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Verification of the Equivalence of an Alternate Definition of the Lie Algebra of a Lie Group

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Verification of the Equivalence of an Alternate Definition of the Lie Algebra Multiplication of a Lie Group

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Abstract

The definition of the multiplication in the Lie algebra of a Lie group in terms of the Lie bracket of left invariant vector fields is completely standard. It is also possible to define a multiplication in more elementary terms by using second partials of the Lie group product. This report provides a verification of the equivalence of the multiplication obtained under these two definitions.

1 Introduction

Standard treatments of Lie groups and their Lie algebras like [1] or [2] take a Lie group as an intrinsically defined smooth manifold and define its Lie algebra as the vector space of left invariant vector fields with the Lie bracket as a multiplication. On the other hand, in [3] a Lie group is taken as a manifold smoothly imbedded in $\Re^n$ by using a result from [4] and its Lie algebra is defined to be the tangent space at the identity with a multiplication determined by mixed second partials of group products at the identity. In [5] in order to simplify the exposition we followed standard practice by taking a Lie group as an intrinsically defined smooth manifold, but followed the suggestion of [3] in defining the Lie algebra multiplication. Thus we could avoid discussing the possibility of imbedding Lie groups in high dimensional Euclidean spaces, left invariant vector fields, and Lie brackets. Due to the need to consider local charts, the definition of the multiplication became slightly more complicated than in [3].

The identification of the space of left invariant vector fields with the tangent space at the identity is completely standard, of course. However, [3] only provides a sketch of a proof that the algebra multiplication found there yields a Lie algebra, and does not considered whether this multiplication coincides with the standard Lie algebra multiplication. That they do coincide follows from a simple calculation, and so the proofs that the standard definition yields a Lie algebra suffice also for the definition in [3] or [5]. We present this calculation here.

2 The Calculation

We begin by presenting the multiplication used in [5], which is derived from [3]. Let G be an n-dimensional Lie group, and let $\{\phi, U\}$ be a local coordinate system for G, such that the identity e belongs to U and $\phi(e) = 0$. Since ϕ and group multiplication are C^∞ functions, there is an open set V in $\Re^n$ containing 0 such that

$$m : (u, v) \in V \times V \to m(u, v) = \phi(\phi^{-1}(u)\phi^{-1}(v)) \in \Re^n$$

is a well defined C^∞ function. Let m_i be the i^{th} component function of m, and define A^i to be the matrix whose jk^{th} entry is $\frac{\partial^2 m_i}{\partial u_j \partial v_k}$ at $(0,0)$, that is,

$$A^i_{jk} = \left. \frac{\partial^2 m_i}{\partial u_j \partial v_k} \right|_{(0,0)}$$

Let ∂_i be the tangent vector in $T_e(G)$, the tangent space to G at e, which operates on a function f, which is C^∞ at e, by $\partial_i f = \left. \frac{\partial f \circ \phi^{-1}}{\partial x_i} \right|_0$. Let $B = \{\partial_i\}_{i=1}^n$ be the basis of these tangent vectors, and let $T_e(G; B)$ be $T_e(G)$ represented in the basis B. Now define

$$[,]^* : (u, v) \in T_e(G) \times T_e(G) \to [u, v]^* \in T_e(G)$$

by defining its component functions in basis B as

$$[,]^*_i : (u, v) \in T_e(G; B) \times T_e(G; B) \to [u, v]^*_i = u^t(A^i - (A^i)^t)v \in T_e(G; B)\ .$$

We wish to show that the multiplication given by $[,]^*$ on $T_e(G)$ coincides with that obtained by identifying $T_e(G)$ with the space of left invariant vector fields and using the standard Lie algebra multiplication given by the Lie bracket. Since each multiplication is bilinear, it suffices to show that the two multiplications agree on the vectors of basis B. Moreover, we need only consider components of the products in basis B. To begin, let v_i be the left invariant vector field whose value at e is ∂_i. Left invariance implies that for any $g \in G$ and any function f, which is C^∞ at e,

$$v_i(g)f(h) = dl_g \circ \partial_i f(h) = \partial_i(f \circ l_g(h)) = \partial_i f(gh) = \left. \frac{\partial}{\partial v_i} f(g\phi^{-1}(v)) \right|_{v=0}\ ,$$

where l_g denotes left multiplication in G by g, h and v are dummy variables, and v_i is the i^{th} component of v. If we let $x_i = v_i \circ \phi$, then $\partial_j x_i = \left. \frac{\partial}{\partial v_j} v_i \circ \phi \circ \phi^{-1} \right|_{v=0} = \delta_{ij}$. It follows that $v(x_i) = a_i$ for any $v = \Sigma a_j \partial_j \in T_e(G)$, so that $v(x_i)$ is i^{th} coordinate of v in basis B. Thus using the standard identification of $T_e(G)$ with the space of left invariant vector fields and the Lie algebra multiplication given by the Lie bracket

$$
\begin{aligned}
&[\partial_j, \partial_k]_i = ([v_j, v_k](e))_i = [v_j, v_k](e)x_i \\
&= \partial_j(\tfrac{\partial}{\partial v_k} v_i \circ \phi(g\phi^{-1}(v))\Big|_{v=0}) - \partial_k(\tfrac{\partial}{\partial v_j} v_i \circ \phi(g\phi^{-1}(v))\Big|_{v=0}) \\
&= \tfrac{\partial}{\partial u_j}(\tfrac{\partial}{\partial v_k} v_i \circ \phi(\phi^{-1}(u)\phi^{-1}(v))\Big|_{v=0})\Big|_{u=0} - \tfrac{\partial}{\partial u_k}(\tfrac{\partial}{\partial v_j} v_i \circ \phi(\phi^{-1}(u)\phi^{-1}(v))\Big|_{v=0})\Big|_{u=0} \\
&= \tfrac{\partial}{\partial u_j}(\tfrac{\partial}{\partial v_k} m_i(u,v)\Big|_{v=0})\Big|_{u=0} - \tfrac{\partial}{\partial u_k}(\tfrac{\partial}{\partial v_j} m_i(u,v)\Big|_{v=0})\Big|_{u=0} = \tfrac{\partial^2 m_i}{\partial u_j \partial v_k}\Big|_{(0,0)} - \tfrac{\partial^2 m_i}{\partial u_k \partial v_j}\Big|_{(0,0)} \\
&= A^i_{jk} - A^i_{kj} = (A^i - (A^i)^t)_{jk} = (\partial_j)^t(A^i - (A^i)^t)\partial_k = [\partial_j, \partial_k]^*_i \ ,
\end{aligned}
$$

so that $[,]$ and $[,]^*$ agree on B component by component, which implies that $[,]$ and $[,]^*$ coincide on all of $T_e(G)$.

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