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Mourad Baïou

Laboratoire SPO, Département d'Informatique,
Université de Bretagne Occidentale, 6 Avenue le Gorgeu,
B. P. 809, 29285 Brest Cedex, France.

Francisco Barahona

IBM Research Division
T.J. Watson Research Center
P.O. Box 218
Yorktown Heights, New York

Ali Ridha Mahjoub

Laboratoire SPO, Département d'Informatique,
Université de Bretagne Occidentale, 6 Avenue le Gorgeu,
B. P. 809, 29285 Brest Cedex, France.

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Mourad Baïou[1]
Francisco Barahona[2]
Ali Ridha Mahjoub[1]

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[1] Laboratoire SPO, Département d'Informatique, Université de Bretagne Occidentale, 6 Avenue le Gorgeu, B. P. 809, 29285 Brest Cedex, France.

[2] IBM T. J. Watson Research Center, Yorktown Heights, NY 10598.

Abstract

Given a graph $G = (V, E)$ with nonnegative weights $x(e)$ for each edge e, a partition inequality is of the form $x\big(\delta(S_1, ..., S_p)\big) \geq ap + b$. Here $\delta(S_1, ..., S_p)$ denotes the multicut defined by a partition $S_1, ..., S_p$ of V. We give a polynomial algorithm for the associated separation problem. This is based on an algorithm for finding the minimum of $x\big(\delta(S_1, ..., S_p)\big) - p$ that reduces to minimizing a symmetric submodular function. This is handled with the recent algorithm of Queyranne.

Key words. Partition inequalities, separation problem, submodular functions, k-connected subgraphs.

AMS subject classifications. 90C27, 05C85.

1. Introduction

Let $G = (V, E)$ be a graph with edge weights $x(e) \geq 0$ for all $e \in E$. Given a partition $S_1, ..., S_p$ of the node set V, we denote by $\delta(S_1, ..., S_p)$ the set of edges with endnodes in different sets of the partition. We use $\delta(S)$ instead of $\delta(S, V \backslash S)$ and we use $x(T)$ to denote $\sum_{e \in T} x(e)$.

Given a and b, an inequality of the type

$$x\big(\delta(S_1, ..., S_p)\big) \geq ap + b \tag{1.1}$$

is called a *partition inequality.* We want to find a violated partition inequality if there is any. This is called the *separation problem.*

Partition inequalities define the dominant of the spanning tree polytope, with $a = 1$ and $b = -1$. As we shall see later, they arise as valid inequalities or facets for combinatorial problems related to k-connectivity.

Without loss of generality we can assume that $a = 1$. The problem with $b \leq -1$ seems to be easier than the case $b > -1$. In order to solve this second case we had to develop a polynomial algorithm for the problem

$$\text{minimize } x\big(\delta(S_1, ..., S_p)\big) - p \tag{1.2}$$

among all partitions with $p \geq 2$. This will also give lower and upper bounds for the k-cut problem that is NP-Hard.

This paper is organized as follows. Sections 2 and 3 are devoted to the case $b = -1$. Section 4 describes Cunningham's approach to this same case. In Section 5 we show that problem (1.2) reduces to minimizing a certain submodular function. In Section 6 we show how Queyranne's algorithm for minimizing symmetric submodular functions applies to our problem. In Section 7 we discuss some applications of these methods. In Section 8 we survey partition inequalities.

2. The case $b \leq -1$

The first separation algorithm for this case was given by Cunningham [9], he reduced this to $|E|$ minimum cut problems. We describe his approach in Section 4. A reduction to $|V|$ min-cut problems was given in [2]. We present this method below, this will be used as a subroutine in the following sections.

Given a graph $G = (V, E)$ the spanning tree polytope $T(G)$ is the convex hull of incidence vectors of spanning trees of G, its dominant is the polyhedron $P(G) = T(G) + \mathbb{R}^E_+$ obtained by adding the nonnegative orthant. It has been proved in [26, 21] that $P(G)$ is defined by

$$\begin{aligned} & x(\delta(V_1, ..., V_p)) \geq p - 1, \text{ for every partition of } V, && (2.1) \\ & x \geq 0. \end{aligned}$$

Jünger and Pulleyblank [19] have given an extended formulation for $P(G)$ as follows. Associate the variables x with the edges and the variables y with the nodes. The system below defines a polyhedron whose projection onto the variables x is $P(G)$. The node r is an arbitrary element of V.

$$x(\delta(S)) + y(S) \geq 2, \text{ if } r \notin S, S \subset V, \tag{2.2}$$

$$x(\delta(S)) + y(S) \geq 0, \text{ if } r \in S, S \subset V, \tag{2.3}$$

$$y(V) = 0. \tag{2.4}$$

So given a vector $\bar{x}$ we can try to find a vector $\bar{y}$ such that $(\bar{x}, \bar{y})$ satisfies (2.2), (2.3), (2.4), or prove that $\bar{y}$ does not exist.

Let

$$f(S) = \begin{cases} 2 - \bar{x}\,(\delta(S)), & \text{if } r \notin S, \\ -\bar{x}\,(\delta(S)), & \text{if } r \in S, \end{cases}$$

for $\emptyset \neq S \subseteq V$. The function $-f$ is submodular.

We are going to solve

$$\left.\begin{aligned} & \text{minimize } y(V) \\ & \text{subject to} \\ & y(S) \geq f(S), \text{ for } S \subseteq V. \end{aligned}\right\} \tag{2.5}$$

As shown by Edmonds [12], the greedy algorithm solves this linear program. This algorithm produces also an optimal solution of the dual problem. This will give us a most violated partition inequality, if there is any. The dual problem is

$$\left.\begin{array}{l}\text{maximize } \sum z_S\, f(S)\\ \text{subject to}\\ \sum\{z_S | u \in S\} = 1, \text{ for all } u \in V,\\ z \geq 0.\end{array}\right\} \tag{2.6}$$

Given a vector y satisfying (2.5), a set S is called *tight* if $y(S) = f(S)$. The function $y(\cdot) - f(\cdot)$ is nonnegative and submodular. So if S and T are tight, and $S \cap T \neq \emptyset$, then $S \cap T$ and $S \cup T$ are also tight.

We start with $y(v_i) = 2\ \forall i$, and decrease the value of each $y(v_i)$ until a set becomes tight. We denote by $\mathcal{F}$ the family of tight sets with a positive dual variable. If we try to add S to $\mathcal{F}$ and there is a set $T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then we replace S and T by $S \cup T$. This is also tight.

Let $V = \{v_1, ..., v_n\}$, the algorithm is below.

Step 0. Set $\bar{y}(v_i) \leftarrow 2$ for $i = 1, ..., n$; $k \leftarrow 1$; $\mathcal{F} \leftarrow \emptyset$.

Step 1. If v_k belongs to a set in $\mathcal{F}$ go to step 3, otherwise
set $\alpha \leftarrow f(\bar{S}) - \bar{y}(\bar{S}) = \max\{f(S) - \bar{y}(S) \,|\, v_k \in S\}$,
$\bar{y}(v_k) \ \leftarrow\ \bar{y}(v_k) + \alpha$,
$\mathcal{F} \leftarrow \mathcal{F} \cup \{\bar{S}\}$.

Step 2. While there are two sets S and T in $\mathcal{F}$ with $S \cap T \neq \emptyset$ do
$\mathcal{F} \leftarrow (\mathcal{F} \backslash \{S, T\}) \cup \{T \cup S\}$.

Step 3. Set $k \leftarrow k + 1$, if $k \leq n$ go to Step 1, otherwise stop.

The vector $\bar{y}$ is built so it satisfies (2.5), the family $\mathcal{F}$ defines a partition of V and $\bar{y}(S) = f(S)$ for every $S \in \mathcal{F}$. We set $\bar{z}_S = 1$, if $S \in \mathcal{F}$, and $\bar{z}_S = 0$ otherwise. We have

$$\bar{y}(V) = \sum\{\bar{y}\,(S) \,|\, S \in \mathcal{F}\} = \sum\{f(S) \,|\, S \in \mathcal{F}\} = \sum\{f(S)\,\bar{z}_S \,|\, S \subseteq V\}.$$

This proves that $\bar{y}$ and $\bar{z}$ are optimal solutions.

If the value of the optimum is 0 then $(\bar{x}, \bar{y})$ satisfies (2.2), (2.3), (2.4), in this case we can pick any partition of V into $V_1, ..., V_p$, add the inequalities in (2.2), (2.3), associated with the sets $\{V_i\}$ and $-y(V) \geq 0$, we obtain a partition inequality. This shows that $\bar{x}$ satisfies all the partition inequalities.

Now assume that the value of the optimum is greater than 0. Let $\tilde{z}$ be a 0-1 vector that satisfies the equations of (2.6), the family $\mathcal{G} = \{S \mid \tilde{z}_S = 1\} = \{S_1, ..., S_p\}$ gives a partition of the set V and

$$\sum f(S)\, \tilde{z}_S = 2(p-1) - 2\,\bar{x}\,(\delta(S_1, ..., S_p)),$$

since $\tilde{z}$ is an optimum of (2.6) it gives a most violated partition inequality. Because of this we have a solution of the separation problem not only for $b = -1$, but also for any $b \leq -1$.

3. Finding tight sets

It remains to show how to compute the number α in step 1. Construct a directed graph $D = (N, A)$, where $N = V \cup \{s, t\}$ and $A = \{(i,j), (j,i) \mid ij \in E\} \cup \{(s,i), (i,t) \mid i \in V\}$. Define

$$\begin{aligned} &\eta(i) = \bar{y}(i), \text{ for } i \in V, i \neq r, \\ &\eta(r) = \bar{y}(r) + 2, \end{aligned}$$

define capacities

$$\begin{aligned} &c(s,i) = -\eta(i),\ c(i,t) = 0, \quad \text{if } \eta(i) < 0,\ i \neq v_k,\ i \in V, \\ &c(i,t) = \eta(i),\ c(s,i) = 0, \quad \text{if } \eta(i) \geq 0,\ i \neq v_k,\ i \in V, \\ &c(s, v_k) = \infty,\ c(v_k, t) = \eta(v_k), \\ &c(i,j) = c(j,i) = \bar{x}(i,j), \quad \text{for } ij \in E. \end{aligned}$$

If $\{s\} \cup T$ induces a cut separating s from t that has capacity λ, we have

$$\bar{y}(T) + \bar{x}(\delta(T)) = \begin{cases} \lambda + \sum\{\eta(v) \mid \eta(v) < 0\} - 2, & \text{if } r \in T \\ \lambda + \sum\{\eta(v) \mid \eta(v) < 0\} & \text{if } r \notin T. \end{cases}$$

Therefore if β is the minimum capacity of a cut separating s from t, then the value α is

$$2 - \beta - \sum\{\eta(v) \mid \eta(v) < 0\}.$$

Suppose now that we have solved problem (2.5), and that we add a new vertex to the graph. We want to show that solving (2.5) again takes just one min-cut calculation. This will be used in the later sections.

Suppose that $\bar{y}$ is the solution of (2.5), for the graph $G = (V, E)$. Suppose that we add the vertex w. Define

$$\bar{\bar{y}}(v) = \bar{y}(v) - \bar{x}(wv),$$

for all $v \in V$, and $\bar{\bar{y}}(w) = 2$. It is easy to see that $\bar{\bar{y}}$ satisfies the inequalities of (2.5), and every set that was tight before will remain tight. Thus the only component that can be modified by the greedy algorithm is $y(w)$, this takes one minimum cut problem.

4. Cunningham's approach

Cunningham [9] used the method below for the case $b \leq -1$. Let f be the rank function of the graphic matroid. It satisfies

$$f(\emptyset) = 0, \tag{4.1}$$

$$f(A) \leq f(B) \text{ if } A \subseteq B \subseteq E \tag{4.2}$$

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \text{ if } A, B \subseteq E. \tag{4.3}$$

Then he proposes to solve

$$\left.\begin{aligned} &\text{maximimize } y(E) \\ &\text{subject to} \\ &y(S) \leq f(S) \text{ if } S \subseteq E, \\ &y \leq x. \end{aligned}\right\} \tag{4.4}$$

For $A \subseteq E$, let $\bar{A} = E \backslash A$. We have

$$y(E) = y(A) + y(\bar{A}) \leq x(A) + f(\bar{A}).$$

Let $\bar{y}$ be obtained by applying the greedy algorithm to (4.4). Then for any $e \in E$ either $y(e) = x(e)$ or e is in a tight set. Let $\bar{A}$ be the union of the tight sets. This is also tight, by submodularity. Thus

$$\bar{y}(E) = \bar{y}(A) + \bar{y}(\bar{A}) = x(A) + f(\bar{A}).$$

This shows that the solution of (4.4) gives the minimum of

$$x(A) + f(\bar{A}).$$

By taking $A = \emptyset$, we have that the minimum is at most $n-1$. So it is equivalent to look for the minimum of

$$x(A) + f(\bar{A}) - (n-1) = x(E) - (n-1) + f(\bar{A}) - x(\bar{A}) \tag{4.5}$$

that is nonpositive.

It is easy to see that it is enough to take sets $S = \gamma(B)$ in (4.4). So $\bar{A}$ is the union of sets of type $\gamma(B)$. We are going to obtain the minimum of

$$\sum_{i=1}^{p} \Big[(|S_i| - 1) - x(\gamma(S_i))\Big] + x(E) - (n-1). \tag{4.6}$$

By taking $S_{p+i} = \{v_i\}$ for $v_i \notin S_j$, $1 \leq j \leq p$, we have

$$x(\delta(S_1, ..., S_k)) - (k-1). \tag{4.7}$$

Applying the greedy algorithm to (4.4) would require $|E|$ iterations. Each time finding a tight set reduces to solving a min-cut problem, so this method requires $|E|$ min-cut problems, cf. [9].

5. The case $b > -1$

We could not find a simple extension of the methods in the preceding sections to the case $b > -1$. When we tried to modify them we would lose submodularity. Finally we decide to study the problem

$$\text{minimize } x(\delta(S_1, ..., S_p)) - p, \tag{5.1}$$

with $p \geq 2$. We call this the *multicut problem.*

This is equivalent to minimize

$$g(S) = x(\delta(S)) - 2 + \min \left\{ x(\delta(T_1, ..., T_k)) - (k-1) \right\},$$

where $S \subset V$ and $\{T_i\}$ is a partition of S. Notice that $\{T_i\}$ could be the trivial partition, i. e. $k = 1$. We want to prove that g is submodular. For that all we need is to prove that

$$h(S) = \min \left\{ x(\delta(T_1, ..., T_k)) - (k-1) \right\}$$

is submodular (as a function of S). As we saw in Section 4, this is $1 - |S|$ plus the value of the optimum of (4.4), where $E = \gamma(S)$.

So we have to consider (4.4) for $E = \gamma(S)$, $E = \gamma(T)$, $E = \gamma(S \cup T)$ and $E = \gamma(S \cap T)$. Suppose that $\bar{y}$ was obtained after applying the greedy algorithm for $E = \gamma(S \cap T)$. We can extend $\bar{y}$ to a solution for $E = \gamma(S \cup T)$, denote this by $\bar{\bar{y}}$. Now denote by $\bar{\bar{y}}_S$ the vector obtained from $\bar{\bar{y}}$ by setting to zero all components not in $\gamma(S)$. We can extend $\bar{\bar{y}}_S$ to a solution for $E = \gamma(S)$. We can proceed similarly for $E = \gamma(T)$. This shows that

$$h(S \cup T) + h(S \cap T) \leq h(S) + h(T).$$

We have proved that the minimum of (5.1) can be obtained by minimizing the submodular function g.

Queyranne [23] gave an algorithm that takes $O(n^3)$ evaluations of f when this is submodular and symmetric. This means that $f(S) = f(\bar{S})$ for all $S \subseteq V$. Since g is not symmetric, we define $g'(S) = h(S) + {}^1\!/_2 x(\delta(S))$ and look for the minimum of $g'(S) + g'(\bar{S})$.

6. Queyranne's algorithm

For a symmetric submodular function f, Queyranne [23] gave the algorithm below, it generalizes the minimum cut algorithm given by Stoer and Wagner [25] and Frank [14]. Following Queyranne's notation we use $S + u$ to denote $S \cup \{u\}$.

Algorithm A

Step 0. Start with $W_0 = \emptyset$, $i = 0$.

Step 1. For all $u \notin W_i$ set $k(u) = f(W_i + u) - f(u)$. Let $k(u_{i+1}) = \min\{k(u)\}$.

Step 2. Set $W_{i+1} \leftarrow W_i + u_{i+1}$, set $i \leftarrow i + 1$. If $i = n$ stop, otherwise go to Step 1.

He proved that

$$f(u_n) = \min\{f(S) \mid S \text{ separates } u_n \text{ and } u_{n-1}\}, \tag{6.1}$$

The next step is to identify u_n and u_{n-1}, apply Algorithm A, and continue until we are left with two elements.

The proof of (6.1) relies on the Lemma below.

Lemma 6.2. [23]. For $1 \leq i \leq n-1$, $y \in V \backslash W_i$ and $X \subseteq W_{i-1}$,

$$f(W_i) + f(y) \leq f(W_i \backslash X) + f(X + y).$$

□

If we apply this algorithm with a function f that is not symmetric, we obtain the minimum of $f(S) + f(\bar{S})$, this follows from Lemma 6.2. In our case we should use the function g' defined in the last section.

Each application of Algorithm A requires $O(n^2)$ evaluations of f. Since Algorithm A is used $n - 1$ times, we need $O(n^3)$ evaluations of f. In our case, one evaluation of the function g' would take $O(n)$ min-cut problems, thus the straightforward implementation of this method would require $O(n^4)$ min-cut problems. However each evaluation in Step 1 is of the type $f(W_i + u)$ where $f(W_i)$ is already known. We have seen at the end of Section 3 that this takes only one min-cut calculation. Thus the entire algorithm requires $O(n^3)$ min-cut problems.

7. Related problems

Let us turn our attention to ratio objectives. The problem

$$\text{minimize } \frac{x\big(\delta(S_1, \ldots, S_p)\big)}{p-1},$$

was studied in [9] and [5]. Chvátal [8] showed that the minimum of

$$\frac{x\left(\delta(S_1, ..., S_p)\right)}{p}$$

is given by a minimum cut. Now consider

$$\text{minimize } \frac{x\left(\delta(S_1, ..., S_p)\right) + a}{p + b}.$$

Cunningham [9] pointed out that his method could be extended to the case $b \leq a - 1$. The general case can be solved using Dinkelbach's algorithm [11] and problem (5.1) as follows . Given a partition $\{T_i\}$, compute

$$\lambda = \frac{x\left(\delta(T_1, ..., T_p)\right) + a}{p + b},$$

and look for the minimum of

$$x\left(\delta(S_1, ..., S_p)\right) + a - \lambda(p + b).$$

If this minimum is negative replace $\{T_i\}$ by the new partition, otherwise stop. This procedure takes at most n iterations cf. [9].

Now consider the *k-cut problem*. We want to minimize

$$x\left(\delta(S_1, ..., S_k)\right)$$

for a fixed value of k. Goldschmidt and Hochbaum [15] gave an $O(n^{k^2})$ algorithm, and they showed that it is NP-Hard if k is part of the input.

For any number λ we can obtain a lower bound

$$f(\lambda) = \min \left\{ x\left(\delta(S_1, ..., S_p)\right) - \lambda(p - k) \right\}, \tag{7.1}$$

and then look for the maximum of f. Then for any value of λ such that the solution of (7.1) has $p > k$, we can obtain an upper bound by shrinking some sets in the partition. It would be interesting to have some information about the quality of these approximations, but this is beyond the scope of this paper.

8. Applications of partition inequalities

Partition inequalities arise in problems involving connectivity requirements. As already mentioned, the dominant of the spanning tree polytope is defined by nonnegativity and

$$x(\delta(V_1, ..., V_p)) \geq p - 1,$$

for every partition of V, [26, 21]. These inequalities have been used inside an algorithm for the *Network Loading Problem* in [3].

Consider the problem of finding a minimum 2-edge connected spanning subgraph. In [20], a family of valid inequalities was introduced as follows. Consider a partition of V into $V_0, V_1, ..., V_p$, and let $F \subseteq \delta(V_0)$ with $|F| = 2k + 1$. If we add the inequalities

$$\begin{aligned} &x(\delta(V_i)) \geq 2, \;\; 1 \leq i \leq p, \\ &-x(e) \geq -1, \;\; e \in F, \\ &x(e) \geq 0, \;\; e \in \delta(V_0) \backslash F, \end{aligned}$$

we obtain

$$2x(\Delta) \geq 2p - 2k - 1.$$

Where $\Delta = \delta(V_1, ..., V_p) \backslash F$. Dividing by 2 and rounding up the right hand side we obtain

$$x(\Delta) \geq p - k. \tag{8.1}$$

If the sets V_i, $1 \leq i \leq p$, are singletons, the separation problem can be solved with the algorithm of Padberg and Rao [22]. If we fix the set F, and we do not restrict the cardinality of V_i, the separation problem can be solved with the method of Section 2.

Since the Travelling Salesman Polytope is a face of the 2-connected subgraph polytope, inequalities (8.1) are also valid for the TSP. We found interesting that we could solve to optimality some small Travelling Salesman problems using only (8.3), (8.2) and (8.1). In Figure 1 we display a fractional vector obtained when solving a TSP with 51 nodes from TSPLIB [24]. We chose a set F among the edges having the value 1 and found a violated partition inequality described below. The dashed lines indicate fractional values, the solid lines indicate the value 1. We draw with bold lines the edges in F. The nodes in the upper half belong to V_0, the other sets V_i are drawn with ellipses and circles. In this case the constraint is $x(\Delta) \geq 7$, however the present vector gives 6.5 for the left hand side. Computational results in the framework of a cutting plane algorithm will be presented in [1].

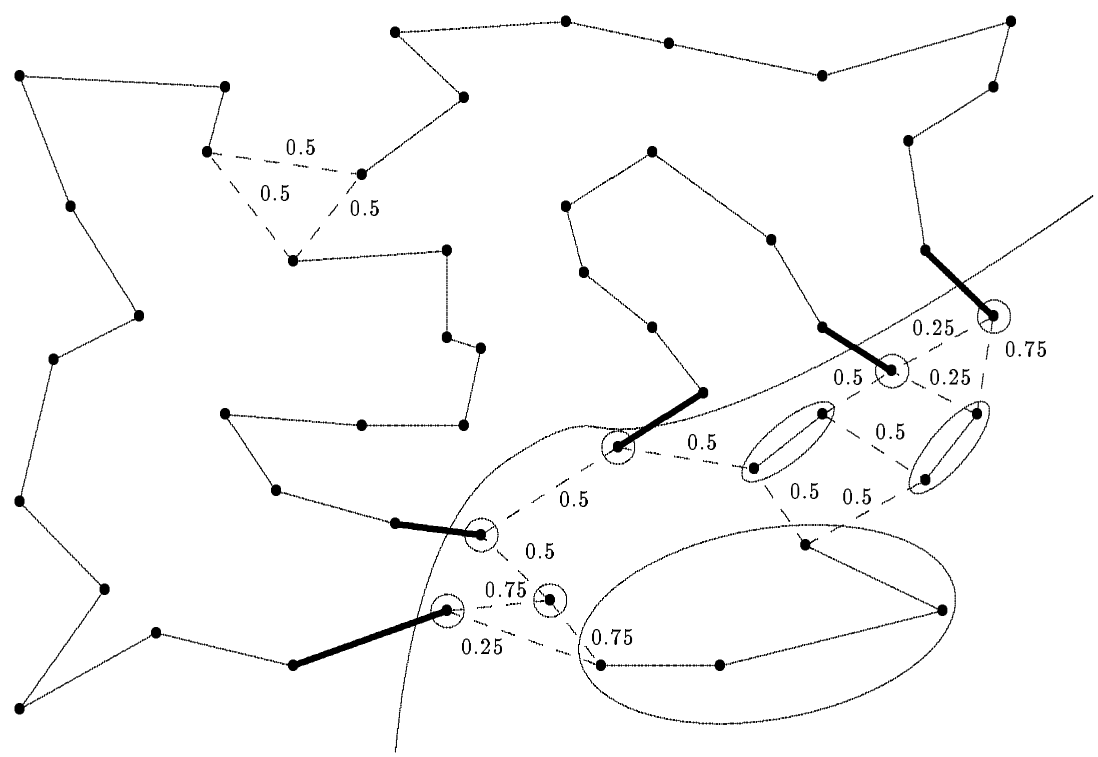

Figure 1.

In [17] Grötschel et *al* consider inequalities of the same type for a more general model. Inequalities (8.1) have also been studied in [4].

Fonlupt and Mahjoub [13] study the extreme points of the polytope $P(G)$ defined by

$$x(\delta(S)) \geq 2, \text{ for } S \subset V, \tag{8.2}$$

$$1 \geq x \geq 0. \tag{8.3}$$

They introduce the notion of *critical* noninteger extreme points of $P(G)$. Roughly speaking, a noninteger extreme point of $P(G)$ is critical if the set of edges with fractional values does not strictly contain the set of fractional edges of another extreme point of $P(G)$. They showed that any critical noninteger extreme point violates an inequality of the type (8.1), and this can be found in polynomial time.

Now consider minimum k-edge connected spanning subgraphs, for $k > 1$. Grötschel et al [17] introduced the constraints

$$x(\delta(V_1, ..., V_p)) \geq \lceil \frac{kp}{2} \rceil. \tag{8.4}$$

Also in [17] they showed that the separation problem for a more general type of partition inequalities is NP-Hard. Cutting plane algorithms along with computational results with these inequalities have been presented in [16, 18]. They used heuristics for the separation problem, and concluded that partition inequalities are very useful in this context. They also pointed out the need for better separation routines.

In [6] Chopra considers the minimum k-edge connected spanning subgraphs when multiple copies of an edge may be used. He introduces the following partition inequalities

$$x(\delta(V_1,\ldots,V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1 \tag{8.5}$$

for all partition $V_1,\ldots,V_p$ of V such that $G(V_i)$ is connected for $i = 1,\ldots,p$. These constraints are valid for the associated polyhedron $P_k(G)$ when the graph G is outerplanar and k is odd. He showed that in this case, $P_k(G)$ is defined by (8.2), (8.5) and nonnegativity. He also conjectured that this result remains true in the more general class of series-parallel graphs. Recently Didi Biha and Mahjoub [10] extended Chopra's result to series-parallel graphs. They showed that (8.2), (8.3) and (8.5) define the k-edge connected subgraph polytope for series-parallel graphs, for all k. The conjecture was also proved independently by Chopra and Stoer [7]. Inequalities (8.5) can be separated with the method of Section 5.

For k-node connected subgraphs the inequalities below have been presented in [17]. Let $Z \subset V$ be a node set with $|Z| = t \leq k-1$ Let $V_1,\ldots,V_p$ be a partition of $V \setminus Z$. Then the following inequalities are valid

$$x(\delta(V_1,...,V_p)) \geq \begin{cases} p-1 & \text{if } k-t=1, \\ \lceil \frac{p(k-t)}{2} \rceil & \text{if } k-t \geq 2. \end{cases}$$

If $k - t = 1$ they can be separated with the method of Section 2. If $k - t$ is positive and even, then they can be separated with the method of Section 5.

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