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Controlling Chaos by Synchronization

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We show how synchronization *à la* Pecora and Carroll allows one to control a seemingly large class of dynamical systems, and, in particular, chaotic ones. The method we propose permits one to stabilize periodic orbits and equilibria, and to target them. In the case that the dynamics are described by differential equations, the method does not require one to consider a first return map.

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Let

$$\begin{aligned} x' &= f(x,y)\,, \\ y' &= g(x,y)\,, \end{aligned} \tag{1}$$

be evolution equations of some smooth dynamical system $\mathcal{S}$, where x and y are respectively m and n-dimensional vectors, and where the prime stands either for differentiation with respect to time (in the case of a vector field) or for the next value (in the case of a map). With t standing for either continuous or discrete time (according to whether we deal with a vector field or a map), and $X(t)$ standing for some m-dimensional smooth function of time, we can construct a time-dependent n-dimensional evolution equation

$$Y' = g(X(t), Y)\,. \tag{2}$$

Let Y_α be an arbitrary initial condition and $Y_\alpha(t)$ be the corresponding solution for (2). If for every pair (Y_α, Y_β), $\parallel Y_\alpha(t) - Y_\beta(t) \parallel \to 0$ as $t \to \infty$ when $(X(t), \cdot)$ is a solution of (1), we say that the evolution equations are *synchronizing* (with *synchronizing variable* x) [1]. If for every pair (Y_α, Y_β), $\parallel Y_\alpha(t) - Y_\beta(t) \parallel \to 0$ as $t \to \infty$ for any $X(t)$, we say that the evolution equations are *absolutely synchronizing* (with *absolutely synchronizing variable* x) [2]. We say that the dynamical system $\mathcal{S}$ is (*absolutely*) *synchronizable* if its evolution equations in some coordinate system are (absolutely) synchronizing. In some cases, like for the dynamical system defined by the Rössler equations that we use below as an example, one might need to restrict the dynamics to some subset of the phase space in order to get a synchronizable dynamical system.

Our goal in this letter is to explain how synchronization allows one to control (chaotic) dynamical systems.

For tutorial reasons, let us first examine what happens if one tries to impose some solution $(x_0(t), y_0(t))$ of (1) by building the system

$$\begin{aligned} U' &= f(x_0(t), V)\,, \\ V' &= g(x_0(t), V)\,. \end{aligned} \tag{3}$$

Assuming $\mathcal{S}$ is synchronizable with synchronizing variable x, for every initial condition V_α we get $\parallel V_\alpha(t) - y_0(t) \parallel \to 0$ as $t \to \infty$, so that also $\parallel U'(t) - x_0'(t) \parallel \to 0$ as $t \to \infty$. However, we can only hope for

$$\parallel U(t) - x_0(t) \parallel \to C \quad \text{as} \quad t \to \infty\,, \tag{4}$$

where the constant C accounts for the initial discrepancy $\| U(0) - x_0(0) \|$ and for the fact that $V_\alpha(t)$ is not $y_0(t)$, with a possibly large difference initially. To cure this asymptotic difference between $U(t)$ and $x_0(t)$, which is clearly unbearable for control purposes (even before we invoke the presence of noise in any natural system), we need to restrict ourselves to a special class of synchronizable dynamical systems. Inspired by the work of Cuomo and Oppenheim [3] (see also [4] and references therein), we say that the equations (1) are *resynchronizing* if x is a synchronizing variable and y is an absolutely synchronizing variable. For instance, (1) is resynchronizing if one of the coordinates of x is synchronizing and one of the coordinates of y is absolutely synchronizing, so that the Lorenz equations are resynchronizing [1, 2, 3].

As an example of the difference between synchronizability and resynchronizability, let us mention that while an open dense subset of the linear maps or flows in $\mathbb{R}^n$, $n > 1$, are synchronizable by a linear change of variable [2], the same methods from [2] allows one to get the following results:

Theorem 1 *An open dense subset of the volume contracting linear flows in $\mathbb{R}^n$, $n > 1$, are resynchronizing after a linear change of variables. A linear flow in $\mathbb{R}^n$, $n > 1$ which is not volume contracting cannot be transformed to a resynchronizing one by a linear change of variables.*

Theorem 2 *An open dense subset of the linear maps in $\mathbb{R}^n$, $n > 1$, whose trace has absolute value strictly smaller than n are resynchronizing after a linear change of variable. A linear map in $\mathbb{R}^n$, $n > 1$, whose trace T satisfies $|T| \geq n$ cannot be transformed to a resynchronizing one by a linear change of variables.*

Assuming now that (1) is resynchronizing, and looking at the system

$$\begin{aligned} W' &= f(W, V)\,, \\ V' &= g(x_0(t), V)\,, \end{aligned} \tag{5}$$

we see that:

$$\begin{cases} \| V(t) - y_0(t) \| \to 0 \quad \text{as} \quad t \to \infty \quad \text{since } x \text{ is synchronizing}\,; \\ \| W(t) - x_0(t) \| \to 0 \quad \text{as} \quad t \to \infty \quad \text{since } y \text{ is absolutely synchronizing}\,. \end{cases}$$

Notice that the convergence we claimed about $\| W(t) - x_0(t) \|$ depends, under standard regularity assumptions, on the fact that $W' = f(W, V)$ admits $W' = f(W, y_0(t))$ as a limiting equation in the sense of [5, 6].

It can then be seen that the requirement we made that y be an absolutely synchronizing variable is stronger than needed, and in fact does not hold in the example presented below. Since, on the other hand, it is not sufficient to assume that y is a synchronizing variable, we chose to present the theory with a sufficient, stronger property already introduced in [2]; a precise discussion of such matters is beyond the scope of this letter. We will thus redefine as *resynchronizing* the pairs (x, y) of variables which are synchronizing enough to get both $\| V(t) - y_0(t) \| \rightarrow 0$ and $\| W(t) - x_0(t) \| \rightarrow 0$ as $t \rightarrow \infty$. For some systems, as in the example below, we could not prove the resynchronizing property, but merely observed it experimentally.

It is clear that the forcing which has to be imposed to stabilize a given orbit is small if and only if the actual trajectory is close to the desired one and in phase with it. This forcing is made apparent by rewriting (5) as

$$\begin{aligned} W' &= f(W, V)\,, \\ V' &= g(W, V) + F(t)\,, \end{aligned} \tag{6}$$

with

$$F(t) = g(x_0(t), V) - g(W, V)\,. \tag{7}$$

If the desired orbit has a dense stable manifold and the simplest use of (7), where $x_0(t)$ specifies the desired orbit, requires a forcing considered as being too big, one can impose appropriate trajectories on this manifold to target the desired orbit. There is then a tradeoff between the length of the branch of stable manifold one wants to use and the maximal strength of forcing one is willing to impose. In particular, for some applications where one wants to stabilize a periodic orbit whose phase is irrelevant, one can use a feedback system allowing one to target the desired orbit more efficiently.

The value for control of the simple observations reported here depends primarily on one's ability to find resynchronizing coordinate systems. A characterization of the relative abundance of resynchronizable dynamical systems among the dissipative systems in $\mathbb{R}^n$ has so far eluded our investigations. However, we notice that we can find resynchronizing coordinates (in some reasonable subset of their phase space) for some systems which previously were not even recognized as being synchronizable [7], as in the example given below. When the resynchronizing coordinates are not the same as the ones that the system was originally presented in, then we can rewrite the forcing $F(t)$ in terms of the original variables. Changing variables in order to simplify the control is a common strategy when working with nonlinear control problems [8].

As an example we consider the Rössler equations

$$
\begin{aligned}
\dot{x} &= -(y+z)\,, \\
\dot{y} &= x+ay\,, \\
\dot{z} &= b+z(x-c)\,,
\end{aligned}
\tag{8}
$$

with $a = b = 0.2$ and $c = 4.7$, which were shown to be synchronizable in [2] (see also [7]). With a random search, we find the change of variables

$$
\begin{aligned}
u &= 0.385x+0.536y-0.940z\,, \\
v &= 0.908x+0.233y+0.641z\,, \\
w &= -0.518x-0.974y-0.554z\,,
\end{aligned}
\tag{9}
$$

such that the pair $(u,(v,w))$ is resynchronizing. Figure 1 illustrates how one reaches the close neighborhood of a periodic orbit by imposing the $u_0(t)$ of this orbit (up to the error in the approximation), while in Figure 2, starting from the same initial condition on v and w, one imposes $u_0(t+\tau)$ were τ is adjusted to diminish the initial forcing. It is interesting to compare the time series of the forcing needed in both cases. We note that in most practical cases the forcing will become small but will not converge to zero because in practice one often only inputs an approximation to a true orbit.

A precise and/or systematic comparison between the methods proposed in this paper and the so-called OGY method [9] is beyond the scope of this letter. We just emphasize here that the synchronization-based method that we propose does not necessarily require one to compute first return maps, except possibly for the purpose of accelerated targeting as we mentioned earlier.

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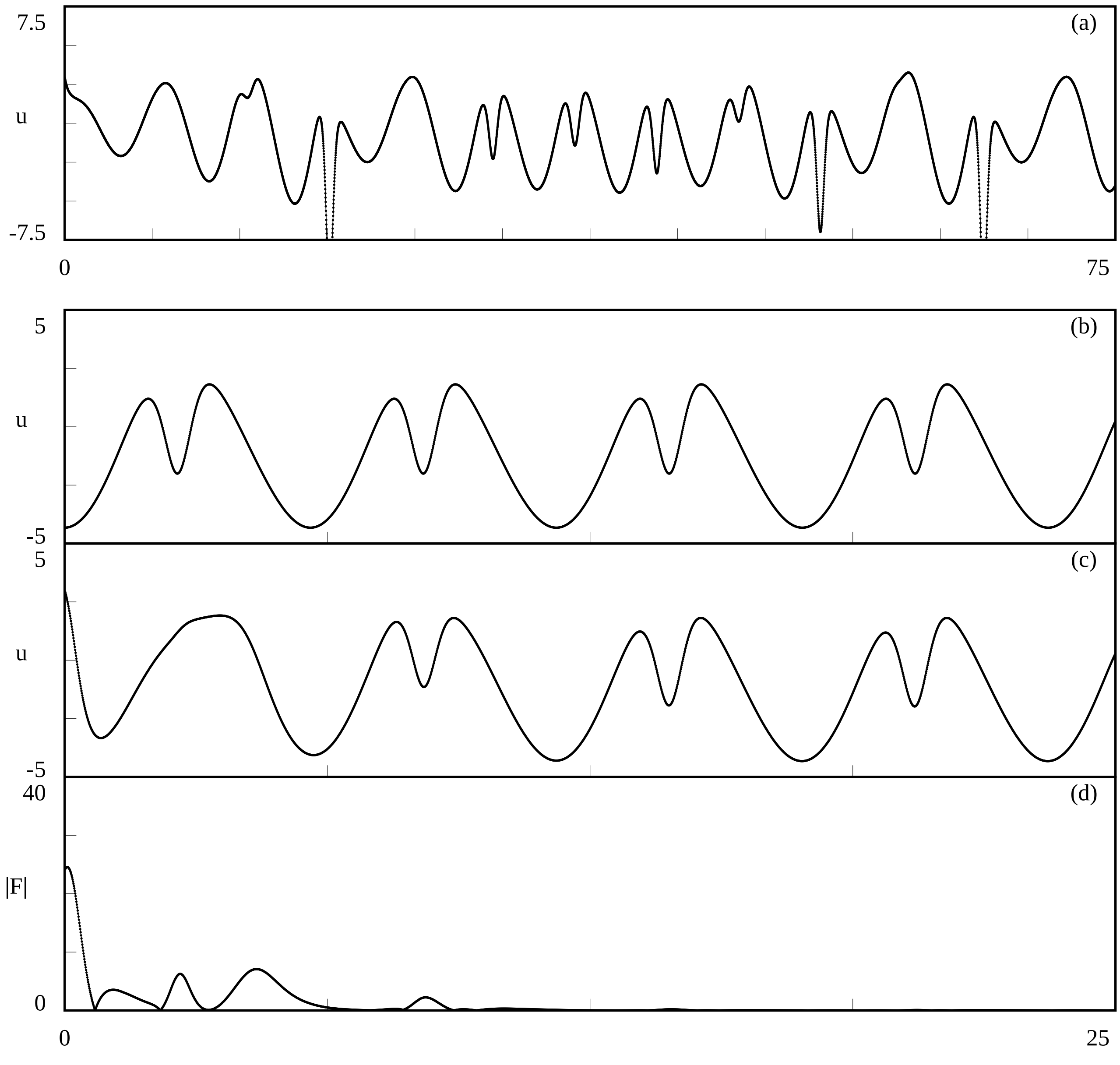

Figure 1: (a) Time series of one coordinate of a typical trajectory of the Rössler system; (b) Time series of one coordinate of a periodic orbit which we will stabilize; (c) Time series of one coordinate from the same initial condition as (a) with the control on; (d) Time series of the magnitude of the forcing corresponding to (c).

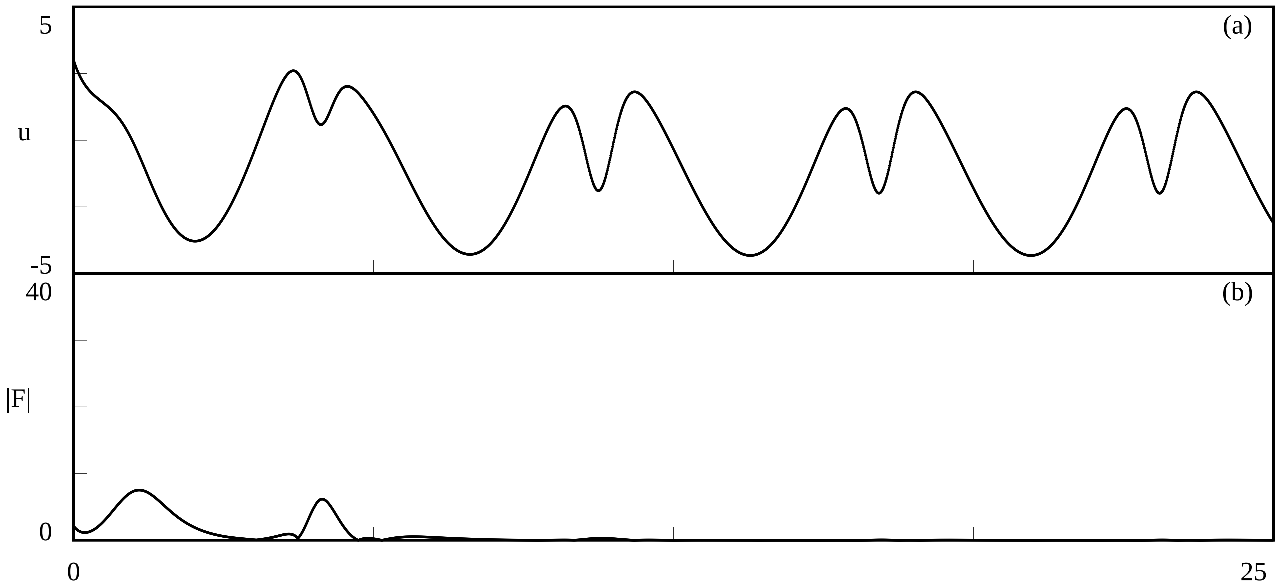

Figure 2: Time series as in Figures 1(c) and 1(d) after choosing the phase of the input so as to diminish the forcing.