$$

Note that the total offered traffic is $n \sum_{ij} r_{ij} = nr$.

For any channel assignment algorithm A, the (normalized) *carried traffic* for the system is represented by a $T \times L$ matrix $C^{A,n} = (c_{ij}^{A,n})$ where $nc_{ij}^{A,n}$, $j \leq l_i$, is the expected number of type i calls that are in progress in cell h_{ij}, under the operation of channel assignment algorithm A, and $c_{ij}^{A,n} = 0$, if $j > l_i$. If we assume each call in progress generates revenue at a rate of unity (say, \$1 per minute), the (normalized) carried traffic corresponds to the expected rate of revenue per channel from the system.

In cellular systems with handoffs, the quantity of interest is not the carried traffic but the *delivered traffic*, which is defined as follows. For any channel assignment algorithm A, the (normalized) *delivered traffic* for the system is represented by a $T \times L$ matrix $D^{A,n} = (d_{ij}^{A,n})$ where $nd_{ij}^{A,n}$, $j \leq l_i$, is the expected number of type i calls that are in progress in cell h_{ij}, and that *will terminate without being dropped*, under the operation of channel assignment algorithm A, and $d_{ij}^{A,n} = 0$, if $j > l_i$. Essentially, in using the delivered traffic as the performance measure of a channel assignment algorithm, we are assuming that dropped calls do not earn any revenue. This is a reasonable assumption from the customer's point of view: he is reimbursed for all dropped calls.

For clarity, we will often write c_{ij} and d_{ij}, suppressing the dependence of the carried and delivered traffic on the channel assignment algorithm and the number of available channels, n.

The summary of the rest of this paper is as follows. We show that there is a function $T_S(R)$, which we call the delivered traffic function, for any cellular system which satisfies the model of Section 3 below which is an upper bound on the (normalized) delivered traffic of any channel assignment algorithm for that system. We also show that there exist channel assignment algorithms whose (normalized) delivered traffic approaches this upper bound arbitarily closely for sufficiently large n.

3. Cellular System Model.

Our model for the cellular system is almost the same as that of [1] and is as follows. We represent the state of an N-cell system by an N-vector $(m_1, \ldots, m_N)$ where m_k is the number of calls in progress in cell k. We denote the set of feasible states of the system when the number of available channels is n by S'_n. Note that if $s_n \in S'_n$ and $0 \leq s'_n \leq s_n$, then $s'_n \in S'_n$. We assume there is a convex subset S' of R^N which serves as an "unscaled model" for the system and which satisfies:

1. If $s \in S'$ and $0 \leq s' \leq s$, then $s' \in S'$.
2. If $s_n \in S'_n$, then $s_n/n \in S'$.
3. For every $s \in S'$, $\lfloor ns \rfloor \in S'_n$. †

† [1] assumes instead that for every $s \in S'$, there is a sequence (s_n) with $s_n \in S'_n$, such that $\lim_{n\to\infty} s_n/n = s$. All the results of this paper hold under this more general

For the 3-cell system, since the only restriction on channel reuse is that the same channel may not be used simultaneously in adjacent cells, it can be shown that,

$$S'_n = \{m = (m_1, m_2, m_3) : m \geq 0,\ m_1 + m_2 \leq n,\ m_1 + m_3 \leq n\},$$

and

$$S' = \{t = (t_1, t_2, t_3) : t \geq 0,\ t_1 + t_2 \leq 1,\ t_1 + t_3 \leq 1\}.$$

4. Indistinguishability of Call Types.

We define the prefixes of the handoff sequence h_i, $l_i > 1$, as the (handoff) sequences h_k, of length $l_k \leq l_i$ such that, $h_{kj} = h_{ij}$, $j = 1, \ldots, l_k$. If h_k is a prefix of h_i, we write $h_k \preceq h_i$ and say h_i is an *extension* of h_k. In the 3-cell example, $(1) \preceq (1,2)$. (Note that a handoff sequence is a prefix of itself.) We *assume* that all prefixes of a handoff sequence are also handoff sequences.

We assume that the system does not have any knowledge of the "future" of a call; it only knows the handoff "history" of a call. Thus, it cannot distinguish between calls of type i and k in cells $h_{k,j} = h_{i,j}$, $j = 1, \ldots, l_k$, if h_i is an extension of h_k. In the 3-cell example, the system cannot distinguish among type (1), type $(1,2)$ and type $(1,3)$ calls in cell 1.

Under the operation of a channel assignment algorithm A, let $nc_{ij}^{A,n} = \lambda_{ij}^{A,n} T_{ij}$, i.e., $\lambda_{ij}^{A,n}$ is the arrival rate of accepted calls of type i in cell h_{ij}. Then the indistinguishability property, which we assume holds axiomatically, can be stated as follows.

Indistinguishability Property. *Under the operation of any channel assignment algorithm A,*

$$\frac{\lambda_{kj}^{A,n}}{\lambda_k} = \frac{\lambda_{ij}^{A,n}}{\lambda_i}, \qquad j = 1, \ldots, l_k, \qquad \text{if } h_k \preceq h_i.$$

This property states that since calls of type i and k cannot be distinguished in cell h_{ij} (h_{kj}), the fraction of accepted (blocked) calls of these two types must be equal for any channel assignment algorithm.

We also observe that the following property holds.

Conservation of Call Arrivals. *Under the operation of any channel assignment algorithm A,*

$$\lambda_i \geq \lambda_{i1}^{A,n} \geq \lambda_{i2}^{A,n} \geq \ldots \geq \lambda_{i,l_i}^{A,n}.$$

The inequalities are due to blocked/dropped calls.

The following lemmas follow from these two properties and the definitions of carried and delivered traffic.

Lemma 1 states that the carrying probabilties, c_{*j}/r_{*j}, of types of calls that are indistinguishable in a cell must be equal in that cell.

Lemma 1. *For any channel assignment algorithm,*

$$\frac{c_{kj}}{r_{kj}} = \frac{c_{ij}}{r_{ij}}, \qquad j = 1, \ldots, l_k, \qquad \text{if } h_k \preceq h_i.$$

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model as well.

Proof. This lemma follows from the indistinguishability property using $nr_{ij} = \lambda_i T_{ij}$ and $nc_{ij} = \lambda_{ij} T_{ij}$. (This can also be viewed as an alternative statement of the indistinguishability property in terms of traffic instead of arrival rates.) ∎

The following lemma relates the delivered traffic to the carried traffic. It states that the delivery probability d_{ij}/r_{ij}, of a type of call (in a cell), equals its carrying probability, c_{il_i}/r_{il_i} in the last cell of its handoff sequence.

Lemma 2. *For any channel assignment algorithm,*

$$d_{ij} = \frac{r_{ij}}{r_{i,l_i}} c_{i,l_i}, \qquad j = 1, \ldots, l_i.$$

Proof.

$$\begin{aligned} d_{ij}/r_{ij} &= \lambda_{i,l_i} T_{ij}/\lambda_i T_{ij} \qquad \text{(definition of } d_{ij}\text{)} \\ &= \lambda_{i,l_i} T_{i,l_i}/\lambda_i T_{i,l_i} \\ &= c_{i,l_i}/r_{i,l_i}. \end{aligned}$$

∎

Lemma 3. *For any channel assignment algorithm, the carried traffic satisfies the following prefix condition:*

$$\frac{c_{k,l_k}}{r_{k,l_k}} \geq \frac{c_{i,l_i}}{r_{i,l_i}}, \qquad \text{if } h_k \preceq h_i.$$

Proof.

$$\begin{aligned} \frac{c_{k,l_k}}{r_{k,l_k}} &= \lambda_{k,l_k} T_{k,l_k}/\lambda_k T_{k,l_k} \\ &= \lambda_{i,l_k}/\lambda_i \qquad \text{(indistinguishability of } h_k \text{ and } h_i \text{ in cell } l_k\text{)} \\ &\geq \lambda_{i,l_i}/\lambda_i \qquad \text{(call conservation)} \\ &= \lambda_{i,l_i} T_{i,l_i}/\lambda_i T_{i,l_i} \\ &= c_{i,l_i}/r_{i,l_i}. \end{aligned}$$

∎

Lemma 3 states that the delivery probability of a call-type cannot be more than the delivery probability of any of its prefixes. In the 3-cell example, this states that the delivery probability of type $(1,2)$ calls cannot be more than those of type (1) calls.

5. Performance Limits: Upper Bound.

We will need a few definitions, which we will illustrate using the 3-cell example, before we are ready to state the main results of our paper.

Let

$$U(R) = \Bigg\{ s = (s_1, s_2, \ldots, s_T) : 0 \leq s_i \leq r_{i,l_i}, \\ \frac{s_k}{r_{k,l_k}} \geq \frac{s_i}{r_{i,l_i}}, \quad \text{if } h_k \preceq h_i \Bigg\}.$$

For the 3-cell example,

$$U(R) = \Bigg\{ s = (s_1, \ldots, s_7) : 0 \leq s \leq (1/9, 1/6, 1/6, 1/18, 1/18, 1/12, 1/12) r, \\ s_1 \geq 2s_4,\ s_1 \geq 2s_5,\ s_2 \geq 2s_6,\ s_3 \geq 2s_7 \Bigg\}.$$

The significance of the space $U(R)$ is stated by the following lemma.

Lemma 4. *For any channel assignment algorithm,*

$$c = (c_{1,l_1}, c_{2,l_2}, \ldots, c_{T,l_T}) \in U(R).$$

Proof. Since the carried traffic is non-negative and at most the offered traffic, $0 \leq c_{i,l_i} \leq r_{i,l_i}$. Combined with Lemma 3, we have,

$$c \in U(R). \qquad \blacksquare$$

Let

$$q_k = \sum_{i: h_k \preceq h_i} r_{i,l_k}.$$

nq_k is the offered traffic of type k and its extensions in cell h_{k,l_k}. For the 3-cell system,

$$q = (2/9, 1/4, 1/4, 1/18, 1/18, 1/12, 1/12)r.$$

Let

$$p_k^{A,n} = \sum_{i: h_k \preceq h_i} c_{i,l_k}^{A,n}.$$

$np_k^{A,n}$ is the carried traffic of type k and its extensions in cell h_{k,l_k}.

The p_k's are related to the q_k's as follows.

Lemma 5.

$$p_k = q_k c_{k,l_k} / r_{k,l_k}.$$

Proof.

$$\begin{aligned} p_k r_{k,l_k} &= \sum_{i: h_k \preceq h_i} c_{i,l_k} r_{k,l_k} \qquad \text{(definition of } p_k\text{)} \\ &= \sum_{i: h_k \preceq h_i} r_{i,l_k} c_{k,l_k} \qquad \text{(using Lemma 1)} \\ &= q_k c_{k,l_k} \qquad \text{(definition of } q_k\text{)}. \qquad \blacksquare \end{aligned}$$

Let

$$S = \left\{ s : t_k = \sum_{i: h_{i,l_i} = k} q_i s_i / r_{i,l_i},\ t \in S' \right\}.$$

For the 3-cell system,

$$\begin{aligned} S &= \{s :\ t_1 = 2s_1 + s_6 + s_7,\ t_2 = 3s_2/2 + s_4,\ t_3 = 3s_3/2 + s_5,\ t \in S'\} \\ &= \{s \geq 0 :\ 2s_1 + 3s_2/2 + s_4 + s_6 + s_7 \leq 1,\ 2s_1 + 3s_3/2 + s_5 + s_6 + s_7 \leq 1\}. \end{aligned}$$

The significance of the space S is stated by the following lemma.

Lemma 6. *For any channel assignment algorithm,*

$$c = (c_{1,l_1}, c_{2,l_2}, \ldots, c_{T,l_T}) \in S.$$

Proof. Under the operation of any channel assignment algorithm, let m_i denote the (random) number of calls in progress in cell i. Consider an arbitrary cell k that is part of handoff sequence j but is not the last cell in this handoff sequence, i.e., $h_{j\alpha} = k$, but $\alpha < l_j$. Since we assume that every prefix of a handoff sequence is also a handoff sequence, there is some other handoff sequence i of length $\alpha = l_i$ such that $h_i \preceq h_j$. Thus, $h_{i,l_i} = k$, i.e., cell k is the last cell in handoff sequence i. Therefore, we can compute the carried traffic in cell k by considering each traffic type, i, for which cell k is the last cell in its handoff sequence and summing over the carried traffic of that type and its extensions (in cell k), as

$$\begin{aligned} E(m_k) &= n \sum_{i:h_{i,l_i}=k} p_i \\ &= n \sum_{i:h_{i,l_i}=k} q_i c_{i,l_i}/r_{i,l_i} \qquad \text{(by Lemma 5).} \end{aligned}$$

Also, since $m \in S'_n \subseteq nS'$ and S' is convex, $E(m) \in nS'$. Therefore, by definition of S, $c \in S$. ∎

Let

$$T_S(R) = \max\left\{\sum_{i=1}^{T} r_i s_i/r_{i,l_i} :\ s \in S \cap U(R)\right\},$$

where $r_i = \sum_j r_{ij}$, be the *delivered traffic function* for the cellular system S. For the 3-cell example,

$$\begin{aligned} T_S(R) &= \max\{s_1 + s_2 + s_3 + 2s_4 + 2s_5 + 2s_6 + 2s_7 : s \in S \cap U(R)\} \\ &= \min(r, r/6 + 6/5, 8/5). \end{aligned}$$

Let $T_S^{A,n}(R) = \sum_{ij} d_{ij}$ be the delivered traffic for the channel assignment algorithm A operating on the system S with the offered traffic matrix R and n available channels. Then the delivered traffic function $T_S(R)$ has the following significance.

Theorem 1. *For any channel assignment algorithm A,*

$$\frac{1}{n} T_S^{A,n}(R) \leq T_S(R) \qquad \text{for } R \geq 0.$$

Proof. For the channel assignment algorithm A, let $c = (c_{1,l_1}, c_{2,l_2}, \ldots, c_{T,l_T})$. By lemmas 4 and 6, $c \in S \cap U(R)$.

$$\begin{aligned} T_S^{A,n}(R) &= n \sum_{ij} d_{ij} \\ &= n \sum_i \frac{c_{i,l_i}}{r_{i,l_i}} \sum_j r_{ij} \qquad \text{(by Lemma 2)} \\ &\leq n \max\left\{\sum_i r_i s_i/r_{i,l_i} : s \in S \cap U(R)\right\} \qquad (\text{since } c \in S \cap U(R)) \\ &= n T_S(R). \end{aligned}$$

∎

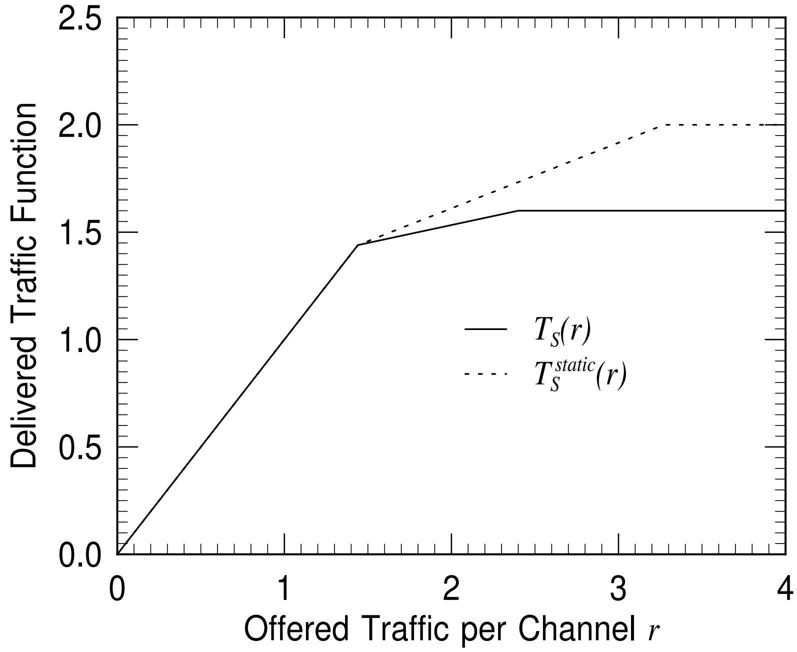

Figure 2. The delivered traffic functions $T_S(r)$ and $T_S^{\text{static}}(r)$ for the 3-cell system of Figure 1.

In the "static" case, Theorem 1 reduces to Theorem 3.1 of [1].

Consider the 3-cell example. Note that the offered traffic (of all types) in cell 1 is $7nr/18$ and in cells 2 and 3 is $11nr/36$. If the offered traffic were the same and there were no handoffs, ($H^T = (1,2,3)$ and $R^T = (7/18, 11/36, 11/36)r$) the delivered/carried traffic function for this system would be

$$T_S^{\text{static}}(r) = \min(r, 11r/36 + 1, 2).$$

For the 3-cell example, the functions $T_S(r)$ and $T_S^{\text{static}}(r)$ are plotted in Figure 2. Note that $T_S(r) = T_S^{\text{static}}(r) = r$ for $r \leq 36/25$ and both $T_S(r) < r$ and $T_S^{\text{static}}(r) < r$ for $r > 36/25$, in agreement with Theorem 7.1 of [1].

6. Asymptotic Tightness of Upper Bound.

So far we have left the statistics of the call arrivals and holding times largely unspecified. We shall show that the upper bound of Theorem 1 is asymptotically tight for traffic models for which new calls are independent of one another and that satisfy the following Asymptotic Traffic Property (ATP) [2] just as in the static case [1,2].

Consider a sequence of single cell systems (indexed by n) with offered traffic r_n Erlangs, number of available channels x_n and a greedy channel assignment algorithm that blocks a call only if no channel is available to carry that call. Let $\tau_n(r_n, x_n)$ denote the carried traffic for the nth one-cell system under the greedy channel assignment algorithm. The traffic statistics are said to satisfy the ATP if

$$\lim_{n\to\infty} \frac{\tau_n(r_n, x_n)}{n} = \min(r, x)$$

for arbitrary sequences r_n and x_n where $\lim_{n\to\infty} r_n/n = r$ and $\lim_{n\to\infty} x_n/n = x$.

The ATP simply states that if the offered traffic is more than the number of available channels, then asymptotically all the available channels are occupied and if the offered traffic is less than the number of available channels, then asymptotically the fraction of blocked/dropped calls is zero.

If $l_k > 1$, let $n\varrho_k^{A,n}$ denote the offered traffic of type k and its extensions in cell h_{k,l_k} under the operation of the channel assignment algorithm A, i.e.,

$$n\varrho_k^{A,n} = \sum_{i:h_k \preceq h_i} \lambda_{i,l_k-1}^{A,n} T_{i,l_k}.$$

(Recall that $\lambda_{ij}^{A,n}$ is the arrival rate of accepted calls of type i in cell h_{ij} under the operation of channel assignment algorithm A.)

Lemma 7. *If $h_i \preceq h_k$ and $l_k = l_i + 1$, for any channel assignment algorithm,*

$$\varrho_k = p_i q_k / q_i.$$

Proof. From Lemma 5, $p_i/q_i = c_{i,l_i}/r_{i,l_i}$. Therefore, it suffices to show that $n\varrho_k r_{i,l_i} = nq_k c_{i,l_i}$.

$$\begin{aligned}
n\varrho_k r_{i,l_i} &= \left(\sum_{j:h_k \preceq h_j} \lambda_{j,l_k-1} T_{j,l_k}\right)(\lambda_i T_{i,l_i}) \\
&= \sum_{j:h_k \preceq h_j} \lambda_{j,l_i} T_{j,l_k} \lambda_i T_{i,l_i} \qquad (\text{using } l_k - 1 = l_i) \\
&= \sum_{j:h_k \preceq h_j} \lambda_{i,l_i} T_{i,l_i} \lambda_j T_{j,l_k} \\
&\qquad\qquad (\text{by indistinguishability of } h_j \text{ and } h_i \text{ in cell } l_i,\ \lambda_{j,l_i}\lambda_i = \lambda_{i,l_i}\lambda_j) \\
&= nc_{i,l_i} q_k.
\end{aligned}$$

∎

Lemma 8. *Let $\hat{s} \in S \cap U(R)$. Then, there exists a fixed channel assignment algorithm F that allocates $\lfloor nq_i \hat{s}_i / r_{i,l_i} \rfloor$ channels to cell h_{i,l_i} for calls of type i and its extensions. Moreover, if the traffic statistics satisfy the ATP,*

$$\lim_{n\to\infty} p_i^{F,n} = q_i \hat{s}_i / r_{i,l_i}.$$

Proof. Under the fixed channel assignment algorithm F, the number of channels allocated to cell k is,

$$f_k = \sum_{i:h_{i,l_i}=k} \lfloor nq_i \hat{s}_i / r_{i,l_i} \rfloor \le \left\lfloor \sum_{i:h_{i,l_i}=k} nq_i \hat{s}_i / r_{i,l_i} \right\rfloor = f'_k.$$

By the definition of S, since $\hat{s} \in S$, the vector $\hat{t} \in S'$ where

$$\hat{t}_k = \sum_{i:h_{i,l_i}=k} q_i \hat{s}_i / r_{i,l_i}.$$

Therefore, by assumption 3 of Section 3, $f'_k \in S'_n$ and hence, $f_k \in S_n$. Thus, the fixed channel assignment algorithm F is feasible.

To compute $\lim_{n\to\infty} p_i^{F,n}$, we use induction on l_i, the length of call-type i. For $l_i = 1$,

$$\begin{aligned}
\lim_{n\to\infty} p_i^{F,n} &= \lim_{n\to\infty} \frac{1}{n}\tau(nq_i, \lfloor nq_i\hat{s}_i/r_{i,l_i}\rfloor) \\
&= \min(q_i, q_i\hat{s}_i/r_{i,l_i}) \qquad \text{(using the ATP)} \\
&= q_i\hat{s}_i/r_{i,l_i} \qquad (\text{since } \hat{s} \in U(R),\ \hat{s}_i \le r_{i,l_i}).
\end{aligned}$$

For $l_i > 1$, let $h_k \preceq h_i$ and let $l_k = l_i - 1$.

$$\begin{aligned}
\lim_{n\to\infty} p_i &= \lim_{n\to\infty} \frac{1}{n}\tau(n\varrho_i^{F,n}, \lfloor nq_i\hat{s}_i/r_{i,l_i}\rfloor) \\
&= \lim_{n\to\infty} \frac{1}{n}\tau(np_k^{F,n}q_i/q_k, nq_i\hat{s}_i/r_{i,l_i}) \qquad \text{(by Lemma 7)} \\
&= \min((q_k\hat{s}_k/r_{k,l_k})q_i/q_k, q_i\hat{s}_i/r_{i,l_i}) \qquad \text{(induction hypothesis and the ATP)} \\
&= q_i\hat{s}_i/r_{i,l_i} \qquad (\text{since } \hat{s} \in U(R),\ \hat{s}_k/r_{k,l_k} \ge \hat{s}_i/r_{i,l_i}).
\end{aligned}$$

∎

The following theorem generalizes Theorem 4.1 of [1] to cellular systems with handoffs.

Theorem 2. *If the traffic statistics satisfy the ATP, there exists a channel assignment algorithm F, such that,*

$$\lim_{n\to\infty} \frac{1}{n} T_S^{F,n}(R) = T_S(R),$$

where $T_S^{F,n}(R)$ is the delivered traffic for the channel assignment algorithm F.

Proof. Let $\hat{s} \in S \cap U(R)$. Consider a fixed channel assignment algorithm F that allocates $nq_i\hat{s}_i/r_{i,l_i}$ channels to cell h_{i,l_i} for calls of type i and its extensions. For this channel assignment algorithm,

$$\begin{aligned}
T_S^{F,n}(R) &= n\sum_{ij} d_{ij}^{F,n} \\
&= n\sum_i c_{i,l_i}^{F,n} r_i/r_{i,l_i} \qquad (\text{using Lemma 2 and } \textstyle\sum_j r_{ij} = r_i) \\
&= n\sum_i (p_i^{F,n} r_{i,l_i}/q_i) r_i/r_{i,l_i} \qquad \text{(using Lemma 5)}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} T_S^{F,n}(R) = \sum_i \hat{s}_i r_i/r_{i,l_i} \qquad \text{(using Lemma 8)}.$$

Since this is valid for arbitrary $\hat{s} \in S \cap U(R)$ we can choose $\hat{s} \in S \cap U(R)$ to maximize $\sum_i \hat{s}_i r_i/r_{i,l_i}$. For this particular choice of $\hat{s}$,

$$\lim_{n\to\infty} \frac{1}{n} T_S^{F,n}(R) = T_S(R).$$

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Consider the 3-cell example and a fixed, greedy channel assignment algorithm that allocates $3n/5$ channels to each of the end-cells (2 and 3) and $2n/5$ channels to the central cell (1) with the restriction that the channels in the end-cells are to be used only for new calls arising in that cell and of the $2n/5$ channels allocated to the central cell, $n/5$ are to be used exclusively for calls handed off from cell 2 and $n/5$ for calls handed off from cell 3. All calls originating in cell 1 are blocked. If $r \geq 12/5$ and if the traffic statistics satisfy the ATP, the fraction of dropped calls will be zero asymptotically and both the carried and delivered traffic for this system approach $8n/5$ which is the best achievable delivered traffic by any channel assignment algorithm for this system (from Theorem 1).

The reader should compare this channel assignment algorithm with the fixed channel assignment algorithm for the static system that is asymptotically optimal for $r \geq 36/11$. This channel assignment algorithm allocates all n available channels to each of the end-cells and no channels to the central cell. The carried traffic for this channel assignment algorithm does approach $2n$ *even when there are handoffs* but the *delivered traffic* only approaches $4n/3 < 8n/5$.

7. Comments and Conclusions.

Note that the fixed channel assignment algorithm of Theorem 2 that is asymptotically optimal for large n is impractical, even if n is large, because it is a function of R. We do not know if there exist channel assignment algorithms, in general, which can be implemented without knowing the offered traffic R and which are asymptotically optimal for every value of R.

As a final point, instead of the delivered traffic, we could maximize any linear function of the carried traffic. Thus we could assign different rates of revenues to different types of calls as was done in [1]. We could also maximize the carried traffic itself which would correspond to assuming that the customer is not refunded for dropped calls (he pays for dropped calls for the time used). For this case, the *carried traffic function* for the 3-cell example is

$$T_S^{\text{carried}}(R) = \min(r, 11r/36 + 1, r/5 + 6/5, 2)$$

which is bounded below by the delivered traffic function $T_S(R)$ and above by the delivered/carried traffic function for the corresponding static system $T_S^{\text{static}}(R)$. It is interesting to note that this function is different from both of them.

Acknowledgements.

I thank Rajiv Ramaswami for his comments on improving the presentation of this paper.

References.

[1] McEliece, R. J., and K. N. Sivarajan, "Maximizing Marginal Revenue in Generalized Blocking Service Networks," *Proc. 1992 Allerton Conference on Communication, Control and Computing.*

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