RC 20512 (7/26/96)
Computer Science/Mathematics 50 pages

IBM Research Report

Connected Lie and Symmetry Subgroups of the Rigid Motions: Foundations and Classification

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Abstract

Both in kinematics and in the geometric specification of parts the classification of a surface by the group of rigid motions that preserve the surface plays an important role. In each case the classification scheme is derived from a more fundamental classification of connected Lie subgroups of the rigid motions. This report provides careful statements and rigorous proofs of the underlying Lie-theoretic results, which lead to the classification schemes, and a cursory exposition of the requisite mathematical background.

1 Introduction

The classification of surfaces based on their symmetry arises naturally both in the kinematic representation of mechanisms and also in the geometric specification of parts. Simple observations about each of these application areas suffice to see how this interest in surface classification occurs.

First consider kinematics. Among other things, kinematics deals with mechanisms. A mechanism is an assembly of rigid parts coupled to each other by kinematic joints. A kinematic joint involves two parts and permits relative motion between the parts by maintaining contact between their boundary elements. A ball-and-socket joint is a typical example of a kinematic joint. If the mutual contact between the part boundaries during the relative motion occurs over a surface (as opposed to over a line or a point), then the kinematic pair is called a "lower-pair". Mechanisms containing lower-pair kinematic joints are widely used in practice and analyzed in engineering. It is generally believed in engineering community that the only possible lower-pairs are [1]:

1. *Spherical pair.* It allows rotations about a point.

2. *Planar pair.* It allows translations along the plane and rotation about an axis perpendicular to the plane.

3. *Cylindrical pair.* It allows rotation about the cylinder axis and a translation along the axis.

4. *Screw pair.* It allows a helical movement in terms of a fixed axial advance per revolution,

5. *Revolute pair.* Both contact surfaces are surfaces of revolution, generated by any profile. It allows a rotation about its axis.

6. *Prismatic pair.* Both contact surfaces are cylinders other than the right circular. It allows a translation parallel to a generatrix.

Now consider geometric part specification. Since all manufactured parts are subjected to variation, a designer must specify not only the nominal part geometry but also what he considers to be allowable variations from the nominal. Such specifications are also called geometric dimensioning and tolerancing. A fundamental building block in part geometric specifications is the notion of a surface feature, which is a finite portion of a surface on the boundary of a nominal part. The part geometry is then described by the intrinsic parameters (for example, a diameter) of these surface features and the relational parameters (such as in positioning a hole with respect to a plane) between the features. Often, some of the primitive features are combined to form a composite feature and for functional reasons are treated as one entity. While six parameters are both necessary and sufficient to uniquely position an arbitrary rigid body relative to some coordinate system in three dimensional space, a smaller number of parameters is required in general to position a particular surface relative to another surface (for example, positioning a cylindrical surface relative to a plane) due to inherent symmetries in these surfaces. This has led to a classification of surfaces [2], on the basis of symmetry, into:

1. Spherical surfaces,

2. Planar surfaces,

3. Cylindrical surfaces,

4. Helicoidal surfaces,

5. Surfaces of revolution,

6. Prismatic surfaces, and
7. Arbitrary surfaces.

How do only these surfaces arise? Why are the lists of both application areas so strikingly similar?

Although the settings and goals of kinematics and geometric part specification differ substantially, the mathematical foundations on which they rely are quite similar. Each models real physical bodies as idealized collections of points in an underlying three-dimensional Euclidean space, E^3. Each is concerned with geometric entities derived from these idealized objects and distances and angles between them. Except for geometrically significant points in the collections, in each case the points can be assumed to be undifferentiated. Euclidean motions, which are transformations of E^3 that preserve Euclidean geometry, are utilized in both areas. In kinematics, the motion of a body from its initial rest position to its final position can be represented by a smooth path of Euclidean motions commencing with the identity transformation. In geometrical part specification, some of the natural variability inherent in manufactured objects can be represented by classes of point sets in E^3 determined by smoothly perturbing the idealized model of the target design by Euclidean motions. In either case, since reflections are not meaningful in the context of real physical bodies, the allowable Euclidean motions are generally restricted to those which can be obtained by combinations of rotations about fixed points and translations, that is, to the rigid motions.

Because a rigid motion may leave a point set unchanged, not all rigid motions lead to nontrivial kinematic motion or design variability. Thus it would be natural that knowledge of the rigid motions leaving a point set unchanged would be of at least passing interest in each of the two application areas. In fact, in both some detailed knowledge of the set of all such transformations is critical. In kinematics it permits the classification of lower-pair joints. In geometrical part specification, it is a primary classification tool for permissable dimensioning and tolerancing schemes. In particular, in the literature of both areas (for example, [2] and [3]) the set of rigid motions leaving a point set unchanged is first identified as a group. The group is either explicitly or implicitly assumed to be a Lie group. The connected component of the identity is identified as the interesting subset of transformations for the application area, because in kinematics the motion of the body is a continuous transition from the initial position, and because in dimensioning and tolerancing the variability is obtained by a continuous perturbation from the design target. This subset of transformations is again explicitly or implicitly assumed to be a Lie group. A list of equivalence classes of connected subgroups of the rigid motions is then presented as uniquely representing all such subgroups. It follows that any point set gives rise to a member of one of these equivalence classes. Equivalence classes of surfaces are then defined in terms of the equivalence classes of the subgroups, that is, two surfaces are defined to be in the same class, if their associated connected subgroups are in the same class. In fact, it is often noted that the list can be pruned, since members of several of the equivalence classes cannot leave invariant point sets of the type required for the application, and the two previously presented lists result.

While from a mathematical perspective many of the claims of the previous paragraph are not obvious, nor even true in full generality, the literature of both areas present the results as unproven claims for the most part (for example, see [2] or [1]). For the purpose of exposition of a small part of a larger theory, when done well, this is all that can and should be expected. It would certainly be sufficient given primary sources for the results. At a minimum, such a source should be easily accessible, provide precise statements of the claims in well defined terms, justify the application of well referenced previous results, and clearly verify any new results. Unfortunately, a completely satisfactory primary source does not exist. Without question, the best of the available references is provided by Hervé in an interesting series of papers ([4], [3], and [5]). However, much of this is available only in French, and even there, possibly because of the computational focus, less attention is paid to theoretical rigor than is required. Other references even lack clarity in their statement and exposition of the claims and offer at best poor heuristic arguments in the place of proofs.

This report aims at rectifying the lack of a primary source. The focus of much of the work is Lie groups and algebras, and, when these are treated in the generality natural for these groups and algebras, this presupposes a more than trivial familiarity with the more basic mathematical structures of groups, topologies, and manifolds. On the other hand, when the groups and algebras are contained in $\Re^n$, as are all those that need be encountered here, great simplifications are often possible, and the necessary background can be treated effectively by employing little more than the tools of multivariable calculus. Many good introductory texts are available on groups, topologies, and manifolds and on Lie groups and algebras. For example, see: [6] or [7] for groups; [8] for topology; [9], [10], or [11] for manifolds; and [9], [12], or [13] for Lie groups and algebras. Although introductory in nature, because each of these focuses on a level of generality natural for its own area, each contain much material not required here, and involves a significant expenditure of effort to master. Thus, in order to make this work accessible to even those with only a background of traditional multivariable calculus (like that obtained from [14], for example), while at the same time preserving the possibility of providing precise definitions, clear statements, and verifiable proofs, we present a short passage from rigid motions to a matrix group R representing them, followed by a rapid exposition of the minimum on groups and manifolds required here. All of the material in the section is completely standard and can be easily found in the more expansive treatments of the references we have just cited. For similar reasons, Section 3 presents a minimal treatment on Lie groups and algebras, stressing the results in the general linear group and algebra, where all further work will take place. In particular, we show that R is a Lie subgroup of the general linear group. The quoted material is again standard and can easily be found in any source like [9] or [13] with two exceptions. First, rather than using left invariant vector fields and Lie brackets to define the multiplication of the Lie algebra of a Lie group, instead we follow the suggestion of [15] and define the multiplication in terms of second partials of the group's multiplication. This allows us to avoid the topic of vector fields in the previous section and also clarifies the relationship between the group and the algebra by using more elementary techniques. That the multiplication, so defined, is equivalent to the traditional one follows from a simple calculation (see [16]). Second, rather than using the standard computational approach involving integral curves to define the exponential map, we use standard uniqueness results to make an existential definition. This again allows us to avoid topics in the previous section. The equivalence of the definition is clear from the standard existence and uniqueness results like Theorem 5.9 of [12]. With the stage now set, the remaining sections present and develop the two main results which are required. In Sections 4 and 5 we find $\mathbf{r}$, the Lie algebra of R and determine all of its subalgebras. We use this to determine all connected Lie subgroups of R, and hence of the rigid motions. Finally, in Section 6 we consider the groups of symmetries of subsets of $\Re^3$. We show under a weak assumption, certainly met by all sets appearing in kinematics or geometric part specification, that the set of rigid motions leaving a point set unchanged is a Lie subgroup of the rigid motions and that its connected component of identity is also such a subgroup. A simple argument then allows us to reduce the set of possible connected subgroups to the set that leads directly to the two classifications of surfaces employed in the application areas.

2 Background

We begin with some simple standard notation. $\Re$ will denote the real numbers, and $\Re^n$ the n-dimensional real vector space. Let e^i be the i^{th} basis element of the standard basis of $\Re^n$, that is, e^i is the vector with all its components being zero except the i^{th} component, which is one. For $v \in \Re^n$ let v_i be the i^{th} component of v. Similarly, if f is a function with values in $\Re^n$, let f_i be its i^{th} component function, and in particular, let x_i be the i^{th} coordinate function, that is, the function such that $x_i(v) = v_i$ for all $v \in \Re^n$. Recall the Kronecker delta, $\delta_{i,j}$, which equals one if $i = j$ and equals zero otherwise. For a matrix v let v_{ij} be its ij^{th} entry. Similarly, if f is a function with its values being $n \times m$ matrices, let f_{ij} be its ij^{th} component function, and in particular, let x_{ij} be the ij^{th} coordinate function, that is, the function such that $x_{ij}(v) = v_{ij}$ for any $n \times m$ matrix v. Let e^{ij} be the $n \times n$ matrix with all entries zero except the ij^{th}

entry, which is one. Let det be the determinant function, and let v^t be the transpose of matrix v. For compact notation define the following four symbols:

$$0_3 = \begin{matrix} 0 & 0 & 0 \end{matrix} \; ; \; 0^3 = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \; ; \; 0_3^3 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \; ; \; I_3^3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \, . \tag{1}$$

E^3 is the standard model we have of the physical world, but for calculations $\Re^n$ is much more hospitable. Fortunately, it is possible to pass from one to the other, as we now sketch[1]. The geometry of E^3 takes points, lines, planes, and space as its primitive objects. By defining some segment to be of unit length, the length of all segments can be defined by proportionality. If we choose some point p on a line l, and a positive direction from p, then the signed length of the segment from p to q allows the points of l to be parameterized by $\Re$. Clearly, p and the unit length segment completely determine l and the parameterization. $\Re$ then parameterizes $O(l)$, the planes orthogonal to line l, by associating each plane with the one point of l it contains. If l_1, l_2, and l_3 are mutually orthogonal lines, and p is any point, then exactly one plane of each of $O(l_1)$, $O(l_2)$, and $O(l_3)$ contain p. Thus, choosing a point p and $B = \{s_1, s_2, s_3\}$, three mutually orthogonal unit length segments with p as an endpoint, yields a bijective map from E^3 onto $\{(p, v) : v \in \Re^3\}$. Such a map is called an affine orthogonal coordinate system with base point, p, and basis, B. A basis can be defined to be right handed by an appeal to the physical space being modelled or, equivalently, by comparison by means of a rigid motion with some arbitrarily chosen basis. Thus by choosing a base point and a right-handed orthonormal basis we can give E^3 an affine orthogonal coordinate system. The change of coordinates from one such coordinate system to another is given by the rigid motion carrying the first base point and basis pair to the second.

By arbitrarily choosing an affine orthogonal coordinate system for E^3 and mapping the base point to the origin of $\Re^3$ we can identify E^3 with $\Re^3$. The distance between points is given as usual by the root of the sum of the squares of the differences in the coordinates. Under this identification the set of rotations of E^3 about the base point are represented by $SO(3, \Re)$, the set of orthogonal 3×3 real matrices whose determinants are equal to one, acting on the 3-vectors of $\Re^3$ by matrix multiplication. To treat translations in E^3 it is convenient to identify $\Re^3$ with the hyperplane P in $\Re^4$ composed of all points whose last coordinate equals one, that is,

$$P = \{(x_1, x_2, x_3, x_4) \in \Re^4 : x_4 = 1\} \tag{2}$$

The distance is given as before by the root of the sum of squares of differences in the coordinates. Now under the identification of P with $\Re^3$, and hence with E^3, $SO(3, \Re)$,

[1]See [17] for complete details.

and hence the rotations of E^3 about the base point, are represented by S, the set of 4×4 real matrices defined by

$$S = \left\{ \begin{pmatrix} r & 0^3 \\ 0_3 & 1 \end{pmatrix} \; : \; r \in SO(3, \Re) \right\} \; , \tag{3}$$

and the translations of E^3 are represented by T, the set of 4×4 real matrices defined by

$$T = \left\{ \begin{pmatrix} I_3^3 & v \\ 0_3 & 1 \end{pmatrix} \; : \; v \in \Re^3 \right\} \; , \tag{4}$$

with the elements of S and T acting by matrix multiplication on the 4-vectors of P. Since a rotation of E^3 about one point can be obtained from a rotation about a second point by translating the first point to the second, rotating, and translating back, and since the rigid motions of E^3 are compositions of rotations and translations, it follows from matrix multiplication that the rigid motions are represented by R, the set of 4×4 real matrices defined by

$$R = \left\{ \begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} \; : \; r \in SO(3, \Re) \text{ and } v \in \Re^3 \right\} \; , \tag{5}$$

again with the elements of R acting by matrix multiplication on the 4-vectors of P. A change in the choice of affine basis for E^3 is equivalent to multiplication of P by some m of R, and if some rigid motion was represented in the first basis by $n \in R$, then it is transformed to mnm^{-1} in the second, as usual.

In the following we will consider at length R endowed with the mathematical structure of a Lie group, but to even define this concept requires more primitive constructs. In particular, we need to be able to speak of a manifold and a group. To define a manifold more precisely is somewhat involved and requires even more primitive objects. On the other hand, a group is much easier to define, and so we begin with its definition.

2.1 Groups

Consider a collection, G, of objects called elements. Say that G is algebraically closed under a binary operation, denoted by $\circ$, on G, if $\circ$ returns values in G, that is, if $\circ : (x, y) \in G \times G \to x \circ y \in G$. If there is an element, e, of G, such that $x \circ e = e \circ x = x$ for all $x \in G$, call e an identity element. If an identity exists, it is easy to show that it is unique. If for some $x \in G$, there is some $y \in G$, such that $x \circ y = y \circ x = e$, call y an inverse of x. If an element x has an inverse, it is easy to show that its inverse is unique, and it is usually denoted by x^{-1}. If $x \circ (y \circ z) = (x \circ y) \circ z$, say that $\circ$ is associative. In these terms, then, a group is a collection of elements that is algebraically closed under an associative binary operation with an identity element and with each element having an inverse. Often, explicit representation of the group operation is suppressed, so that we will write xy rather than $x \circ y$, if the operation is clear from context.

If f is a one-to-one map from a group G with binary operation $\circ$ onto a group $G^{'}$ with binary operation $\circ^{'}$ such that $f(x \circ y) = f(x) \circ^{'} f(y)$ for all x and y in G, then f is an isomorphism, and G and H are isomorphic. For the purposes of group theory, isomorphic groups are equivalent. An isomorphism mapping a group onto itself is called an automorphism. The map of group G given by $f : x \in G \rightarrow yxy^{-1} \in G$ for some $y \in G$ is called the inner automorphism determined by y. Since $f(xx^{'}) = yxx^{'}y^{-1} = yxy^{-1}yx^{'}y^{-1} = f(x)f(x^{'})$ for all $x, x^{'} \in G$, an inner automorphism is an automorphism.

Any vector space with vector addition is a group. In particular, then, $\Re^n$ with vector addition is a group. As a primary example of a group consider the set of $n \times n$ invertible real matrices which is denoted by $Gl(n, \Re)$. Under matrix multiplication the identity matrix I serves as an identity. By assumption each element of $Gl(n, \Re)$ has an inverse in $Gl(n, \Re)$. Matrix multiplication is associative. The product of two invertible real matrices is again an invertible real matrix, since $(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = I = x(yy^{-1})x^{-1} = (xy)(y^{-1}x^{-1})$ for all $x, y \in Gl(n, \Re)$, so that matrix multiplication is a binary operation on $Gl(n, \Re)$. Thus $Gl(n, \Re)$ with matrix multiplication forms a group, called the general linear group.

Other examples of groups can come from certain subsets of a group. If H is a subset of a group G such that H is algebraically closed under the group operation and H contains the inverse of each of its elements, then H is a group. In this case H is said to be a subgroup of G. If $g \in G$ and B is a subset of G, let $gB = \{gb : b \in B\}$ and $Bg = \{bg : b \in B\}$. If B and C are subsets of G, then let $BC = \{bc : b \in B, c \in C\}$. If H is a subgroup of G, then H is said to be normal, if $gH = Hg$ for all $g \in G$. If H is a normal subgroup of G, and K is a subgroup of G, then $HG = GH$ is a subgroup of G, and we say that HG is the product of H and K.

We now provide several concrete examples of subgroups. Consider $SO(3, \Re)$ in $Gl(3, \Re)$. If x is orthogonal, then $x^t(x^t)^t = x^t x = I$, so that x^t is orthogonal. Since $det(x) = det(x^t)$, if $det(x) = 1$, then $det(x^t) = 1$. Thus if $x \in SO(3, \Re)$, then $x^t = x^{-1} \in SO(3, \Re)$, so that $SO(3, \Re)$ contains the inverses of its elements. If x and y are orthogonal, then $(xy)(xy)^t = xyy^t x^t = I$, so that xy is orthogonal. Since $det(xy) = det(x)det(y)$, if $det(x) = det(y) = 1$, then $det(xy) = 1$. Thus if $x, y \in SO(3, \Re)$, then $xy \in SO(3, \Re)$, so that $SO(3, \Re)$ is algebraically closed in $Gl(3, \Re)$. It follows that $SO(3, \Re)$ is a subgroup of $Gl(3, \Re)$. We can use $SO(3, \Re)$ to show that R is a subgroup of $Gl(4, \Re)$. If $r \in SO(3, \Re)$ and $v \in \Re^3$, then by matrix multiplication

$$\begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} \begin{pmatrix} r' & v' \\ 0_3 & 1 \end{pmatrix} = \begin{pmatrix} rr' & rv' + v \\ 0_3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} \begin{pmatrix} r^t & -r^t v \\ 0_3 & 1 \end{pmatrix} = I \ . \quad (6)$$

Thus R is an algebraically closed subset of $Gl(4, \Re)$ and contains inverses, so it is a subgroup of $Gl(4, \Re)$. Equation 6 implies in turn that S and T are subgroups of R, and hence also of $Gl(4, \Re)$. It also implies that T is normal in R, so that $R = ST$, that is, R is the product of S and T. The equation even implies that

$$\Phi : r \in SO(3, \Re) \to \begin{pmatrix} r & 0^3 \\ 0_3 & 1 \end{pmatrix} \in S \quad (7)$$

is an isomorphism.

2.2 Manifolds

Now we turn to the second structure we need, the manifold. Multivariable calculus has $\Re^n$ as its natural setting. Multivariable calculus talks about smooth functions and their derivatives, about linear approximations to sets and functions between sets. Manifolds generalize $\Re^n$ and serve as a setting for a generalized multivariable calculus. A manifold can be defined intuitively as a set that locally seems like $\Re^n$, when viewed properly, with all these views fitting together nicely. Intuitively, but a bit fuzzily. To make it more precise we need to define several more primitive structures: a topological space, a locally Euclidean space, and a C^∞ differentiable structure. A topological space will formalize what is meant by locally. A locally Euclidean space will formalize what is meant by seems like $\Re^n$, when viewed properly. A C^∞ differentiable structure will formalize what is meant by the views fitting together nicely. We begin with the most primitive, the topological space.

First, a topological space is a set, X, of objects, called points, and a collection, τ, of subsets of X such that X is in τ, the empty set is in τ, the union of the subsets of any subcollection of τ is in τ, and the intersection of the subsets of any finite subcollection of τ is again in τ. In this case, τ is called a topology, and the elements of τ are called the open sets of the topology. The complement of a open set is said to be closed. Since the union of open sets is open, the intersection of closed sets is closed. This implies that by intersecting all closed sets containing any set we obtain a smallest closed set containing the set, called the closure of the set. For a set F in topological space we will denote its closure by $\overline{F}$. If there is a collection of countably many open sets in τ with the property that any set in τ can be obtained as the union of sets in the collection, τ is said to be second countable. If for any two points, p and p', in X there are disjoint open sets, O and O', with $p \in O$ and $p' \in O'$, then τ is said to be Hausdorff. A subset of a topological space is said to be connected, if it cannot be written as the union of two disjoint open sets. Thus, for example, a set composed of two distinct points in a Hausdorff space is never connected. A function mapping a topological space X into a topological space Y is said to be continuous, if the inverse image of every open set in Y is open in X. Since a closed set is the complement of an open set, this implies that the inverse image of every closed set in Y is closed in X. It follows directly from the definitions that the image of a connected set under a continuous function is connected.

To present a first example of a topological space, let the distance between points in $\Re^n$ be the root of the sum of the squares of the differences in the coordinates, and let the open ball about the point p in $\Re^n$ of radius ϵ be those points in $\Re^n$ a distance from p of less than ϵ. If we let τ be all sets obtained as the union of collections of open balls, then $\Re^n$ with τ forms a topological space. It is easy to see that it is Hausdorff, and since there are only countably many open balls with rational radius about points with rational coordinates, it follows that it is also second countable. In what follows $\Re^n$ will always assumed to have this topology, which is called the usual topology.

In $\Re^n$ many topological considerations are simplified and hence more intuitively appealing. A mapping from $\Re^n$ into $\Re^m$ is continuous under the topological definition in terms of open sets, if and only if it is continuous by the standard ϵ-δ definition. A sequence of points, $\{p_n\}$, is said to converge to point, p, if for any $\epsilon > 0$ there is a k such that p_n is closer to p than ϵ for all $n > k$. If a set contains a sequence of points that converges to a point, p, then p is said to be a cluster point of the set. In $\Re^n$, a closed set is one which contains all its cluster points. In particular, a set containing a single point is closed. A set is said to be bounded, if it is contained in some ball of radius ρ about the origin. Any bounded sequence contains a convergent subsequence, so that any bounded sequence in a closed set contains a subsequence that converges to a point in the set.

To present a second example of a topological space, let Y be a subset of a topological space X. If we define a set to be open in Y if it is the intersection of Y with an open set of X, then Y becomes a topological space. If X is Hausdorff or second countable, so is Y. Y is said to have inherited its topology from X or to have the relative topology, and we speak of relatively open or relatively closed sets. A relatively open set of an open set is open, and a relatively closed set of a closed set is closed. Unless otherwise stated, a subset of a topological space will always be assumed to have this topology in what follows.

A continuous function with a continuous inverse is called a homeomorphism, and when such a mapping exists between two topological spaces they are said to be homeomorphic. A Hausdorff topological space X is said to be locally Euclidean of dimension d, if for each point in the space there is an open set containing it, which is homeomorphic with an open set in $\Re^d$. If ϕ is one of these homeomorphisms mapping the open set U of X to an open set of $\Re^d$, $\{\phi, U\}$ is call a local coordinate system. If $\{\phi_i, U_i\}$ for $i = 1, 2$ are local coordinate systems of X, and U_1 and U_2 are not disjoint, then $\phi_1(U_1 \cap U_2)$ is an open set in $\Re^d$, and $\phi_2\phi_1^{-1}$ is a continuous map from $\phi_1(U_1 \cap U_2)$ into $\Re^d$ called a transition function. A transition function thus is a local change of coordinates. A real-valued function f is said to be C^k at a point $p \in \Re^n$, if applying any k partial derivatives to f at p results in a continuous function, and f is said to be C^∞ at p, if it is C^k at p for all k. A function mapping a subset E of $\Re^n$ into $\Re^m$ is said to be a C^∞ function on E, if its coordinate functions are C^∞ at all points of some open set containing E. Since a transition function maps an open set in $\Re^d$ into $\Re^d$, we can define the collection of local coordinate systems giving X the structure of a locally Euclidean space to be a C^∞ differentiable structure, if each of the transition functions is C^∞. A collection of local coordinate systems that forms a C^∞ differentiable structure is maximal, if it contains all local coordinate systems of X that could be added to the collection, while preserving the differentiability of the transition functions. We can always pass from a C^∞ differentiable structure to a maximal one by defining it to contain all allowable additions.

Now we define a d-dimensional C^∞ manifold (or manifold for short) to be a second countable topological space, which is locally Euclidean of dimension d, and which has a maximal C^∞ differentiable structure on it. A 2-dimensional manifold is often called a surface.

$\Re^n$ serves as a trivial example of a manifold. It is a second countable, Hausdorff, topological space, and choosing $\Re^n$ itself as an open set and the identity map as a homeomorphism gives $\Re^n$ a local coordinate system making it locally Euclidean. The one transition function that results is again the identity which is C^∞. Maximality is obtained by assuming the addition of all compatible local coordinate systems.

If we can identify the points of some set with the points of a manifold, we can obtain a new example of a manifold from an old one. For example, by using the coordinate vectors of a vector space in some basis of the vector space, we can identify the points of an n-dimensional vector space with the points of $\Re^n$, and in so doing give the vector space the structure of a manifold. Thus, if we identify $\mathbf{gl}(n, \Re)$, the set of $n \times n$ real matrices, with $\Re^{n^2}$, we can endow $\mathbf{gl}(n, \Re)$ with the structure of an n^2-dimensional manifold.

An open set of a manifold endowed with the relative topology is a manifold of the same dimension, when the C^∞ differentiable structure of the manifold is restricted to the open set. In particular, any open subset of $\Re^n$ is a n-dimensional manifold. As a concrete example consider $Gl(n, \Re)$ in $\mathbf{gl}(n, \Re)$. A matrix is invertible, if and only if its determinant is nonzero. The determinant is a polynomial function on $\mathbf{gl}(n, \Re)$, and, since polynomials in any $\Re^n$ can be shown by the usual ϵ-δ techniques to be continuous, the determinant is a continuous function on $\Re^{n^2}$. The sets $\{x \in \Re \ : \ x < 0\}$ and $\{x \in \Re \ : \ x > 0\}$ are open in $\Re$, so $Gl(n, \Re)$ is an open set of $\mathbf{gl}(n, \Re)$, and hence is a manifold of dimension n^2.

An m-dimensional manifold M and an n-dimensional manifold N can be combined to form a manifold of dimension $m+n$, called the product manifold and denoted $M \times N$. It is defined in terms of ordered pairs of elements from M and N, that is, $M \times N = \{(p,q) : p \in M \ , \ q \in N\}$. It is topologized by defining all sets of the form $U \times V = \{(p,q) \in M \times N : p \in U \ , \ q \in V\}$ to be open in $M \times N$, where U is open in M and V is open in N. Local coordinate systems in $M \times N$ are defined in terms of those for M and N. Thus if $\{\phi, U\}$ is a local coordinate system of M, and $\{\psi, V\}$ is a local coordinate system of N, then $\{\phi \times \psi, U \times V\}$ is a local coordinate system of $M \times N$, where $\phi \times \psi : (p,q) \in M \times N \to (\phi(p), \psi(q)) \in \Re^{m+n}$ in the obvious manner.

On a manifold the notion of a subset being connected has a more intuitive interpretation. A path or curve between two points p and q in a topological space is a continuous map, c, from the closed interval, [0,1] in $\Re$, into the topological space with $c(0) = p$ and $c(1) = q$. A subset of a topological space is said to be pathwise connected, if there is always a path in the subset connecting any two points. On a manifold, connectedness is equivalent to pathwise connectedness. This implies that if a point x belongs to two connected sets A and B, then by passing through x we can always construct a path from any point in A to any one in B, so that the union of A and B is again connected. By taking the union of all connected sets containing x we obtain a largest connected set containing x, which is called the connected component of x. By using local coordinate systems it easy to show that a connected component is open and also closed in the manifold.

Several manifold theoretic definitions are required in subsequent sections. We end this section by considering them.

C^∞ functions on manifolds are defined in terms of the composition with homeomorphisms of local coordinate systems being C^∞ on subsets of $\Re^n$. To be precise, let a function f map a subset E of m-dimensional manifold M into an n-dimensional manifold N. For any local coordinate system $\{\phi, U\}$ of M, and any local coordinate system $\{\psi, V\}$ of N, let $K(\phi, \psi) = \phi(f^{-1}(V) \cap U \cap E)$. $K(\phi, \psi)$ is a subset of $\Re^m$, and $\psi f \phi^{-1}$ is defined on it and maps it into $\Re^n$. We say that f is a C^∞ function on E in M, if for all local coordinate systems $\{\phi, U\}$ of M, and for all local coordinate systems $\{\psi, V\}$ of N, $\psi f \phi^{-1}$ is a C^∞ function on $K(\phi, \psi)$. The set of all C^∞ functions on E in M mapping E into N is denoted $C^\infty(E, M; N)$, or $C^\infty(E; N)$, if M is obvious from context. The union of all $C^\infty(E, M; N)$ over all N, that is, when the range of the function is not important, is denoted $C^\infty(E, M)$, or $C^\infty(E)$, again if M is obvious from context. The composition of C^∞ functions is a C^∞ function. A function f mapping an open set U of $\Re^m$ into $\Re^n$, such that each of the coordinate functions of f is a polynomial (or rational) function in the coordinates of $\Re^m$, provides a simple example of a C^∞ function, that is, $f \in C^\infty(U, \Re^m; \Re^m)$. A function $f \in C^\infty(M; N)$ with a C^∞ inverse is called a diffeomorphism, and M and N are said to be diffeomorphic. The composition of diffeomorphisms is a diffeomorphism. For the purposes of manifold theory, diffeomorphic manifolds are equivalent.

Submanifolds play the role for manifolds that subgroups play for groups. They can be defined in many ways, but a very simple elementary definition will suffice here. Let m and n be non-negative integers with $m \leq n$. Let Π^n_m be the map of $\Re^n$ onto $\Re^m$ given by mapping a point to its first m coordinates. For $\{\phi, U\}$ a local coordinate system of an n-dimensional manifold define $Z(\{\phi, U\}, m)$ to be those points $p \in U$ for which the last $n - m$ coordinates of $\phi(p)$ are all zero. For $\{\psi, V\}$ a local coordinate system of an m-dimensional manifold say that $\{\psi, V\}$ is a slice of $\{\phi, U\}$, if $V = Z(\{\phi, U\}, m)$ and $\psi(p) = \Pi^n_m(\phi(p))$ for all $p \in V$. In these terms, a manifold M is a submanifold of a manifold N, if for each point in M there is a local coordinate system of M that is a slice of a local coordinate system of N. If $\{\psi, V\}$ and $\{\phi, U\}$ can be chosen so that $Z(\{\phi, U\}, m) = M \cap U = V$, we will say that M is an imbedded submanifold of N. Since U is open in N, and the union of open sets is open, M being an imbedded submanifold of N implies that M is relatively open in N. Submanifold M is an imbedded submanifold of manifold N, if and only if M has the relative topology inherited from N. For general submanifolds $M \cap U$ need not equal V and can be quite chaotic, in fact. This implies that in some sense along their own coordinate directions all submanifolds are topologically well behaved with respect to the manifold that contains them, but only for imbedded submanifolds is this claim justified for all the directions of the manifold that contains them.

Trivially, any manifold is a submanifold of itself. Almost as trivially, an open subset of a manifold is an imbedded submanifold. Since a connected component of a manifold is open, it is an imbedded submanifold. If L is a submanifold of manifold M, and M is a submanifold of manifold N, then L is a submanifold of N. A similar statement is true for imbedded submanifolds.

Let M be a submanifold of manifold N. If f is a C^∞ function on M, then for any $p \in M$ there is an open set U in M containing p, and a function $F \in C^\infty(N)$ such that $f(q) = F(q)$ for all $q \in U$. In this sense, we can say that locally in M a C^∞ function on M is the restriction of a C^∞ function on N. If M is an imbedded submanifold of N, we can strengthen this to say that for any $p \in M$ there is an open set V in N containing p, and a function $F \in C^\infty(N)$ such that $f(q) = F(q)$ for all $q \in M \cap V$. Now we can say that locally in N a C^∞ function on M is the restriction of a C^∞ function on N. Finally, if M is a closed imbedded submanifold of N, we can fully strengthen this to say that a C^∞ function on M is the restriction of a C^∞ function on N. Looking at C^∞ functions and their restrictions in the other direction is much easier. If $F \in C^\infty(N)$, then its restriction to any submanifold M belongs to $C^\infty(M)$.

An m-dimensional affine subspace, K, of $\Re^n$ gets the structure of an m-dimensional manifold by identifying it with $\Re^m$, so that it is immediate that K is an imbedded submanifold of $\Re^n$. The identification then becomes a diffeomorphism. Thus P defined in Equation 2 gives a trivial example of a 3-dimensional imbedded submanifold of $\Re^4$. Only slightly less trivially for manifolds M and N with $p \in M$ the C^∞ map $f : q \in N \to (p,q) \in M \times N$ gives a diffeomorphism from N onto the imbedded submanifold $p \times N = \{(x,y) \in M \times N : x = p\}$ of $M \times N$.

Verifying that something is a manifold by applying first principles can be difficult. Verifying that it is an imbedded submanifold can be worse. Actually, since many standard results dealing with submanifolds exist, it is rarely necessary to appeal to first principles. For example, let $F = \{f_i\}_{i=1}^k \subset C^\infty(N,\Re)$. Say that the functions of F are independent at $p \in N$, if $\{\nabla\phi^{-1} \circ f_i|_{\phi(p)}\}$ is a set of independent vectors in $\Re^n$, for some local coordinate system, $\{\phi, U\}$, of N and ∇ defined as the usual gradient operator on functions from $\Re^n$ to $\Re$. In these terms we have[2]

Proposition 1 *Let N be an n-dimensional manifold. Let M be the points of N, where the functions of $F = \{f_i\}_{i=1}^k \subset C^\infty(N,\Re)$ are independent and each function of F vanishes. If M is not empty, M is a imbedded submanifold of N of dimension n-k.*

If $k = 1$ and N is $\Re^3$, this proposition reduces to the claim that the solution set of a C^∞ function on $\Re^3$ is an imbedded 2-dimensional submanifold, a surface, where the gradient does not vanish. Thus quadric surfaces, tori, spline surfaces are examples of imbedded submanifolds. More generally, the proposition implies that algebraic or real analytic varieties give rise to examples of any dimension.

Manifolds have linear approximations. At each point, p, of an n-dimensional manifold, M, there is a vector space of dimension n, which best approximates M. This space is called the tangent space to M at p and is denoted by $T_p(M)$. Intuitively, a tangent space should be in some sense the space spanned by the directions of all possible infinitesimal motions within the manifold at the point. Several common techniques are used to define $T_p(M)$. One of the most elementary employs directional derivatives at p, and in so doing captures at least some of the intuitive sense of directions of all possible infinitesimal motions. For this purpose, begin by assuming that $\{\phi, U\}$ is a local coordinate system for M with $p \in U$. If V is the domain of f for some $f \in C^\infty(p, M; \Re)$, then $f \circ \phi^{-1} : \phi(U \cap V) \subset \Re^n \to \Re$, so we can define the operator ∂_i on functions $f \in C^\infty(p, M; \Re)$ by setting $\partial_i f = \frac{\partial f \circ \phi^{-1}}{\partial x_i}\Big|_{\phi(p)}$. Extend the set of operators $B = \{\partial_i\}_{i=1}^n$ to form a vector space by defining $(\lambda_1 v_1 + \lambda_2 v_2) f = \lambda_1(v_1 f) + \lambda_2(v_2 f)$ for operators $v_1, v_2 \in B$ and $\lambda_1, \lambda_2 \in \Re$. Define $T_p(M)$ to be this space of operators. Since $\partial_i x_j \circ \phi = \frac{\partial x_j}{\partial x_i}\Big|_{\phi(p)} = \delta_{i,j}$, the vectors of B are independent. Thus B is a basis of $T_p(M)$, and the dimension of $T_p(M)$ is n. Under this definition, tangent vectors at m are linear operators given by directional differentiation on C^∞

[2] See Theorem 1.38 of [9] for example.

functions at m. Any other definition yields an equivalent space, and in particular, a different choice of local coordinate system leads to the same space with different basis vectors. In fact, if $\{\phi, U\}$ and $\{\phi', U'\}$ are two local coordinate systems containing the point p, then the Jacobian matrix

$$J = \left.\frac{\partial \phi' \circ \phi^{-1}}{\partial x}\right|_{\phi(p)} \tag{8}$$

from multivariable calculus serves as the change of basis matrix from basis B to the basis B', derived from $\{\phi', U'\}$. In particular, if we use $T_p(M; B)$ to denote the vectors of M_p represented by their coordinate vectors with respect to a basis B and $T_p(M; B')$ correspondingly for B', then the change of basis takes the form

$$c \in T_p(M; B) \to Jc \in T_p(M; B') \,. \tag{9}$$

Let $s : t \in (-1, 1) \subset \Re \to s(t) \in \phi(U)$ be a curve in $\phi(U)$ for some local coordinate system $\{\phi, U\}$ of a manifold M. Let $s_i(t)$ be its i^{th} component function. If $s \in C^\infty(0, \Re; \Re)$, and $s(0) = \phi(p)$, then

$$\frac{d}{dt} f \circ \phi^{-1} \circ s\bigg|_0 = \Sigma_i \left.\frac{\partial f \circ \phi^{-1}}{\partial x_i}\right|_{\phi(p)} \left.\frac{ds_i}{dt}\right|_0 = \Sigma_i \left.\frac{ds_i}{dt}\right|_0 \partial_i f$$

for all $f \in C^\infty(p, M; \Re)$. Thus s gives rise to some $v \in T_p(M)$. In fact, by choosing the values of $\frac{ds}{dt}$ at 0 any $v \in T_p(M)$ can be generated in this manner.

The manifold $\Re^n$ has a common simple interpretation of its tangent spaces. We can always choose $\{id, \Re^n\}$, where id is the identity map, as a local coordinate system containing any $p \in \Re^n$. Under this choice the vectors of basis $B = \{\partial_i\}$ of $T_p(\Re^n)$ satisfy $\partial_i f = \left.\frac{\partial f}{\partial x_i}\right|_p$ for all $f \in C^\infty(p, \Re^n; \Re)$. Thus ∂_i is just the usual directional derivative in $\Re^n$ along e_i. But now this implies that if c is the coordinate vector of some $v \in T_p(\Re^n)$, then if we treat c as a coordinate vector of a vector in $\Re^n$ with its usual basis $\{e_i\}$, then $v(f)$ is given by the usual directional derivative in the direction c, that is, $v(f) = \left.\frac{\partial f}{\partial c}\right|_p$ for all $f \in C^\infty(p, \Re^n; \Re)$. Thus, we can identify $T_p(\Re^n)$ with $\Re^n$, itself. Since we can identify any n-dimensional vector space with $\Re^n$ by using the corresponding coordinate vectors, we can do the same for any vector space.

The tangent spaces of submanifolds of a manifold can be considered to be subspaces of the corresponding tangent spaces of the manifold in a natural manner. Let M be an m-dimensional submanifold of N, and for $p \in M$ let $\{\psi, V\}$ be a local coordinate system of M that contains p and that is a slice of $\{\phi, U\}$, a local coordinate system of N. Identify $\Re^m$ with the first m coordinates of $\Re^n$. Each ∂_i in the basis of $T_p(M)$ corresponds to a partial derivative along a coordinate direction of $\Re^m$. Since this is also a coordinate direction of $\Re^n$, it also corresponds to a basis vector $\partial_i^{'}$ of $T_p(N)$. Any $f \in C^\infty(p, M; \Re)$ can be extended near p to a function $f^{'} \in C^\infty(p, N; \Re)$ such that $f(q) = f^{'}(q)$ on some open set of M containing p. It then follows that $\partial_i f = \partial_i^{'} f^{'}$, so that for any $f \in C^\infty(p, M; \Re)$ we can define $\partial_i^{'} f = \partial_i^{'} f^{'}$, where $f^{'}$ is some extension of f near p. Thus, we can identify ∂_i with $\partial_i^{'}$, and hence $T_p(M)$ with a subspace of $T_p(N)$.

Combining the last two paragraphs we find that if M is a submanifold of $\Re^n$, then $T_p(M)$ can be identified with a subspace of $\Re^n$ for any $p \in M$. In particular, if M is an open subset of $\Re^n$, then $T_p(M)$ can be identified with $\Re^n$, itself. The affine subspace $p + T_p(M)$ is the space which often occurs in multivariable calculus as the notion of a tangent space.

Any C^∞ map between manifolds can be approximated by its differential, a linear map between tangent spaces of the manifolds. Let M and N be manifolds of dimension m and n, respectively, and let f be a C^∞ map from M to N. Let $\{\phi, U\}$ be a local coordinate system for M with some $p \in U$ and $\{\psi, V\}$ a local coordinate system for N with $f(p) \in V$. Since $\psi \circ f \circ \phi^{-1}$ is a map from an open set of $\Re^m$ to $\Re^n$, multivariable calculus yields the $m \times n$ Jacobian matrix,

$$K = \left. \frac{\partial \psi \circ f \circ \phi^{-1}}{\partial x} \right|_{\phi(p)} . \tag{10}$$

Let B be the basis for $T_p(M)$ associated with $\{\phi, U\}$ and C be the basis for $T_{f(p)}(N)$ associated with $\{\psi, V\}$. Let $df(p)$, the differential of f at p, be the linear map from $T_p(M)$ to $T_{f(p)}(N)$ defined by

$$df(p) : c \in T_p(M; B) \rightarrow Kc \in T_{f(p)}(N; C) . \tag{11}$$

Since we have used explicit choices of local coordinate systems in this definition, it would seem that $df(p)$ is dependent on these choices. However, let $B^{'}$ be the basis for $T_p(M)$ associated with $\{\phi^{'}, U^{'}\}$, a second local coordinate system for M with $p \in U^{'}$, and $C^{'}$ the basis for $T_{f(p)}(N)$ associated with $\{\psi^{'}, V^{'}\}$, a second local coordinate system for N with $f(p) \in V^{'}$. Let J be the change of basis matrix given by Equation 8 transforming $T_p(M; B)$ to $T_p(M; B^{'})$ and L that transforming $T_{f(p)}(N; C)$ to $T_{f(p)}(N; C^{'})$. Now in bases $B^{'}$ and $C^{'}$ Equation 11 becomes

$$df(p) : c \in T_p(M; B^{'}) \rightarrow LKJ^{-1}c \in T_{f(p)}(N; C^{'}) . \tag{12}$$

On the other hand, if we were to define $df(p)$ in terms of $\{\phi^{'}, U^{'}\}$ and $\{\psi^{'}, V^{'}\}$, then we would begin by obtaining the Jacobian matrix $K^{'} = \frac{\partial \psi^{'} \circ f \circ \phi^{'-1}}{\partial x}$ at $\phi^{'}(p)$. Since $\psi^{'} \circ f \circ \phi^{'-1} = \left(\psi^{'} \circ \psi^{-1}\right) \circ \psi \circ f \circ \phi^{-1} \circ \left(\phi \circ \phi^{'-1}\right)$, the chain rule of multivariable calculus yields $K' = LKJ^{-1}$. Now defining $df(p)$ using K' in Equation 11, that is, defining $df(p)$ in terms of $\{\phi', U'\}$ and $\{\psi', V'\}$, leads directly to Equation 12. Thus $df(p)$ is well defined and in particular is independent of the choice of local coordinate system. It is common to suppress explicit mention of the point, where the differential is defined, if it is clear from context, and to write df rather than $df(p)$.

If f is a C^∞ map from an open set of $\Re^m$ to $\Re^n$, then by choosing $\{id, \Re^m\}$ and $\{id, \Re^n\}$ as the local coordinate systems it is easy to see that df reduces to the usual multivariable calculus notion of a differential, given by the Jacobian, which linearly maps $\Re^m$ to $\Re^n$. If M is a submanifold of manifold M', N a submanifold of manifold N', and $f \in C^\infty(p, M; N)$, we can always extend f near p in M to a function $f' \in C^\infty(p, M'; N)$. It will then follow that df' restricted to $T_p(M)$ yields df. Thus, if M is a submanifold of $\Re^m$, N a submanifold of $\Re^n$, and $f \in C^\infty(p, M; N)$, then df can be represented as the restriction to the subspace $T_p(M)$ of $\Re^n$ of a linear map of $\Re^m$ to $\Re^n$ given by the Jacobian of f'.

3 Lie Groups and Their Algebras

The central mathematical construct of this work is the Lie group. The central mathematical tool which we will employ is the Lie algebra of a Lie group. We begin this section by considering them in general and providing the standard results and ideas which we will apply later.

A Lie group G is a group endowed with a manifold structure such that the group operations are C^∞ functions, that is, such that the product $prod : (g, h) \in G \times G \to gh$ and the inverse $inv : g \in G \to g^{-1}$ are C^∞ functions. In fact, it is not even really necessary to assume that inv is C^∞, since this follows from $prod$ being C^∞ and the implicit function theorem.

If G and H are Lie groups, then a diffeomorphism from G to H, which is also an isomorphism, is called a Lie group isomorphism. In this case, G and H are said to be Lie group isomorphic. If Φ is a Lie group isomorphism between G and H, then since Φ is a diffeomorphism, G and H are equivalent as manifolds. Moreover, since Φ is an isomorphism, G and H are equivalent as groups. Thus it is natural that Lie groups that are Lie group isomorphic can be considered as equivalent in Lie group theory.

A Lie group isomorphism that maps a Lie group onto itself is called a Lie group automorphism. As an example, consider an inner automorphism i_g of a Lie group G with $g \in G$, defined by

$$i_g : h \in G \to ghg^{-1} \in G \ .$$

Since i_g is clearly a group isomorphism, we need only show that i_g is a diffeomorphism. To see this, note that since the map $h \in G \to (g, h) \in G \times G$ and the group multiplication, $prod$, are C^∞, their composition, $L_g : h \in G \to gh \in G$, is C^∞.

Similarly, $R_g : h \in G \to hg \in G$ is C^∞. Now since $i_g = L_g \circ R_{g^{-1}}$, it follows that i_g is C^∞. Similarly $i_{g^{-1}}$ is C^∞. Thus both i_g and its inverse $i_{g^{-1}}$ are C^∞, and so i_g is a diffeomorphism. Thus any inner automorphism is a Lie group automorphism.

Trivially, $\Re^n$ with vector addition and its usual vector space topology is a Lie group. Less trivially, $Gl(n, \Re)$ serves as a primary example of a Lie group. It is a group under the usual matrix multiplication with the $n \times n$ identity matrix I as its identity element. It inherits a manifold structure as an open set of $\mathbf{gl}(n, \Re)$ endowed with its usual vector space topology. Multiplication in the group is a polynomial function. The inverse of a matrix can be obtained by ratios of minors and the determinant, which are polynomial functions, so that obtaining inverses is a rational function. Now since polynomial and rational functions in $\Re^n$ are C^∞ functions, $Gl(n, \Re)$ is a Lie group.

Analogous to the subgroups and submanifolds of groups and manifolds are the Lie subgroups. H is a Lie subgroup of a Lie group G, if H is a submanifold of G that is also a subgroup of G. Since the restriction to a submanifold of a C^∞ function on a manifold is a C^∞ function on the submanifold, it follows that the group operations of the Lie group are C^∞ functions on a Lie subgroup, so that a Lie subgroup is a Lie group. As with subgroups and submanifolds, if G is a Lie subgroup of H, which is a Lie subgroup of K, then G is a Lie subgroup of K.

As an example, for a Lie group G consider its identity component, G_e, that is, the connected component of G containing its identity, e. If $c(t)$ and $d(t)$ are paths in G connecting e to points p and q, then $c(t)d(t)$ and $(c(t))^{-1}$ are paths connecting e to points pq and p^{-1}. Since connectedness and pathwise connectedness are equivalent in manifolds, G_e contains the products and inverses of its elements, so that it is a group. Since a component is an open set of a manifold, G_e is an imbedded submanifold of G. Formalizing this we get the following.

Proposition 2 *The identity component of a Lie group is an imbedded Lie subgroup of the group.*

Verifying that a subset of a Lie group is a subgroup can be quite simple, at least, compared to verifying that it is a submanifold. Fortunately, the following can be demonstrated.

Proposition 3 *If H is a (topologically) closed subgroup of a Lie group G, then H is an imbedded Lie subgroup of G.*

A Lie subgroup need not be a closed set, so that the result will not apply in general. Fortunately, however, most subgroups encountered here will be closed, so it will suffice to show that they are Lie subgroups. For example, consider $SO(3, \Re)$ in $Gl(3, \Re)$. It is a subgroup of $Gl(3, \Re)$. A matrix u in $Gl(3, \Re)$ is in $SO(3, \Re)$, if and only if the determinant of u is one, and u^t, the transpose of u, satisfies $uu^t = I$. Thus $SO(3,\Re)$ in $Gl(3,\Re)$ can be characterized as those u in $Gl(3,\Re)$ simultaneously satisfying the 10 equations given by $det(u) = 1$ and $x_{ij}(uu^t) = \delta_{i,j}$, Since the 10 functions are rational or polynomial in the coordinates of $Gl(3,\Re)$, they are continuous. Since a singleton set is closed, and each of the functions is continuous, the solution set to each of these equations is closed in $Gl(3,\Re)$. Since each is closed, the intersection of the 10, that is, $SO(3,\Re)$, is closed in $Gl(3,\Re)$. From the proposition we conclude that $SO(3,\Re)$ is an imbedded Lie subgroup of $Gl(3,\Re)$.

As a second example consider R, defined in Equation 5, which represents the rigid motions. A matrix u in $Gl(4,\Re)$ is in R, if and only if its upper left 3×3 minor belongs to $SO(3,\Re)$, which we again characterize by 10 rational or polynomial equations, (this time in the coordinates of $Gl(4,\Re)$), if the first three entries of its last row are zero, and if the last entry of its last row is one. Thus, as in the case of $SO(3,\Re)$ in $Gl(3,\Re)$, R is the intersection of closed sets in $Gl(4,\Re)$, and hence is closed. It now follows again from the proposition that R is an imbedded Lie subgroup of $Gl(4,\Re)$. In a similar manner, we see that S and T are imbedded Lie subgroups of R, and hence of $Gl(4,\Re)$.

Recall that Φ of Equation 7 is an isomorphism of $SO(3,\Re)$ onto S. It extends easily to a map from $Gl(3,\Re)$ into $\mathbf{gl}(4,\Re)$. The extension is polynomial in the coordinates of $Gl(3,\Re)$, so that it is a C^∞ function on $Gl(3,\Re)$, and hence Φ is a C^∞ function on $SO(3,\Re)$. Similarly, the function mapping a matrix in $Gl(4,\Re)$ onto its upper left 3×3 minor is a C^∞ function on $Gl(4,\Re)$. Its restriction to S gives a C^∞ inverse for Φ, which makes Φ a diffeomorphism. Thus Φ is a diffeomorphism, and hence Φ is a Lie group isomorphism of $SO(3,\Re)$ onto S.

As should be expected, Lie subgroups are preserved by Lie group isomorphisms:

Proposition 4 *Let Φ be a Lie group isomorphism of Lie group G onto Lie group G'. H is a Lie subgroup of G, if and only if $\Phi(H)$ is a Lie subgroup of G'.*

Since Φ is a diffeomorphism, H is connected, if and only if $\Phi(H)$ is.

Now we turn to Lie algebras. We begin with several definitions. Recall that for (real) vector spaces V and W a mapping f from V into W is said to be linear, if

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

for all $u, v \in V$ and $\alpha, \beta \in \Re$. For a (real) vector space V let $V \times V$ be the set of ordered pairs defined by

$$V \times V = \{(u, v) : u, v \in V\}.$$

Say that a function

$$\Psi : (u, v) \in V \times V \to \Psi(u, v) \in V$$

is a bilinear operator on V, if it is linear in both its parameter values, that is, if

$$\Psi(\alpha u + \beta v, \gamma w + \delta x) = \alpha\gamma\Psi(u, w) + \alpha\delta\Psi(u, x) + \beta\gamma\Psi(v, w) + \beta\delta\Psi(v, x)$$

for all $u, v, w, x \in V$ and $\alpha, \beta, \gamma, \delta \in \Re$.

A Lie algebra is a (real) vector space V with a multiplication defined on it, which is a bilinear operator, denoted by $[,]$, that satisfies

$$[u, v] = -[v, u]$$

and

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$$

for all $u, v, w \in V$. The first of these conditions is called anticommutativity and the second is called the Jacobi identity.

If $\mathbf{h}$ is a subspace of a Lie algebra $\mathbf{g}$ such that the algebra's multiplication is closed on it, that is, such that $[,] : \mathbf{h} \times \mathbf{h} \to \mathbf{h}$, then $\mathbf{h}$ is said to be a subalgebra of $\mathbf{g}$. In this case, the restriction of $[,]$ to the subalgebra is an anticommutative bilinear operator on the subalgebra that satisfies the Jacobi identity, so that a subalgebra of a Lie algebra is again a Lie algebra.

If Φ is an invertible linear map from a Lie algebra $\mathbf{g}$ onto a Lie algebra $\mathbf{h}$ that satisfies

$$\Phi([u, v]) = [\Phi(u), \Phi(v)]$$

for all $u, v \in \mathbf{g}$, then Φ is an algebra isomorphism, and $\mathbf{g}$ and $\mathbf{h}$ are said to be algebra isomorphic. Algebra isomorphic Lie algebras are completely equivalent from the point of view of Lie algebra theory.

Consider some examples. We can always make a vector space into a Lie algebra by defining the product of any two vectors to be the zero vector. Such a Lie algebra is said to be trivial. In particular, $\Re^n$ with this multiplication is a Lie algebra. Less trivially, if we define

$$[,] : (u, v) \in \mathbf{gl}(n, \Re) \times \mathbf{gl}(n, \Re) \to [u, v] = uv - vu \in \mathbf{gl}(n, \Re), \tag{13}$$

with uv and vu being the normal matrix multiplication in $\mathbf{gl}(n, \Re)$, then it is easy to show that $[,]$ is an anticommutative bilinear operator on $\mathbf{gl}(n, \Re)$ that satisfies the Jacobi identity. Thus $\mathbf{gl}(n, \Re)$ with the multiplication given by $[,]$ is an example of a Lie algebra for any n. If we define

$$\mathbf{so}(n, \Re) = \{u \in \mathbf{gl}(n, \Re) : u^t = -u\}, \tag{14}$$

then $\mathbf{so}(n, \Re)$ is the subspace of $n \times n$ skew symmetric matrices. Since $[u, v]^t = u^t v^t - v^t u^t = vu - uv = -[u, v]$, it follows that $\mathbf{so}(n, \Re)$ is a subalgebra of $\mathbf{gl}(n, \Re)$.

These algebras and their subalgebras will, in fact, be the only Lie algebras, which we will encounter or need.

The differential and group structures of an n-dimensional Lie group G can be used in a canonical manner to define the multiplication of an n-dimensional Lie algebra, which is referred to as the Lie algebra of G. Several techniques can be utilized to provide this definition. To present one of the most elementary, let $\{\phi, U\}$ be a local coordinate system for G, such that the identity e belongs to U and $\phi(e) = 0$. Since ϕ and group multiplication are C^∞ functions, there is an open set V in $\Re^n$ containing 0 such that

$$m : (u, v) \in V \times V \to m(u, v) = \phi(prod(\phi^{-1}(u), \phi^{-1}(v))) \in \Re^n \tag{15}$$

is a well defined C^∞ function. Since $m(u, 0) = u$, it follows that $\frac{\partial m_j}{\partial u_i} = \delta_{i,j}$ at $(u, 0)$, so that $\frac{\partial^2 m_i}{\partial u_j \partial u_k} = 0$ at $(0, 0)$. Similarly from $m(0, v) = v$ it follows that $\frac{\partial m_j}{\partial v_i} = \delta_{i,j}$ at $(0, v)$ and that $\frac{\partial^2 m_i}{\partial v_j \partial v_k} = 0$ at $(0, 0)$. Thus if we let A^i be the matrix whose jk^{th} entry is $\frac{\partial^2 m_i}{\partial u_j \partial v_k}$ at $(0, 0)$, then the second order Taylor expansion of $m_i(u, v)$ takes the form

$$m_i(u, v) = u_i + v_i + u^t A^i v + \varepsilon_3(u, v) \tag{16}$$

with $\varepsilon_3(u, v)$ being the third order error function. Now if B is the basis of $T_e(G)$ derived from $\{\phi, U\}$, define

$$[,] : (u, v) \in T_e(G) \times T_e(G) \to [u, v] \in T_e(G)$$

by defining its component functions in basis B as

$$[,]_i : (u, v) \in T_e(G; B) \times T_e(G; B) \to [u, v]_i = u^t(A^i - (A^i)^t)v \in T_e(G; B)\ . \tag{17}$$

This multiplication gives $T_e(G)$ the structure of a Lie algebra, which is independent of the choice of local coordinate system. $T_e(G)$ with this multiplication is referred to as the Lie algebra of G.[3]

Several remarks are in order. First, if G is commutative, that is, if $gh = hg$ for all $g, h \in G$, then A^i is symmetric for all i, so the Lie algebra of G is the trivial one with the product of any elements equal to zero. Next, note that Equation 15 means that m is just a representation of the group's multiplication near the identity in a local coordinate system. Since inv is C^∞ and $g(e) = e$, both g and g^{-1} belong to $\phi^{-1}(V)$ for all g in some open set $W \subset G$ containing e. On this set W by Equation 16

$$0 = m(\phi(g), \phi(g^{-1})) = \phi(g) + \phi(g^{-1}) + \varepsilon_2(\phi(g), \phi(g^{-1})) \tag{18}$$

[3][16] shows that the definition of the multiplication appearing here agrees with the standard definition of the multiplication of the Lie algebra of a Lie group obtained from the Lie bracket of left invariant vector fields. The claims then follow from standard results.

with $\varepsilon_2(u,v)$ being the second order error function. Thus, this and Equation 16 imply that to a first order approximation near the identity the group operations, *prod* and *inv*, of a Lie group reduce to summing and subtracting coordinates in any local coordinate system.

Consider some examples. $\Re^n$ with vector addition is a Lie group. We can identify $T_0(\Re^n)$ with $\Re^n$. Since vector addition is commutative, the Lie algebra of $\Re^n$ is $\Re^n$ with the trivial product. $Gl(n,\Re)$ is more interesting. We can identify the tangent space at I with $\mathbf{gl}(n,\Re)$. If we define $\phi : g \in Gl(n,\Re) \to g - I \in \mathbf{gl}(n,\Re)$, then after identifying $\mathbf{gl}(n,\Re)$ with $\Re^{n^2}$ we can use $\{\phi, Gl(n,\Re)\}$ as a local coordinate system. Since $\phi(I) = 0$, we can use $\{\phi, Gl(n,\Re)\}$ to evaluate m of Equation 15 finding

$$m(u,v) = \phi(\phi^{-1}(u)\phi^{-1}(v)) = (u+I)(v+I) - I = u + v + uv\ .$$

To find A^{ij} of Equation 16 in the standard basis $\{e_{ij}\}$ of $\mathbf{gl}(n,\Re)$ we observe that

$$(uv)_{ij} = \Sigma_p u_{ip} v_{pj} = \Sigma_{pq,rs} A^{ij}_{pq,rs} u_{pq} v_{rs} = u^t A^{ij} v,$$

where $A^{ij}_{pq,rs} = \delta_{p,i}\delta_{q,r}\delta_{j,s}$. Now since

$$u^t (A^{ij})^t v = \Sigma_{pq,rs} A^{ij}_{rs,pq} u_{pq} v_{rs} = \Sigma_{pq,rs} \delta_{r,i}\delta_{s,p}\delta_{j,q} u_{pq} v_{rs} = \Sigma_p u_{pj} v_{ip} = (vu)_{ij}\ ,$$

it follows from Equation 17 that using the multiplication of Equation 13 makes $\mathbf{gl}(n,\Re)$ into the Lie algebra of $Gl(n,\Re)$.

Since the multiplication of the Lie algebra of a Lie group is defined in terms of the group's multiplication, it should not be surprising that much of the structure of the group is mirrored in the algebra. In fact, a Lie algebra is primarily of interest, because of its intimate relationship with its group. Often we can obtain information about a Lie group by obtaining more accessible information about its Lie algebra. Many times this relationship is exploited by using a certain C^∞ map to which we now turn. If G is a Lie group with Lie algebra $\mathbf{g}$, then $f \in C^\infty(\mathbf{g}, G)$ is called an exponential map, if $f(0) = e$, *df* at 0 is the identity map, and $f((\alpha+\beta)v) = f(\alpha v)f(\beta v)$ for all $\alpha, \beta \in \Re$ and $v \in \mathbf{g}$. In these terms we have

Proposition 5 *Each Lie group has a unique exponential map. There is an open set U of the Lie algebra of the group, which contains the zero vector, and an open set V of the group, which contains the identity, such that the exponential map is a diffeomorphism from U onto V.*

A Lie group's unique exponential map is uniformly denoted by exp.

The choice of the name of the exponential map is well motivated and justified by this primary example. Consider the map defined on $\mathbf{gl}(n,\Re)$ by

$$\exp : v \in \mathbf{gl}(n,\Re) \to \exp(v) = \sum_{i=0}^{\infty} v^i/i!\ , \tag{19}$$

where v^0 is defined to be I for all $v \in \mathbf{gl}(n, \Re)$. Since the series converges uniformly on any bounded subset of $\mathbf{gl}(n, \Re)$, it follows that $\exp \in C^\infty(\mathbf{gl}(n, \Re))$. Clearly, $\exp(0) = I$, and $d\exp$ at 0 is the identity on $\mathbf{gl}(n, \Re)$. For all $\alpha, \beta \in \Re$ and $v \in \mathbf{g}$ we have

$$
\begin{aligned}
(\sum_{i=0}^{\infty} \frac{(\alpha v)^i}{i!})(\sum_{i=0}^{\infty} \frac{(\beta v)^i}{i!}) &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{(\alpha v)^j}{j!} \frac{(\beta v)^{i-j}}{(i-j)!} \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) \alpha^j \beta^{i-j} \frac{v^i}{i!} = \sum_{i=0}^{\infty} (\alpha + \beta)^i \frac{v^i}{i!} \, ,
\end{aligned}
$$

so that $\exp(\alpha v)\exp(\beta v) = \exp((\alpha + \beta)v)$. Since $\exp(v)\exp(-v) = \exp(v - v) = \exp(0) = I$, $\exp(v)$ is invertible, that is, $\exp(v) \in Gl(n, \Re)$. It follows that exp is the exponential map of $Gl(n, \Re)$.

The exponential map can sometimes be used to show that a subgroup is a Lie subgroup and that a subspace of a Lie algebra is a subalgebra. This is accomplished by the following partial converse to the last result.

Proposition 6 *Let G be a Lie group with Lie algebra $\mathbf{g}$. Let U be an open set of $\mathbf{g}$ containing the zero vector, and let V be an open set of G containing the identity, such that the exponential map,* exp, *is a diffeomorphism from U onto V. If there is a subspace $\mathbf{h}$ of $\mathbf{g}$ and a subgroup H of G, such that $\exp(\mathbf{h} \cap U) = H \cap V$, then H is a Lie subgroup of G, and $\mathbf{h}$ is the Lie algebra of H.*

Since there is always an open set of the zero vector of the Lie algebra and an open set of the identity of the Lie group upon which the exponential map is a diffeomorphism, the last result implies that the image under the exponential map of any subspace of the Lie algebra, which is a subgroup, is a Lie subgroup. The definition of the exponential implies that the image of any line through the zero vector of the Lie algebra is a subgroup, so it follows that it is a Lie subgroup. We formalize these observations in a corollary.

Corollary 1 *Let G be a Lie group with Lie algebra $\mathbf{g}$. If $\mathbf{h}$ is a subspace of $\mathbf{g}$, such that $\exp(\mathbf{h})$ is a subgroup of G, then $\exp(\mathbf{h})$ is a Lie subgroup of G, and $\mathbf{h}$ is its Lie algebra. The image of any line through the zero vector of $\mathbf{h}$ under the exponential map is a Lie subgroup of G with the line as its Lie algebra.*

As an example of the structural similarities of Lie groups and their algebras and how the exponential map can be used to exploit them we have the following result. In it Lie subgroups of a Lie group, subalgebras of its Lie algebra, and the exponential map are shown to fit together pleasingly:

Proposition 7 *Let* $\mathbf{g}$ *be the Lie algebra of a Lie group* G*. If* H *is a Lie subgroup of* G*, then the Lie algebra of* H *is a subalgebra of* $\mathbf{g}$*. If* $\mathbf{h}$ *is a subalgebra of* $\mathbf{g}$*, then* G *has a Lie subgroup that has* $\mathbf{h}$ *as its Lie algebra. If* $\mathbf{h}$ *is the Lie algebra of* H*, then the restriction to* $\mathbf{h}$ *of the exponential map of* G *is the exponential map of* H*, and* $\mathbf{h}$ *is the inverse image of* H *under the exponential map of* G*.*

There may in fact be many subgroups of a Lie group that have the same subalgebra of the group's Lie algebra as their Lie algebra. However, for connected subgroups the situation is simpler.

Proposition 8 *If* $\mathbf{g}$ *is the Lie algebra of a Lie group* G*, then each subalgebra of* $\mathbf{g}$ *is the Lie algebra of a unique connected Lie subgroup of* G*.*

Propositions 7 and 8 are consistent, because a Lie group and its identity component have the same Lie algebra, and the image of the exponential map is connected, and hence always contained in the identity component.

We close this section by presenting a result on the relationship between Lie isomorphic groups and their algebras.

Proposition 9 *Let* G *be a Lie group with Lie algebra* $\mathbf{g}$ *and* H *a Lie group with Lie algebra* $\mathbf{h}$*. If* f *is a Lie group isomorphism from* G *onto* H*, then* df *at* e *is an algebra isomorphism from* $\mathbf{g}$ *onto* $\mathbf{h}$*.*

4 $SO(3, \Re)$ and Its Connected Lie Subgroups

In this section we treat $SO(3, \Re)$. We can consider it an example of application of the tools discussed previously, but the results also serve later in the classification of the connected Lie subgroups of R. We begin by showing that $\mathbf{so}(3, \Re)$ is its Lie algebra. Although we will not use it, precisely the same proof shows that $\mathbf{so}(n, \Re)$ is the Lie algebra of $SO(n, \Re)$. Next we find the subalgebras of $\mathbf{so}(3, \Re)$. Here we employ inner automorphisms to simplify some calculation. As a byproduct we obtain a simple representation of each 1-dimensional algebra. Finally we use these algebras to obtain the connected Lie subgroups of $SO(3, \Re)$. The exponential map will be the tool that allows us to pass from the subalgebras to the subgroups.

To begin recall that $SO(3, \Re)$ is a Lie subgroup of $Gl(3, \Re)$, so its Lie algebra is a subalgebra of $\mathbf{gl}(3, \Re)$. Thus we already know the multiplication on $T_I(SO(3, \Re))$, its tangent space at its identity, and we need only determine the subspace itself.

By definition the inverse of any $u \in SO(3, \Re)$ is its transpose u^t. For notational purposes let τ denote the transpose map, that is, $\tau : u \in SO(3, \Re) \to u^t$, and as usual let inv be the inverse map, that is, $inv : u \in SO(3, \Re) \to u^{-1}$. In these terms we have $\tau(u) = inv(u)$ for all $u \in SO(3, \Re)$. Since τ maps $SO(3, \Re)$ onto $SO(3, \Re)$

and maps I to I, the differential of τ, $d\tau$, maps $T_I(SO(3,\Re))$ to $T_I(SO(3,\Re))$. Now τ extends easily to $\tau^{'}: u \in \mathbf{gl}(3,\Re) \to u^t$ and is linear there, so that $\tau^{'}$ is its own differential, that is, $d\tau^{'}: v \in \mathbf{gl}(3,\Re) \to \tau^{'}(v) = v^t$. Since $SO(3,\Re)$ is a submanifold of $\mathbf{gl}(3,\Re)$, the restriction of $d\tau^{'}$ to $T_I(SO(3,\Re))$ yields $d\tau$. By Equation 18 $dinv$: $v \in T_I(SO(3,\Re)) \to -v$. Thus $v^t = d\tau(v) = dinv(v) = -v$ for all $v \in T_I(SO(3,\Re))$, so that $T_I(SO(3,\Re)) \subset \mathbf{so}(3,\Re)$.

To obtain the opposite inclusion, let $v \in \mathbf{so}(3,\Re)$. By Equation 19 we have that $(\exp(v))^t = \exp(v^t)$, so that $\exp(v)(\exp(v))^t = \exp(v)\exp(v^t) = \exp(v)\exp(-v) = \exp(v-v) = I$, that is, $\exp(v)$ is orthogonal. Since $\exp(v)$ is orthogonal, its determinant is ± 1. Since the image of $\mathbf{so}(3,\Re)$ under exp is connected, and contains I whose determinant is 1, it follows that $det(\exp(v)) = 1$, so that $\exp(v) \in SO(3,\Re)$. This in turn implies that $v \in T_I(SO(3,\Re))$ by Proposition 7. Since v was arbitrary, $\mathbf{so}(3,\Re) \subset T_I(SO(3,\Re))$, yielding that $\mathbf{so}(3,\Re) = T_I(SO(3,\Re))$, as claimed. Since $\mathbf{so}(3,\Re)$ is 3-dimensional, so is $SO(3,\Re)$.

If $g \in SO(3,\Re)$, then by Proposition 9 the differential of the inner automorphism

$$i_g : u \in SO(3,\Re) \to gug^{-1} = gug^t$$

is an algebra isomorphism of $\mathbf{so}(3,\Re)$. Since i_g easily extends to a linear transformation of $\mathbf{gl}(3,\Re)$, di_g coincides with i_g, so that this algebra isomorphism is given by

$$\mathbf{i}_g : v \in \mathbf{so}(3,\Re) \to gvg^t \, .$$

If $v \in \mathbf{so}(3,\Re)$, then $det(v) = 0$, since $det(v) = det(v^t) = det(-v) = -det(v)$. This implies that there exists a nonzero $w \in \Re^3$ with $vw = 0$. If $g \in SO(3,\Re)$ is a transformation taking w to a multiple of e_3, then gvg^t annihilates e_3, so that $\mathbf{i}_g(v) = gvg^t$ has the form

$$\begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{20}$$

If, as in the applications, we view $SO(3,\Re)$ (or $\mathbf{so}(3,\Re)$) as representations of transformations of $\Re^3$, then i_g (or $\mathbf{i}_g$) corresponds to nothing more than the effect on the representations of an orthogonal change of basis of $\Re^3$. Thus we may say that after a change of basis any $v \in \mathbf{so}(3,\Re)$ can be assumed to have the form of Equation 20.

Now we determine the subalgebras of $\mathbf{so}(3,\Re)$. We will proceed by increasing dimension. The first two are easy. Trivially, the zero vector is the only 0-dimensional subalgebra. Almost as trivially, the anticommutativity of Lie algebras implies that any 1-dimensional subspace, that is, any line through the zero vector, is a subalgebra. We now will complete our catalog of subalgebras by showing that any subalgebra of dimension at least two must in fact be 3-dimensional, and so coincides with $\mathbf{so}(3,\Re)$, itself. For this purpose let $\mathbf{h}$ be a 2-dimensional subalgebra of $\mathbf{so}(3,\Re)$. Since the dimension of an algebra is unaffected by an isomorphism, we can assume, possibly after an orthogonal change of basis, that $\mathbf{h}$ contains a matrix u of the form found in Equation 20. Since $\mathbf{h}$ is a vector space, all multiples of u are in $\mathbf{h}$, so we may even assume that $a = 1$ there. Since a vector space contains linear combinations of its vectors, we may assume, by subtracting a multiple of u, if necessary, that $\mathbf{h}$ contains a nonzero matrix v with $v_{21} = v_{12} = 0$. Since $\mathbf{h}$ is a subalgebra, $w = [u,v] \in \mathbf{h}$. To summarize then we may assume that

$$u = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & -c \\ b & c & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & -b \\ -c & b & 0 \end{pmatrix} \tag{21}$$

belong to $\mathbf{h}$. Since v is nonzero, both b and c cannot equal zero, but this implies that u, v, and w are linearly independent. Thus $\mathbf{h}$ is a vector subspace of dimension at least three in 3-dimensional $\mathbf{so}(3,\Re)$, and so must coincide with $\mathbf{so}(3,\Re)$, as claimed. We collect these simple results about subalgebras of $\mathbf{so}(3,\Re)$ in the following.

Lemma 1 *If $\mathbf{h}$ is a subalgebra of $\mathbf{so}(3,\Re)$, then $\mathbf{h}$ is the zero matrix alone, or a line through the zero matrix, or $\mathbf{so}(3,\Re)$, itself. If $\mathbf{h}$ and $\mathbf{h}'$ are any two of the 1-dimensional subalgebras, and A and A' are the lines annihilated by $\mathbf{h}$ and $\mathbf{h}'$, respectively, and g is a rotation carrying A to A', then the inner automorphism, i_g, of $SO(3,\Re)$ yields an algebra isomorphism $\mathbf{i}_g$ of $\mathbf{h}$ onto $\mathbf{h}'$.*

Now we turn to the connected Lie subgroups of $SO(3,\Re)$. To determine them we apply several results from Section 3. By Propositions 7 and 8 each subalgebra of $\mathbf{so}(3,\Re)$ is the Lie algebra of a unique connected Lie subgroup of $SO(3,\Re)$, and the Lie algebra of each connected Lie subgroup of $SO(3,\Re)$ is a subalgebra of $\mathbf{so}(3,\Re)$. Thus by identifying the connected Lie subgroups associated with the subalgebras of Lemma 1 we will determine all the connected Lie subgroups of $SO(3,\Re)$.

We consider the action of the exponential map on the algebras of Lemma 1 proceeding by increasing dimension. By definition the image of the zero vector under the exponential map is the identity, so of course the only 0-dimensional connected Lie subgroup of $SO(3,\Re)$ is the identity alone.

Among the 1-dimensional algebras, first consider the one given by $\mathbf{h} = \{\alpha u : \alpha \in \Re\}$, where u is defined as in Equation 21. By Corollary 1 the image of $\mathbf{h}$ under the exponential map of $SO(3,\Re)$ the connected Lie subgroup with Lie algebra $\mathbf{h}$. Denote this Lie subgroup by H. Since $SO(3,\Re)$ is a Lie subgroup of $Gl(3,\Re)$, Proposition 7 implies that the restriction of Equation 19 to $\mathbf{so}(3,\Re)$ is the exponential of $SO(3,\Re)$.

A simple calculation now yields

$$\exp : \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{h} \rightarrow \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \ .$$

Since α is an arbitrary real, H is immediately recognizable as the group of rotations about the $z-$axis.

Now consider any 1-dimensional algebra, $\mathbf{h}^{'}$, and let $H^{'}$ be the connected Lie subgroup of $SO(3,\Re)$ with $\mathbf{h}^{'}$ as its Lie algebra. By Lemma 1 there is an inner automorphism, i_g, of $SO(3,\Re)$, such that $\mathbf{i}_g$ is an algebra isomorphism of $\mathbf{h}$ onto $\mathbf{h}^{'}$. Corollary 1 and the uniform convergence of exp imply that

$$H^{'} = \exp(\mathbf{h}^{'}) = \exp(\mathbf{i}_g(\mathbf{h})) = \exp(g\mathbf{h}g^{-1}) = g\exp(\mathbf{h})g^{-1} = i_g(\exp(\mathbf{h})) = i_g(H) \ .$$

Since i_g is nothing more than the effect of an orthogonal change of basis, it follows that $H^{'}$ is the group of rotations about the axis given by the action of g on the $z-$axis. Thus each 1-dimensional connected Lie subgroup is the group of rotations about some axis through the origin. Conversely, if $H^{'}$ is the group of rotations about axis A through the origin, and if g is a rotation carrying A to the $z-$axis, then $i_g(H) = H^{'}$. Since i_g is a Lie group automorphism of $SO(3,\Re)$, and H is a connected Lie subgroup of $SO(3,\Re)$, it follows that $H^{'}$ is a connected Lie subgroup of $SO(3,\Re)$.

To finish we must find the connected Lie subgroup corresponding to the one 3-dimensional algebra, $\mathbf{so}(3,\Re)$, itself. Since each element of $SO(3,\Re)$ is a rotation about some axis, it is an element of some 1-dimensional subgroup. This 1-dimensional subgroup provides a path connecting the element to the identity, so $SO(3,\Re)$ is connected. Collecting these results we obtain the following.

Lemma 2 *If H is a Lie subgroup of $SO(3,\Re)$, then H is the identity alone, or the group of rotations about any fixed line through the origin, or $SO(3,\Re)$, itself. If H and $H^{'}$ are any two of the 1-dimensional subgroups with axis of rotation A and $A^{'}$, respectively, and g is a rotation carrying A to $A^{'}$, then the inner automorphism, i_g, of $SO(3,\Re)$ is a Lie group isomorphism of H onto $H^{'}$.*

5 R and Its Connected Lie Subgroups

Recall that R, defined in Equation 5, represents the rigid motions as matrices in $Gl(4,\Re)$. In Section 3 it was shown to be a Lie group. In this section we will treat R in a manner analogous to the manner $SO(3,\Re)$ was treated in the last section. In particular, we will first find its Lie algebra. Next, we will determine all subalgebras of this Lie algebra. To close we will use the exponential map and the knowledge about these subalgebras to obtain all the connected Lie subgroups of R.

Recall that R is the product of S and T, defined in Equations 3 and 4. From Section 3 we know that S and T are Lie subgroups of R, which is a Lie subgroup of $Gl(4,\Re)$, so the Lie algebras of S and T are subalgebras of the Lie algebra of R, which is a subalgebra of $\mathbf{gl}(4,\Re)$. Thus, we know the multiplication on the Lie algebras of R, S, and T to be that of $\mathbf{gl}(4,\Re)$ and need only identify the subspaces of $\mathbf{gl}(4,\Re)$ corresponding to them.

Identifying the Lie algebra of S is easy. Map Φ of Equation 7 is a Lie group isomorphism of $SO(3,\Re)$ onto S. Proposition 9 implies that $d\Phi$ is an algebra isomorphism of $\mathbf{so}(3,\Re)$ onto the Lie algebra of S. Since Φ extends easily to an affine map from $\mathbf{gl}(3,\Re)$ into $\mathbf{gl}(4,\Re)$, its differential is just its linear part there. Upon restricting to $\mathbf{so}(3,\Re)$ it follows that

$$\mathbf{s} = \left\{ \begin{pmatrix} v & 0^3 \\ 0_3 & 0 \end{pmatrix} \in \mathbf{gl}(4,\Re) : v \in \mathbf{so}(3,\Re) \right\}$$

is the Lie algebra of S. It is equally easy to find the Lie algebra of T. If we define

$$\Psi : v \in \Re^3 \to \begin{pmatrix} I_3^3 & v \\ 0_3 & 1 \end{pmatrix} \in T \ ,$$

then Equation 6 implies that Ψ is a Lie algebra isomorphism of $\Re^3$ onto T. From an argument similar to that we just gave to obtain $\mathbf{s}$ it follows that

$$\mathbf{t} = \left\{ \begin{pmatrix} 0_3^3 & v \\ 0_3 & 0 \end{pmatrix} \in \mathbf{gl}(4,\Re) : v \in \Re^3 \right\}$$

is the Lie algebra of T. Since S and T are Lie subgroups of R, the Lie algebra of R contains $\mathbf{s}$ and $\mathbf{t}$. In fact, since it is a vector space, it must contain the sums of all vectors in these spaces, so it must contain

$$\mathbf{r} = \left\{ \begin{pmatrix} r & v \\ 0_3 & 0 \end{pmatrix} \in \mathbf{gl}(4,\Re) : r \in \mathbf{so}(3,\Re), v \in \Re^3 \right\} \ .$$

Since $\mathbf{so}(3,\Re)$ is a subalgebra of $\mathbf{gl}(3,\Re)$, by Equation 13 the multiplication of $\mathbf{gl}(4,\Re)$ restricted to $\mathbf{r}$ satisfies

$$[,] : \left(\begin{pmatrix} r & v \\ 0_3 & 0 \end{pmatrix}, \begin{pmatrix} s & u \\ 0_3 & 0 \end{pmatrix} \right) \in \mathbf{r} \times \mathbf{r} \to \begin{pmatrix} rs - sr & ru - sv \\ 0_3 & 0 \end{pmatrix} \in \mathbf{r} \ , \qquad (22)$$

so that $\mathbf{r}$ is a subalgebra of the Lie algebra of R. By Proposition 7 R has Lie subgroup, R', with $\mathbf{r}$ being the Lie algebra of R'. Since $\mathbf{s}$ and $\mathbf{t}$ are subalgebras of $\mathbf{r}$, and since

S and T are connected, Proposition 8 implies that S and T are Lie subgroups of R'. But now since R is the product of S and T, R' contains R, so that $R = R'$, and hence $\mathbf{r}$ is the Lie algebra of R. Since $\mathbf{s}$ and $\mathbf{t}$ are each 3-dimensional algebras, $\mathbf{r}$ is of dimension 6, so that R is a 6-dimensional group.

Earlier we saw that an inner automorphism of $SO(3,\Re)$ becomes an algebra isomorphism when applied to $\mathbf{so}(3,\Re)$. It follows from a completely analogous argument that an inner automorphism of R induces an algebra isomorphism on $\mathbf{r}$, so that, in particular, inner automorphism

$$i_g : u \in R \rightarrow gug^{-1} \in R \text{ for } g \in R \tag{23}$$

yields the algebra isomorphism

$$\mathbf{i}_g : \mathbf{v} \in \mathbf{r} \rightarrow g\mathbf{v}g^{-1} \in \mathbf{r} \text{ for } g \in R\ . \tag{24}$$

Applying Equation 6 and simple calculation yield the more explicit forms

$$i_g : \begin{pmatrix} s & w \\ 0_3 & 1 \end{pmatrix} \in R \rightarrow \begin{pmatrix} rsr^t & -rsr^tv + rw + v \\ 0_3 & 1 \end{pmatrix} \text{ for } g = \begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} \in R \tag{25}$$

and

$$\mathbf{i}_g : \begin{pmatrix} s & w \\ 0_3 & 0 \end{pmatrix} \in \mathbf{r} \rightarrow \begin{pmatrix} rsr^t & -rsr^tv + rw \\ 0_3 & 0 \end{pmatrix} \text{ for } g = \begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} \in R \tag{26}$$

for the automorphism and the algebra isomorphism. If B and C are subsets of R, and there is some $g \in R$ such that $i_g(B) = C$, then we will say that B and C are i-isomorphic, or that C is the i-isomorphic image of B. Similarly, if $\mathbf{b}$ and $\mathbf{c}$ are subsets of $\mathbf{r}$, and there is some $g \in R$ such that $\mathbf{i}_g(\mathbf{b}) = \mathbf{c}$, then we will say $\mathbf{i}_g$ is an i-isomorphism and that $\mathbf{b}$ and $\mathbf{c}$ are i-isomorphic, or that $\mathbf{c}$ is the i-isomorphic image of $\mathbf{b}$.

Choosing r as in the discussion of the previous section leading to Equation 20 and Equation 26 imply any $\mathbf{v} \in \mathbf{r}$ is i-isomorphic to a matrix of the form

$$\begin{pmatrix} 0 & -a & 0 & w_1 \\ a & 0 & 0 & w_2 \\ 0 & 0 & 0 & w_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{27}$$

If, as in the applications, we view R (or $\mathbf{r}$) as representations of linear transformations of $\Re^3$, then i_g corresponds to nothing more than the effect on the representations of an affine orthogonal change of basis of $\Re^3$. Thus we may say that after a change of basis any $v \in \mathbf{r}$ can be assumed to have the form of Equation 27.

5.1 The Subalgebras of $\mathbf{r}$

We now consider the subalgebras of R. As was the case for the subalgebras of $\mathbf{so}(3,\Re)$ we will proceed by increasing dimensions. Employing inner automorphisms will again aid certain calculations. We will use one additional construct: projections onto $\mathbf{s}$. We discuss this construct, before beginning our catalog of algebras.

Since $\mathbf{r}$ is the sum of complementary spaces $\mathbf{s}$ and $\mathbf{t}$, any $\mathbf{u} \in \mathbf{r}$ can be uniquely expressed as $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathbf{s}$ and $\mathbf{w} \in \mathbf{t}$. For any subset $\mathbf{f}$ of $\mathbf{r}$ define $\pi_{\mathbf{s}}(\mathbf{f})$, the projection of $\mathbf{f}$ onto $\mathbf{s}$, by setting

$$\pi_{\mathbf{s}} : \mathbf{f} \subset \mathbf{r} \to \{s : s \in \mathbf{s}, t \in \mathbf{t}, s + t \in \mathbf{r}\} \subset \mathbf{s} .$$

For any subalgebra $\mathbf{h}$ of $\mathbf{r}$ Equation 22 implies that $\pi_{\mathbf{s}}(\mathbf{h})$ is a subalgebra of $\mathbf{s}$. If $\mathbf{i}_g$ is an i-isomorphism, then Equation 26 and the results of Section 4 imply that $\pi_{\mathbf{s}}(\mathbf{i}_g(\mathbf{h})) = \mathbf{i}_g(\pi_{\mathbf{s}}((\mathbf{h}))$ and that $\mathbf{i}_g$ is an isomorphism of $\pi_{\mathbf{s}}(\mathbf{h})$ onto a subalgebra of $\mathbf{s}$, so that $\pi_{\mathbf{s}}(\mathbf{h})$ is of the same dimension as $\mathbf{i}_g(\pi_{\mathbf{s}}((\mathbf{h}))$.

It follows from Equation 26 that $\pi_{\mathbf{s}}(v) = 0$, if and only if $\pi_{\mathbf{s}}(\mathbf{i}_g(v)) = 0$ for any $v \in \mathbf{h}$. Thus if $\pi_{\mathbf{s}}(\mathbf{h}) \neq 0$, then there is a $g \in R$ and a $v \in \mathbf{h}$ such that $\mathbf{i}_g(v)$ has the form of Equation 27 with $a = 1$. If we now define

$$g' = \begin{pmatrix} 1 & 0 & 0 & w_2 \\ 0 & 1 & 0 & -w_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

then a simple calculation yields that $\mathbf{i}_{g'}(\mathbf{i}_g(v)) = \mathbf{i}_{g'g}(v)$ has the form

$$\mathbf{u}(\mu) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

for some $\mu \in \Re$. In sum, we have

Lemma 3 *If $\mathbf{h}$ is a subalgebra of $\mathbf{r}$ with $\pi_{\mathbf{s}}(\mathbf{h})$ of dimension greater than zero, then there is a $g \in R$, such that the subalgebra $\mathbf{i}_g(\mathbf{h})$ contains a vector of the form of Equation 28.*

Because $\pi_{\mathbf{s}}(\mathbf{h})$ is a projection of $\mathbf{h}$, the dimension of $\pi_{\mathbf{s}}(\mathbf{h})$ is less than or equal to that of $\mathbf{h}$. Since $\mathbf{s}$ is algebra isomorphic to $\mathbf{so}(3,\Re)$, it follows that $\pi_{\mathbf{s}}(\mathbf{h})$ is algebra isomorphic to some subalgebra of $\mathbf{so}(3,\Re)$. Thus Lemma 1 can be used to describe the subalgebras determined by $\pi_{\mathbf{s}}$. In particular, $\pi_{\mathbf{s}}(\mathbf{h})$ must be a 0-, 1-, or 3-dimensional subalgebra of $\mathbf{s}$. We will use the dimension of these derived algebras to determine subcases of subalgebras of the same dimension below.

To begin our catalog of subalgebras, we note that of course the only 0-dimensional algebra is the zero matrix alone. In the following sections we consider the more interesting cases.

5.1.1 The 1-dimensional subalgebras

The anticommutativity of Lie algebra multiplication implies that the 1-dimensional subalgebras are lines through the zero matrix and that any such line is a 1-dimensional subalgebra. If $\mathbf{h}$ is such an algebra, we distinguish two cases: when the dimension of $\pi_{\mathbf{s}}(\mathbf{h})$ is zero and when it is one.

If the dimension of $\pi_{\mathbf{s}}(\mathbf{h})$ is zero, then $\mathbf{h}$ is a subalgebra of $\mathbf{t}$. It follows that $\mathbf{h} = \{\alpha\mathbf{w} : \alpha \in \Re\}$ for some nonzero $\mathbf{w} = \begin{pmatrix} 0_3^3 & w \\ 0_3 & 0 \end{pmatrix} \in \mathbf{t}$. If $\mathbf{i}_g$ is an i-isomorphism derived from i_g, then we have from Equation 26 that

$$\mathbf{i}_g(\mathbf{h}) = \left\{\alpha \begin{pmatrix} 0_3^3 & rw \\ 0_3 & 0 \end{pmatrix} : \alpha \in \Re\right\} \text{ for } g = \begin{pmatrix} r & v \\ 0_3 & 1 \end{pmatrix} .$$

From this we can conclude that $\mathbf{h}$ is i-isomorphic to any other 1-dimensional subalgebra of $\mathbf{t}$, and if $\mathbf{h}^{'}$ is any other 1-dimensional subalgebra of $\mathbf{r}$, which is not a subalgebra of $\mathbf{t}$, then $\mathbf{h}$ and $\mathbf{h}^{'}$ are not i-isomorphic. In particular, each 1-dimensional subalgebra with 0-dimensional projection onto $\mathbf{s}$ is i-isomorphic to the subalgebra defined by

$$\mathbf{t}(1) = \left\{ \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha \in \Re \right\} . \tag{29}$$

Now we consider the case of a 1-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$. To begin for $\mu \in \Re$ let $\mathbf{m}(1;\mu) = \{\alpha\mathbf{u}(\mu) : \alpha \in \Re\}$ where $\mathbf{u}(\mu)$ is defined by Equation 28, that is, let

$$\mathbf{m}(1;\mu) = \left\{ \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha \in \Re \right\} . \tag{30}$$

$\mathbf{m}(1;\mu)$ is a 1-dimensional subalgebra with 1-dimensional projection onto $\mathbf{s}$ for all μ. To investigate when $\mathbf{m}(1;\mu)$ and $\mathbf{m}(1;\mu^{'})$ are i-isomorphic, let $\mathbf{i}_g$ be an i-isomorphism of $\mathbf{m}(1;\mu)$ onto $\mathbf{m}(1;\mu')$ for $g = \begin{pmatrix} r & d \\ 0_3 & 1 \end{pmatrix}$, and let us write $\mathbf{u}(\mu)$ as $\begin{pmatrix} u & w_\mu \\ 0_3 & 0 \end{pmatrix}$ and $\mathbf{u}(\mu')$ as $\begin{pmatrix} u & w_{\mu'} \\ 0_3 & 0 \end{pmatrix}$ with the obvious definitions of u, w_μ, and $w_{\mu'}$. Since $\mathbf{i}_g(\mathbf{m}(1;\mu)) = \mathbf{m}(1;\mu')$, it follows that $\pi_{\mathbf{s}}(\mathbf{i}_g(\mathbf{u}(\mu))) \subset \pi_{\mathbf{s}}(\mathbf{m}(1;\mu'))$, so that $rur^t = \pi_{\mathbf{s}}(\mathbf{i}_g(\mathbf{u}(\mu))) = \lambda u$ from Equation 26. If L_1 and L_2 are orthogonal lines through the origin of $\Re^3$, then it is easy to see that any rotation about the origin can be written as the product of a rotation about L_1, a rotation about L_2 of angle less than π, and a second rotation about L_1. In particular, we can write r as

$$r = r_1 r_2 r_3 = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\alpha_2 < \pi$. Since $r_3 u r_3^t = u$, it follows from $r_1 r_2 r_3 u (r_1 r_2 r_3)^t = rur^t = \lambda u$ that $r_1 r_2 u (r_1 r_2)^t = \lambda u$, and since $r_1^t \lambda u r_1 = \lambda u$, it then follows that $r_2 u r_2^t = \lambda u$. A simple calculation implies that $\lambda = 1$ and $\alpha_2 = 0$, so that

$$g = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & d_1 \\ \sin\beta & \cos\beta & 0 & d_2 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some $\beta \in \Re$. Thus by Equation 26 again

$$\mathbf{i}_g(\mathbf{u}(\mu)) = \begin{pmatrix} 0 & -1 & 0 & -d_2 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{m}(1;\mu') \,. \tag{31}$$

The definition of $\mathbf{m}(1;\mu')$ and this last equation imply that $\mu = \mu'$. Thus $\mathbf{m}(1;\mu)$ and $\mathbf{m}(1;\mu')$ are i-isomorphic, only when $\mu = \mu'$. We note in passing that $\mathbf{m}(1;0)$ is a subalgebra of $\mathbf{s}$ and that for no other μ is $\mathbf{m}(1;\mu)$ a subalgebra of $\mathbf{s}$ or $\mathbf{t}$.

Now if $\mathbf{h}$ is any 1-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$, then by Lemma 3 $\mathbf{h}$ is i-isomorphic to $\mathbf{m}(1;\mu)$ for some μ. By the preceeding this μ is unique. Conversely, $\mathbf{i}_g(\mathbf{m}(1;\mu))$ is a 1-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$ for any i-isomorphism $\mathbf{i}_g$ and any $\mu \in \Re$.

5.1.2 The 2-dimensional subalgebras

Again we distinguish two subcases of 2-dimensional subalgebras dependent upon whether the projection of the subalgebra onto $\mathbf{s}$ is of dimension one or two.

As before, if $\mathbf{h}$ is a 2-dimensional subalgebra with a 0-dimensional projection onto $\mathbf{s}$, then $\mathbf{h}$ is a subalgebra of $\mathbf{t}$. It follows that $\mathbf{h} = \left\{ \begin{pmatrix} 0_3^3 & v \\ 0_3 & 0 \end{pmatrix} : v \in V \right\}$ for some 2-dimensional subspace V of $\Re^3$. By Equation 22 the multiplication on $\mathbf{t}$ is trivial, so any 2-dimensional subspace $\mathbf{t}$ is a subalgebra. Thus if U is a 2-dimensional subspace of $\Re^3$, then $\mathbf{h}' = \left\{ \begin{pmatrix} 0_3^3 & v \\ 0_3 & 0 \end{pmatrix} : v \in U \right\}$ is a 2-dimensional subalgebra with a 0-dimensional projection onto $\mathbf{s}$. If $r \in \mathbf{so}(3, \Re)$ represents a rotation carrying V onto U, and $g = \begin{pmatrix} r & 0^3 \\ 0_3 & 1 \end{pmatrix} \in R$, then $\mathbf{i}_g$ is an $\imath$-isomorphism of $\mathbf{h}$ onto $\mathbf{h}'$ by Equation 26. Thus any 2-dimensional subspace of $\mathbf{t}$ is a subalgebra and is $\imath$-isomorphic to any other. In particular, any 2-dimensional subalgebra with a 0-dimensional projection onto $\mathbf{s}$ is an $\imath$-isomorphic image of the subalgebra defined by

$$\mathbf{t}(2) = \left\{ \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \Re \right\} . \tag{32}$$

Now we consider the case of a 2-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$. We begin by considering the possibility of an algebra of this type which contains matrix $\mathbf{u}(\mu)$ for some $\mu \in \Re$ defined by Equation 28. Since the projection of the subalgebra onto $\mathbf{s}$ is 1-dimensional, for each $\mathbf{w}$ in the subalgebra there is a $\lambda(\mathbf{w}) \in \Re$, such that $\pi_{\mathbf{s}}(\mathbf{w}) = \lambda(\mathbf{w})\pi_{\mathbf{s}}(\mathbf{u}(\mu))$. Since the subalgebra is a 2-dimensional vector space, it must contain some $\mathbf{w}$ such that $\mathbf{u}(\mu)$ and $\mathbf{w}$ are linearly independent, and in fact such that $\mathbf{v} = \mathbf{w} - \lambda(\mathbf{w})\mathbf{u}(\mu) \in \mathbf{t}$ and $\mathbf{u}(\mu)$ are linearly independent vectors contained in the subalgebra. To summarize the subalgebra contains

$$\mathbf{u}(\mu) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

which are linearly independent vectors. But the product of the vectors

$$[\mathbf{u}(\mu), \mathbf{v}] = \begin{pmatrix} 0 & 0 & 0 & -b_2 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

must also belong to the subalgebra. $[\mathbf{u}(\mu), \mathbf{v}]$ is a third independent vector, unless $b_1 = b_2 = 0$. Since the subalgebra is 2-dimensional, we must have $b_1 = b_2 = 0$, which reduces the equation to the statement that $[\mathbf{u}(\mu), \mathbf{v}]$ yields the zero matrix. This and anticommutativity imply that the span of $\mathbf{u}(\mu)$ and $\mathbf{v}$ is a subalgebra with the trivial product. Thus

$$\mathbf{m}(2) = \left\{ \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \Re \right\} \tag{33}$$

is the 2-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$ that contains $\mathbf{u}(\mu)$. But $\mathbf{m}(2)$ is independent of μ and clearly contains $\mathbf{u}(\mu)$ for all μ.

If we now turn to the more general situation, where $\mathbf{h}$ is a 2-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$, then Lemma 3 implies the existence of an automorphism $\imath_g$, such that $\mathbf{i}_g(\mathbf{h})$ contains $\mathbf{u}(\mu)$ for some μ. From what we have just shown, then, $\mathbf{i}_g(\mathbf{h}) = \mathbf{m}(2)$. Thus any 2-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$ is the $\imath$-isomorphic image of $\mathbf{m}(2)$.

5.1.3 The 3-dimensional subalgebras

Now we turn to the 3-dimensional subalgebras of $\mathbf{r}$. As before we partition into subcases dependent on the dimension of the projection of the algebra onto $\mathbf{s}$. Here we have projections of dimension zero, one and three.

We begin, as usual, with 3-dimensional subalgebras with 0-dimensional projections. We note as before that if a subalgebra has a 0-dimensional projection, then it must be a subalgebra of $\mathbf{t}$. However, since $\mathbf{t}$ is 3-dimensional, this implies that the subalgebra must coincide with $\mathbf{t}$.

We begin consideration of 3-dimensional subalgebras with 1-dimensional projections onto $\mathbf{s}$ by discussing the possibility of having one that also contains $\mathbf{u}(\mu)$ for some $\mu \in \Re$. Since the projection of the subalgebra onto $\mathbf{s}$ is 1-dimensional, for each $\mathbf{v}$ in the subalgebra there is a $\lambda(\mathbf{v}) \in \Re$, such that $\pi_{\mathbf{s}}(\mathbf{v}) = \lambda(\mathbf{v})\pi_{\mathbf{s}}(\mathbf{u}(\mu))$. Since the subalgebra is a 3-dimensional vector space, it must contain some $\mathbf{v}^{''}$ and $\mathbf{w}^{''}$ such that $\mathbf{u}(\mu)$, $\mathbf{v}^{''}$, and $\mathbf{w}^{''}$ are linearly independent, and in fact such that $\mathbf{u}(\mu)$, $\mathbf{v}^{'} = \mathbf{v}^{''} - \lambda(\mathbf{v}^{''})\mathbf{u}(\mu) \in \mathbf{t}$, and $\mathbf{w}^{'} = \mathbf{w}^{''} - \lambda(\mathbf{w}^{''})\mathbf{u}(\mu) \in \mathbf{t}$ are linearly independent vectors contained in the subalgebra. To summarize

$$\mathbf{u}(\mu) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v}^{'} = \begin{pmatrix} 0 & 0 & 0 & v_1^{'} \\ 0 & 0 & 0 & v_2^{'} \\ 0 & 0 & 0 & v_3^{'} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{w}^{'} \begin{pmatrix} 0 & 0 & 0 & w_1^{'} \\ 0 & 0 & 0 & w_2^{'} \\ 0 & 0 & 0 & w_3^{'} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are three linearly independent vectors in the subalgebra. In order to show that $w_3^{'} = 0$ we assume the contrary, that $w_3^{'} \neq 0$, and establish a contradiction. If we define

$\mathbf{v} = \mathbf{v}' - v_3'\mathbf{w}'/w_3'$ and $\mathbf{w} = \mathbf{w}'/w_3'$, then it must follow that

$$\mathbf{u}(\mu) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 0 & 0 & 0 & w_1 \\ 0 & 0 & 0 & w_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are three linearly independent vectors contained in the subalgebra. Since a subalgebra contains products of its elements,

$$[\mathbf{u}(\mu), \mathbf{v}] = \begin{pmatrix} 0 & 0 & 0 & -v_2 \\ 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

belongs to the subalgebra. But $\mathbf{u}(\mu)$, $\mathbf{v}$, $\mathbf{w}$, and $[\mathbf{u}, \mathbf{w}]$ form a set of four linearly independent vectors, unless $v_1 = v_2 = 0$. Since a 3-dimensional subalgebra cannot contain four linearly independent vectors, we have that the assumption that $w_3' \neq 0$ implies that $v_1 = v_2 = 0$. This in turn implies that $\mathbf{v}$ is the zero matrix, which is impossible, since $\mathbf{u}(\mu)$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent vectors. Since the contradiction followed from assuming that $w_3' \neq 0$, we have the contrary, that is, $w_3' = 0$. An identical argument yields that $v_3' = 0$. Thus, if it exists, the subalgebra must be the span of $\mathbf{u}(\mu)$, $\mathbf{v}'$, and $\mathbf{w}'$ with $w_3' = v_3' = 0$, so that it coincides with

$$\mathbf{m}(3;\mu) = \left\{ \begin{pmatrix} 0 & -\alpha & 0 & \beta \\ \alpha & 0 & 0 & \gamma \\ 0 & 0 & 0 & \mu\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \Re \right\}. \tag{34}$$

However, Equation 22 implies that $\mathbf{m}(3;\mu)$ is a subalgebra, so $\mathbf{m}(3;\mu)$ is the only 3-dimensional subalgebra with 1-dimensional projection onto $\mathbf{s}$ that contains $\mathbf{u}(\mu)$.

If $\mathbf{i}_g$ is an i-isomorphism from $\mathbf{m}(3;\mu)$ to $\mathbf{m}(3;\mu')$, then an argument identical to that which led to Equation 31 of Section 5.1.1 implies that $\mu = \mu'$, so that $\mathbf{m}(3;\mu)$ is i-isomorphic to $\mathbf{m}(3;\mu')$, only if $\mu = \mu'$. This and Lemma 3, as has been the case before, now imply that any 3-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$ is the i-isomorphic image of a $\mathbf{m}(3;\mu)$ for some unique μ, and conversely that any i-isomorphic image of some $\mathbf{m}(3;\mu)$ is such an algebra.

Before discussing the final case of 3-dimensional subalgebras, we present the following lemma, which bears directly on the case, and which will also be of use later.

Lemma 4 *If* $\mathbf{h}$ *is a subalgebra of* $\mathbf{r}$ *with* $\pi_{\mathbf{s}}(\mathbf{h}) = \mathbf{s}$, *then* $\mathbf{h}$ *is* i*-isomorphic to a subalgebra* $\mathbf{h}'$ *of* $\mathbf{r}$ *with* $\mathbf{s} \subset \mathbf{h}'$.

Proof. Since $\pi_{\mathbf{s}}(\mathbf{h}) = \mathbf{s}$, three linearly independent vectors of the form

$$\mathbf{u} = \begin{pmatrix} 0 & -1 & 0 & u_1 \\ 1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & u_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 & 0 & -1 & v_1 \\ 0 & 0 & 0 & v_2 \\ 1 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 0 & 0 & 0 & w_1 \\ 0 & 0 & -1 & w_2 \\ 0 & 1 & 0 & w_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{35}$$

must be contained in $\mathbf{h}$. Since $\mathbf{h}$ is an algebra, $\mathbf{h}$ contains $[\mathbf{v},\mathbf{w}]$, $[\mathbf{u},\mathbf{v}]$, and $[\mathbf{w},\mathbf{u}]$. Either $[\mathbf{v},\mathbf{w}]$, $[\mathbf{u},\mathbf{v}]$, and $[\mathbf{w},\mathbf{u}]$ are not linearly independent of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, or at least one of $[\mathbf{v},\mathbf{w}]$, $[\mathbf{u},\mathbf{v}]$, and $[\mathbf{w},\mathbf{u}]$ is linearly independent of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. We consider these two possibilities separately and begin by assuming that $[\mathbf{v},\mathbf{w}]$, $[\mathbf{u},\mathbf{v}]$, and $[\mathbf{w},\mathbf{u}]$ are not linearly independent of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Since

$$[\mathbf{v},\mathbf{w}] = \begin{pmatrix} 0 & -1 & 0 & -w_3 \\ 1 & 0 & 0 & v_3 \\ 0 & 0 & 0 & w_1 - v_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$[\mathbf{v},\mathbf{w}]$ is linearly dependent on $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, only if $u_1 = -w_3$, $u_2 = v_3$, and $u_3 = w_1 - v_2$, which implies that

$$\mathbf{u} = \begin{pmatrix} 0 & -1 & 0 & u_1 \\ 1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & u_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 & 0 & -1 & v_1 \\ 0 & 0 & 0 & v_2 \\ 1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \text{and } \mathbf{w} = \begin{pmatrix} 0 & 0 & 0 & u_3 + v_2 \\ 0 & 0 & -1 & w_2 \\ 0 & 1 & 0 & -u_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similar reductions based on the assumption that $[\mathbf{u},\mathbf{v}]$ and $[\mathbf{w},\mathbf{u}]$ in turn are not independent of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ imply that

$$\mathbf{u} = \begin{pmatrix} 0 & -1 & 0 & u_1 \\ 1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 & 0 & -1 & v_1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \text{and } \mathbf{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & v_1 \\ 0 & 1 & 0 & -u_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we let

$$g = \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & 0 & -u_1 \\ 0 & 0 & 1 & -v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and define $\mathbf{u}' = \mathbf{i}_g(\mathbf{u})$, $\mathbf{v}' = \mathbf{i}_g(\mathbf{v})$, and $\mathbf{w}' = \mathbf{i}_g(\mathbf{w})$, then

$$\mathbf{u}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{v}' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \text{and } \mathbf{w}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & v_1 \\ 0 & 1 & 0 & -u_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that $\mathbf{s} \subset \mathbf{i}_g(\mathbf{h})$. Thus choosing $\mathbf{h}' = \mathbf{i}_g(\mathbf{h})$ satisfies the claim of the lemma in this case.

Now we consider the second possibility, that is, we assume that at least one of $[\mathbf{v}, \mathbf{w}]$, $[\mathbf{u}, \mathbf{v}]$, and $[\mathbf{w}, \mathbf{u}]$ is linearly independent of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, if even one of the products is linearly independent of the three vectors, then $\mathbf{h}$ is of dimension at least four. Since $\mathbf{t}$ is 3-dimensional, and both $\mathbf{h}$ and $\mathbf{t}$ are contained in 6-dimensional $\mathbf{r}$, their intersection, $\mathbf{h} \cap \mathbf{t}$, must contain some nonzero matrix $\mathbf{a}$, and hence must contain $\{\lambda \mathbf{a} : \lambda \in \Re\}$, the line through $\mathbf{a}$. If we let

$$\mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{a}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are a basis of $\mathbf{t}$. By Equation 26 we may choose an $\imath$-isomorphism $\mathbf{i}_g$, such that $\mathbf{i}_g(\{\lambda \mathbf{a} : \lambda \in \Re\}) = \{\lambda \mathbf{a}_1 : \lambda \in \Re\}$. Consider subalgebra $\mathbf{g} = \mathbf{i}_g(\mathbf{h})$. By construction $\mathbf{a}_1 \in \mathbf{g}$. Since $\pi_{\mathbf{s}}(\mathbf{h}) = \mathbf{s}$, it follows that $\pi_{\mathbf{s}}(\mathbf{g}) = \mathbf{s}$, so that $\mathbf{g}$ contains three linearly independent vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ of the form given by Equation 35. Since $\mathbf{g}$ is a subalgebra, $\mathbf{g}$ contains $[\mathbf{u}, \mathbf{a}_1] = \mathbf{a}_2$ and $[\mathbf{v}, \mathbf{a}_1] = \mathbf{a}_3$. Thus $\mathbf{g}$ is 6-dimensional, so that $\mathbf{h}$, its $\imath$-isomorphic image, is also 6-dimensional. But since $\mathbf{r}$ is 6-dimensional, it follows that $\mathbf{h} = \mathbf{r}$. Since $\mathbf{r}$ certainly satisfies the claim of the lemma, the proof is completed. ■

As an immediate corollary of the proof of this lemma we have following result.

Corollary 2 *Let* $\mathbf{h}$ *be a subalgebra of* $\mathbf{r}$ *with* $\pi_{\mathbf{s}}(\mathbf{h}) = \mathbf{s}$. *If there exists nonzero* $v \in \mathbf{h} \cap \mathbf{t}$, *then* $\mathbf{h} = \mathbf{r}$.

Now to complete discussions of 3-dimensional subalgebras, we consider the case of a subalgebra $\mathbf{h}$ with a 3-dimensional projection onto $\mathbf{s}$. Since the projection onto $\mathbf{s}$ is 3-dimensional, $\pi_{\mathbf{s}}(\mathbf{h}) = \mathbf{s}$, so that the requirements of Lemma 4 are met. Thus $\mathbf{h}$ is $\imath$-isomorphic to a subalgebra $\mathbf{h}'$ of $\mathbf{r}$ with $\mathbf{s} \subset \mathbf{h}'$. Since $\mathbf{h}$ is 3-dimensional, so is $\mathbf{h}'$. Since $\mathbf{h}'$ is 3-dimensional, it must coincide with its 3-dimensional subalgebra $\mathbf{s}$. Thus any 3-dimensional subalgebra with a 3-dimensional projection onto $\mathbf{s}$ is $\imath$-isomorphic to $\mathbf{s}$, and, of course, conversely, any $\imath$-isomorphic image of $\mathbf{s}$ is such a subalgebra.

5.1.4 The higher dimensional subalgebras and a summary

Since little needs be treated for subalgebras of dimension greater than three, we will consider them together in this section.

Start by noting that if a subalgebra is 4-dimensional, then it must intersect each of the 3-dimensional subalgebras, $\mathbf{s}$ and $\mathbf{t}$, of 6-dimensional $\mathbf{r}$ in at least a line. This implies that the projection of the subalgebra onto $\mathbf{s}$ is at least 1-dimensional. Since the intersection of the subalgebra with $\mathbf{t}$ is nonzero, Corollary 2 implies that the projection cannot be 3-dimensional. It follows that we need only consider the case where the projection of the subalgebra onto $\mathbf{s}$ is 1-dimensional.

We begin by considering the possibility of having a 4-dimensional subalgebra with a 1-dimensional projection onto $\mathbf{s}$ that contains $\mathbf{u}(0)$. Since the projection of the subalgebra onto $\mathbf{s}$ is 1-dimensional, for each $\mathbf{v}$ in the subalgebra there is a $\lambda(\mathbf{v}) \in \Re$, such that $\pi_{\mathbf{s}}(\mathbf{v}) = \lambda(\mathbf{v})\pi_{\mathbf{s}}(\mathbf{u}(0))$. Since the subalgebra is a 4-dimensional vector space, it must contain matrices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, such that $\mathbf{u}(0)$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. The linear independence of the four vectors implies that $\mathbf{u}(0)$, $\mathbf{v}_1 - \lambda(\mathbf{v}_1)\mathbf{u}(0) \in \mathbf{t}$, $\mathbf{v}_2 - \lambda(\mathbf{v}_2)\mathbf{u}(0) \in \mathbf{t}$, and $\mathbf{v}_3 - \lambda(\mathbf{v}_3)\mathbf{u}(0) \in \mathbf{t}$ are linearly independent. But since $\mathbf{t}$ is 3-dimensional, this implies that the subalgebra must coincide with $\mathbf{m}(4) = \{\alpha\mathbf{u}(0) + \mathbf{v} : \alpha \in \Re, \mathbf{v} \in \mathbf{t}\}$, from which it follows that

$$\mathbf{m}(4) = \left\{ \begin{pmatrix} 0 & -\alpha & 0 & \beta \\ \alpha & 0 & 0 & \gamma \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \Re \right\}. \tag{36}$$

We note that $\mathbf{m}(4)$ is a subalgebra by Equation 22 and that $\mathbf{u}(\mu) \in \mathbf{m}(4)$ for all $\mu \in \Re$. Lemma 3 and what we have proven here imply that any 4-dimensional subalgebra of $\mathbf{r}$ is i-isomorphic to $\mathbf{m}(4)$, and, of course, any i-isomorphic image of $\mathbf{m}(4)$ is a 4-dimensional subalgebra.

To close we note that any 5-dimensional subspace of $\mathbf{r}$ must intersect each of the 3-dimensional subalgebras, $\mathbf{s}$ and $\mathbf{t}$, of 6-dimensional $\mathbf{r}$ in at least a 2-dimensional subspace. If the 5-dimensional subspace were a subalgebra, then this would force its projection onto $\mathbf{s}$ to be 3-dimensional, so that Corollary 2 would force the 5-dimensional subspace to coincide with 6-dimensional $\mathbf{r}$, which is absurd. Thus there can be no 5-dimensional subalgebras, and of course $\mathbf{r}$, itself, is the only possible 6-dimensional subalgebra.

We collect many of the results on subalgebras of $\mathbf{r}$ in the following proposition.

Proposition 10 *A subalgebra of* $\mathbf{r}$ *is i-isomorphic to one and only one of the following subalgebras: the zero matrix, alone;* $\mathbf{s}$*;* $\mathbf{t}(1)$*,* $\mathbf{t}(2)$*, or* $\mathbf{t}$*, itself;* $\mathbf{m}(1;\mu)$*,* $\mathbf{m}(2)$*,* $\mathbf{m}(3;\mu)$*, or* $\mathbf{m}(4)$*; or* $\mathbf{r}$*, itself.*

5.2 The Subgroups of R

In the previous section we obtained descriptions of the subalgebras of $\mathbf{r}$, the Lie algebra of R. In this section we will obtain equivalent descriptions of connected Lie subgroups of R.

We begin by determining the connected Lie subgroups associated with the subalgebras of Proposition 10. In many cases this is trivial. By definition exp maps the zero matrix to the identity. Earlier we showed that $\mathbf{s}$ was the Lie algebra of S. S is closed and connected, because it is Lie group isomorphic to $SO(3,\Re)$, which is closed and connected. We also know that $\mathbf{t}$ is the Lie algebra of T and that T is closed and connected. Since $\mathbf{t}$ is a subalgebra of $\mathbf{gl}(4,\Re)$, the restriction of the exponential of $\mathbf{gl}(4,\Re)$ to $\mathbf{t}$ is the exponential of $\mathbf{t}$. Thus by Equation 19

$$\exp : \mathbf{v} \in \mathbf{t} \to I + \mathbf{v} \in R \, . \tag{37}$$

If we use the last equation and define

$$T(1) = \exp(\mathbf{t}(1)) = \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \alpha \in \Re \right\}$$

and

$$T(2) = \exp(\mathbf{t}(2)) = \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \alpha, \beta \in \Re \right\} \, ,$$

then Equation 6 implies that $T(1)$ and $T(2)$ are subgroups, and hence Lie subgroups by Corollary 1. Moreover, $T(1)$ and $T(2)$ are obviously closed and connected. The Lie algebra of R is $\mathbf{r}$. Any $g \in R$ can be obtained as a sum $g = u + v$ with $u \in S$ and $v \in \mathbf{t}$. Since S is connected, there is a path $b(t)$ in S connecting u to I. Since $\mathbf{t}$ is connected, there is a path $c(t)$ in $\mathbf{t}$ connecting v to the zero matrix. It follows that $b(t) + c(t)$ is a path in R connecting g to I, so that R is connected. Thus only the connected Lie subgroups associated with $\mathbf{m}(1;\mu)$, $\mathbf{m}(2)$, $\mathbf{m}(3;\mu)$, and $\mathbf{m}(4)$ remain to be determined.

The first collection is easy. If for each $\mu \in \Re$ we define

$$M(1;\mu) = \exp(\mathbf{m}(1;\mu)) \, ,$$

then $M(1;\mu)$ is a Lie subgroup by Corollary 1. As the image of a line it is connected. To obtain a more explicit description we note that since $\mathbf{m}(1;\mu)$ is a subalgebra of $\mathbf{gl}(4,\Re)$, the restriction of the exponential of $\mathbf{gl}(4,\Re)$ to $\mathbf{m}(1;\mu)$ is the exponential of $\mathbf{m}(1;\mu)$. Thus for any $\mu \in \Re$ Equation 19 and a simple calculation yield

$$\exp : \alpha \mathbf{u}(\mu) \in \mathbf{m}(1;\mu) \to \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & \mu\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \in R \, , \tag{38}$$

so that

$$M(1;\mu)=\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 & 0\\ \sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & 1 & \mu\alpha\\ 0 & 0 & 0 & 1\end{pmatrix} : \alpha\in\Re\right\} .$$

From this for any $u \in R$ it is easy to see that we have $u \in M(1;0)$, if and only if u_{13}, u_{14}, u_{23}, u_{24}, and u_{34} are all zero and $u_{33}=1$; and that for $\mu\neq 0$ we have $u\in M(1;\mu)$, if and only if u_{13}, u_{14}, u_{23}, u_{24} all equal zero, $u_{33}=1$, $u_{11}=\cos(u_{34}/\mu)$, and $u_{21}=\sin(u_{34}/\mu)$. Since the coordinate functions and the trigonometric functions are continuous, it follows that $M(1;\mu)$ is topologically closed in R for all $\mu\in\Re$.

To obtain the remaining Lie groups, let us define

$$M(2)=\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 & 0\\ \sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & 1 & \beta\\ 0 & 0 & 0 & 1\end{pmatrix} : \alpha,\beta\in\Re\right\} ,$$

$$M(3;\mu)=\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 & \beta\\ \sin\alpha & \cos\alpha & 0 & \gamma\\ 0 & 0 & 1 & \mu\alpha\\ 0 & 0 & 0 & 1\end{pmatrix} : \alpha,\beta,\gamma,\in\Re\right\} ,$$

and

$$M(4)=\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 & \beta\\ \sin\alpha & \cos\alpha & 0 & \gamma\\ 0 & 0 & 1 & \delta\\ 0 & 0 & 0 & 1\end{pmatrix} : \alpha,\beta,\gamma,\delta\in\Re\right\} .$$

It follows from from Equation 6 that $M(2)$, $M(3;\mu)$, and $M(4)$ are subgroups of R. $M(2)$, $M(3;\mu)$, and $M(4)$ can easily be shown to be closed in R by characterizing them as the solution sets of equations in the coordinate functions and trigonometric functions, as was done for $M(1;\mu)$. Thus each is a Lie subgroup of R by Proposition 3. We will show that $M(2)$, $M(3;\mu)$, and $M(4)$ are each connected and that $\mathbf{m}(2)$, $\mathbf{m}(3;\mu)$, and $\mathbf{m}(4)$ are their Lie algebras.

To show that $M(2)$ is connected, we define $\mathbf{t}^{'}(1)=\{\mathbf{v}\in\mathbf{t}:\mathbf{v}_{14}=\mathbf{v}_{24}=0\}$, and note that we can decompose any $g\in M(2)$ as $g=u+\mathbf{v}$ with $u\in M(1;0)$ and $\mathbf{v}\in\mathbf{t}^{'}(1)$. Since $M(1;0)$ is connected, there is a curve $b(t)$ in $M(1;0)$ connecting u to I, and since $\mathbf{t}^{'}(1)$ is a line, there is a curve $c(t)$ in $\mathbf{t}^{'}(1)$ connecting $\mathbf{v}$ to the zero matrix. Thus, after reparameterizing $c(t)$, if necessary, $b(t)+c(t)$ is a curve in $M(2)$ connecting g to I, and so $M(2)$ is connected.

To determine the tangent space to $M(2)$ at I we use this same decomposition of elements of $M(2)$. If $a(t)$ is any curve in $M(2)$ passing through I, we can decompose

$a(t)$ as $a(t) = b(t) + c(t)$, where $b(t)$ is a curve in $M(1;0)$ passing through I and $c(t)$ is a curve in $\mathbf{t}^{'}(1)$ passing through the zero matrix. Conversely, if $b(t)$ is a curve in $M(1;0)$ passing through I and $c(t)$ is a curve in $\mathbf{t}^{'}(1)$ passing through the zero matrix, then by reparameterizing $c(t)$, if necessary, we obtain a curve $a(t) = b(t) + c(t)$ in $M(2)$ passing through I. These decompositions of curves imply that the tangent space to $M(2)$ at I is the span of the tangent space to $M(1;0)$ at I and the tangent space to $\mathbf{t}^{'}(1)$ at the zero matrix. The tangent space to $M(1;0)$ at I is $\mathbf{m}(1;0)$. Since $\mathbf{t}^{'}(1)$ is a subspace, the tangent space to it at any of its points can be identified with $\mathbf{t}^{'}(1)$, itself. Thus the tangent space to $M(2)$ at I is the span of $\mathbf{m}(1;0)$ and $\mathbf{t}^{'}(1)$, but this is easily seen to be $\mathbf{m}(2)$. We may then conclude that $M(2)$ is the unique connected Lie subgroup of R with $\mathbf{m}(2)$ as its Lie algebra. If we define

$$T^{'}(1) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} : \alpha \in \Re \right\} ,$$

Equation 6 implies that $T^{'}(1)$ is a normal subgroup of $M(2)$ and that $M(2) = T^{'}(1)M(1;0) = M(1;0)T^{'}(1)$.

Since we can decompose any $g \in M(3;\mu)$ as $g = u + v$ with $u \in M(1;\mu)$ and $v \in \mathbf{t}(2)$, a completely analogous treatment allows us to show that $M(3;\mu)$ is a connected Lie subgroup of R and that $\mathbf{m}(3;\mu)$ is its Lie algebra. Equation 6 implies that $T(2)$ is a normal subgroup of $M(3;\mu)$ and that $M(3;\mu) = T(2)M(1;\mu) = M(1;\mu)T(2)$. Since we can decompose any $g \in M(4)$ as $g = u + v$ with $u \in M(1;0)$ and $v \in \mathbf{t}$, the same type of analogous treatment implies that $M(4)$ is a connected Lie subgroup of R and that $\mathbf{m}(4)$ is its Lie algebra. Equation 6 implies that T is a normal subgroup of $M(4)$ and that $M(4) = TM(1;0) = M(1;0)T$.

Now if H is a connected Lie subgroup of R, then $\mathbf{h}$, the Lie algebra of H, is a subalgebra of $\mathbf{r}$, the Lie algebra of R. By Proposition 10 there is a $g \in R$, such that $\mathbf{i}_g(\mathbf{h}) = \mathbf{h}^{'}$, where $\mathbf{h}^{'}$ is one of the algebras listed in the proposition. Since i_g is an automorphism, $i_g(H)$ is a connected Lie subgroup of R by Proposition 4, and $\mathbf{h}^{'}$ is its Lie algebra by Proposition 9. Thus by Proposition 8 $i_g(H)$ must equal one of the groups we have previously discussed in this section.

Collecting these results on connected Lie subgroups of R yields the following.

Proposition 11 *A connected Lie subgroup of R is i-isomorphic to one and only one of the following connected Lie subgroups: the identity alone; S; $T(1)$, $T(2)$, or T, itself; $M(1;\mu)$, $M(2)$, $M(3;\mu)$, or $M(4)$; or R, itself. $M(2) = T^{'}(1)M(1;0) = M(1;0)T^{'}(1)$, $M(3;\mu) = T(2)M(1;\mu) = M(1;\mu)T(2)$, and $M(4) = TM(1;0) = M(1;0)T$.*

Since each of the subgroups in this list is a closed subgroup, and an i-isomorphism is a diffeomorphism of R, the following is an immediate corollary.

Corollary 3 *Every connected Lie subgroup of R is topologically closed.*

Recall that by choosing an affine orthogonal coordinate system for E^3 we identified E^3 with $\Re^3$ and then identified $\Re^3$ with the subspace P of $\Re^4$ given by Equation 2. Under these identifications we then identified the rigid motions acting on E^3 with the matrices of R acting on P. As is customary, let us call the first coordinate axis of $\Re^3$, the x-axis, the second the y-axis, and the third the z-axis, so that the xy-coordinate plane is well defined.

We now wish to describe the rigid motions represented by the groups of the last proposition and discuss the action of inner automorphisms on them. The identity group in R corresponds to the identity as a rigid motion, and by definition an automorphism must leave the identity fixed.

By construction T corresponds to the rigid motions given by translations. In particular, if $t = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in T$, then $t : p \in P \to p + v$, so that t corresponds to the rigid motion of translation by v. Thus $T(1)$ corresponds to the translations along the x-axis, $T^{'}(1)$ to the translations along the z-axis, $T(2)$ to the translations in the xy-coordinate plane. Let i_g be an inner automorphism of R. Since T is a normal subgroup of R, $i_g(T) = T$, and we need only consider the effect of i_g on subspaces of T. Since $R = ST$, it follows that $i_g = i_u \circ i_t$ for some $u \in S$ and $t \in T$. The multiplication on T is commutative, so that i_t is the identity on T, and we need only consider the effect of i_u. By Equation 6 i_u carries $T(1)$ to a subgroup representing translations in a line through the origin, and conversely, any such set of translations is the image of i_u for an appropriately chosen u. Similarly, $T(2)$ is mapped to a subgroup representing translations in a plane through the origin, and any such set of translations is the image of $T(2)$ under an inner automorphism.

By construction S corresponds to the rotations about the base point of the chosen affine orthogonal coordinate system, that is, the point identified with the origin of $\Re^3$. If i_g is an inner automorphism of R acting on S, then since $R = TS$, it follows that $g = tu$ for some $t \in T$ and $u \in S$, so that $i_g = i_t \circ i_u$. Since i_u is an inner automorphism of S, we need only consider the effect of i_t, which maps S to the representations of rotations about the fixed point given by the image of the origin under t.

If $h \in M(1; \mu)$, then from what we have already done, h represents a rotation of α radians about the z-axis followed by a translation of $\mu\alpha$ times the z-axis basis vector. $i_g(h)$ is a rotation of α radians about the image of the z-axis, the line L, followed by a translation along L of length $\mu\alpha$. Such a transformation can be called a screw transformation along L of pitch μ. Thus $M(1; \mu)$ represents the screw transformations along the z-axis of pitch μ, and $i_g(M(1; \mu))$ is the set of screw transformations along along L of pitch μ. Conversely, the set of screw transformations along any line L of pitch μ is represented by $i_g(M(1; \mu))$ for any g, which maps the z-axis to L. We note that for $\mu = 0$, the rigid motions represented by $M(1;0)$ degenerate to rotations about the z-axis, and those represented by $i_g(M(1;0))$ to rotations about L.

Since $M(2)$, $M(3;\mu)$, and $M(4)$ are products of groups that we have already considered, the rigid motions that they represent can be easily described in terms of compositions of their factors.

By choosing some an affine orthogonal coordinate system for E^3 we can use the identification of the rigid motions with R to give the rigid motions the structure of a Lie group. As we have seen, any other choice of affine orthogonal coordinate system only changes R, and hence the identification, by an inner automorphism. This implies that we can restate our recent considerations as follows.

Theorem 1 *Let an affine orthogonal coordinate system of E^3 be used to identify the rigid motions and R, and use this identification to give the rigid motions the structure of a Lie group. A connected Lie subgroup of the rigid motions is one and only one of the following connected Lie subgroups: the identity alone; the group of rotations about an arbitrary but fixed point; the group of translations in an arbitrary but fixed line; the group of translations in an arbitrary but fixed plane; the group of translations; the screw group of pitch μ about an arbitrary but fixed line; the product of the rotation group about an arbitrary but fixed line and the group of translations along the same line; the product of the screw group of pitch μ about an arbitrary but fixed line and the group of translations along the plane orthogonal to the line; the product of the rotation group about an arbitrary but fixed line and the group of translations; the group of rigid motions, itself.*

Since the set of rigid motions inherits its Lie group structure from R, Corollary 3 implies that each of its Lie subgroups is topologically closed.

We should mention in closing this section that the notion of the equivalence of Lie groups and algebras used here differs from the usual notion of equivalence customarily employed in the study of Lie groups and algebras. We used a notion embodied by i-isomorphisms, which is natural in this setting, because it corresponds to an affine orthogonal change of basis of the underlying space. In the study of Lie groups and algebras, where the desire for intrinsic knowledge of the groups and algebras is the primary motivation, the notion of equivalence embodied by Lie group or algebra isomorphism is almost always more appropriate. Under this less restrictive notion of equivalence many of the Lie algebras of Proposition 10 and the Lie groups of Proposition 11 are equivalent. For example, all of the 1-dimensional subgroups except $M(1,0)$ are Lie group isomorphic, and hence all their Lie algebras are also isomorphic.

6 Symmetry Subgroups

As was mentioned in the introduction, interest in groups of rigid motions that leave point sets invariant arises naturally in applications. If some simple classification scheme for the groups that occur can be provided, it can be used in theoretical and algorithmic developments. This potential utility is motivation for the interest within the application areas for classification results like Proposition 11 or Theorem 1. However, for the potential to be fulfilled and the results applied, we must know that the results are applicable, that is, we must know that the groups of interest are Lie subgroups of the rigid motions. We address this and related questions in this section. To be more precise throughout this section we will assume that an affine orthogonal basis has been chosen and used to identify E^3 with $\Re^3$ or P defined in Equation 2 and to identify the rigid motions with R. Now let F be a subset of $\Re^3$, and say that $r \in R$ is a symmetry or automorphism of F if $r(F) = F$, and denote the collection of all symmetries of F by $Aut(F)$. In this section then we ask when we can know that $Aut(F)$ is a Lie subgroup of R.

The group theoretic side offers little difficulty. If r and r' belong to $Aut(F)$, then $r(r'(F)) = r(F) = F$, so $rr' \in Aut(F)$. Since R is a group, r has an inverse r^{-1}, and $r^{-1}(F) = r^{-1}(r(F)) = (r^{-1}r)(F) = I(F) = F$, so $r^{-1} \in Aut(F)$. Thus $Aut(F)$ is a subgroup of R. This much is clear. What is less clear is when $Aut(F)$ is a Lie subgroup of R. In general, this need not be the case. But for many subsets F, including those encountered in applications, we can show that this is the case.

We will proceed by imposing conditions on F, which will allow us to show that $Aut(F)$ is closed, and then by applying Proposition 3 we can conclude that $Aut(F)$ is a Lie subgroup.

We begin with a simple lemma.

Lemma 5 *If F is a closed subset of $\Re^3$, then $Aut(F)$ is closed in R.*

Proof. We use the identification of $\Re^3$ with P to identify F with a set in P. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of rigid motions in $Aut(F)$ that converges to some $r \in R$. We will show that $Aut(F)$ is closed by showing that $r \in Aut(F)$, that is, by showing that $r(F) = F$. We will show this in two steps: first by demonstrating the inclusion $r(F) \subseteq F$, and then using this inclusion to prove the opposite one.

By considering coordinates, it can easily be seen that the map

$$f : (g, x) \in R \times P \to g(x) \in P$$

is continuous. This continuity implies that

$$r(x) = f(r, x) = f(Lim_{n\to\infty} r_n, x) = Lim_{n\to\infty} f(r_n, x) = Lim_{n\to\infty} r_n(x)$$

for all $x \in P$. Since $r_n \in Aut(F)$ for all n, if $y \in F$, then $r_n(y) \in F$. Now, because F is closed, $Lim_{n\to\infty} r_n(y) \in F$. Thus $r(y) \in F$, and hence $r(F) \subseteq F$.

To obtain the opposite inclusion we begin by noting that since $Lim_{n\to\infty} r_n = r$, the continuity of taking inverses in Lie groups implies that $Lim_{n\to\infty} r_n^{-1} = r^{-1}$. Since each $r_n \in Aut(F)$, which is a group, it follows that $r_n^{-1} \in Aut(F)$. Thus, by applying the inclusion that we have just demonstrated, $r^{-1}(F) \subseteq F$. Applying r to each side yields $F = rr^{-1}(F) \subseteq r(F)$.

Now we have shown that $F \subseteq r(F)$ and that $r(F) \subseteq F$, so that $r(F) = F$, and hence $r \in Aut(F)$. Thus $Aut(F)$ is closed, as claimed. ■

This lemma would suffice for a large part of the sets encountered in applications. For example, any finite collection of points, lines, planes, spheres and cylinders satisfy the assumptions of the lemma. A more complete solution is provided by the next chain of results. Recall that $\overline{B}$ denotes the closure of B and that $B \setminus C$ denotes the set $\{x \in B : x \notin C\}$ for any sets B and C.

Lemma 6 *If F is a subset of $\Re^3$, then $Aut(F) \subseteq Aut(\overline{F})$ and $Aut(F) \subseteq Aut(\overline{F} \setminus F)$.*

Proof. First we handle the claim that $Aut(F) \subseteq Aut(\overline{F})$. Let $r \in Aut(F)$. Let x be in $\overline{F}$. It follows that there is a sequence of $x_n \in F$ with $Lim_{n\to\infty} x_n = x$. By continuity $r(x) = Lim_{n\to\infty} r(x_n)$, so that $r(x_n)$ is a convergent sequence. Since $r \in Aut(F)$, it follows that $r(x_n) \in F$ for all n, so that $Lim_{n\to\infty} r(x_n) \in \overline{F}$. Thus $r(x) \in \overline{F}$, which implies that $r(\overline{F}) \subseteq \overline{F}$. But r^{-1} is also in $Aut(F)$, so that $r^{-1}(\overline{F}) \subseteq \overline{F}$. Applying r to both sides of the inclusion statement yields $\overline{F} \subseteq r(\overline{F})$. Thus $r(\overline{F}) = \overline{F}$, so that $r \in Aut(\overline{F})$. Since r was an arbitrary element of $Aut(F)$, we conclude that $Aut(F) \subseteq Aut(\overline{F})$, as claimed.

Now we handle the second claim that $Aut(F) \subseteq Aut(\overline{F} \setminus F)$. Again let $r \in Aut(F)$, and let $x \in \overline{F} \setminus F$. By what we have already proven, $r \in Aut(\overline{F})$, so that $r(\overline{F}) = \overline{F}$. This implies that $r(x) \in \overline{F}$. Since $r \in Aut(F)$, we have that $r(F) = F$, which implies that $r(x) \notin F$. Thus $r(x) \in \overline{F} \setminus F$, so that $r(\overline{F} \setminus F) \subseteq \overline{F} \setminus F$. Since $Aut(F)$ is a group, $r^{-1} \in Aut(F)$, so that $r^{-1}(\overline{F} \setminus F) \subseteq \overline{F} \setminus F$. As before, applying r to each side yields $\overline{F} \setminus F \subseteq r(\overline{F} \setminus F)$, and we conclude that $Aut(F) \subseteq Aut(\overline{F} \setminus F)$, as claimed. ■

Proposition 12 *If F is a subset of $\Re^3$ such that $\overline{F} \setminus F$ is closed, then Aut(F) is a closed Lie subgroup of the rigid motions.*

Proof. Let $r \in R$ be in $\overline{Aut(F)}$. It suffices to prove that $r \in Aut(F)$.

By definition $\overline{F}$ is closed, and by assumption $\overline{F} \setminus F$ is closed, so $Aut(\overline{F})$ and $Aut(\overline{F} \setminus F)$ are closed by Lemma 5. $Aut(F) \subseteq Aut(\overline{F})$ and $Aut(F) \subseteq Aut(\overline{F} \setminus F)$ by Lemma 6. It follows that $\overline{Aut(F)} \subseteq Aut(\overline{F})$ and that $\overline{Aut(F)} \subseteq Aut(\overline{F} \setminus F)$. In particular, $r \in Aut(\overline{F})$ and $r \in Aut(\overline{F} \setminus F)$. Thus since F and $\overline{F} \setminus F$ are disjoint, and since r is injective, $r(F)$ and $r(\overline{F} \setminus F) = \overline{F} \setminus F$ are disjoint. It now follows that

$$F \cup (\overline{F} \setminus F) = \overline{F} = r(\overline{F}) = r(F \cup (\overline{F} \setminus F)) = r(F) \cup r(\overline{F} \setminus F) = r(F) \cup (\overline{F} \setminus F)$$

implies that $F = r(F)$. Thus $r \in Aut(F)$, so that $Aut(F)$ is closed, as claimed. ■

For a subset F of $\Re^3$, the set $\overline{F} \setminus F$ is the set of points outside of F that can be approached by a sequence of points in F. Let us call this the limit set of F. The assumption of the last proposition is then that F has a closed limit set. This mild assumption is met by the sets likely to be encountered in applications. For example, we have the following.

Proposition 13 *If F is an imbedded submanifold of $\Re^3$, then F has a closed limit set.*

Proof. If F is an imbedded m-dimensional submanifold of $\Re^3$, then for each $x \in F$ there is a local coordinate system $\{\phi, U(x)\}$ of $\Re^3$ with $x \in U(x)$ and a local coordinate system $\{\psi, V(x)\}$ of F with $x \in V(x)$, such that $\{\psi, V(x)\}$ is a slice of $\{\phi, U(x)\}$ with $F \cap U(x) = V(x)$. Since $U(x)$ is open, if $y \in U(x) \cap \overline{F}$, then there is a sequence of $y_n \in U(x) \cap F$, such that $Lim_{n \to \infty} y_n = y$. Since $y_n \in U(x) \cap F = V(x)$, the last 3-$m$ coordinates of $\phi(y_n)$ equal zero by the definition of a slice. Continuity implies that the last 3-m coordinates of $\phi(y)$ equal zero, so $y \in V(x) \subseteq F$, again by the definition of a slice. Thus $\overline{F} \setminus F$ and $U(x)$ are disjoint. Taking the union of the sets $U(x)$ over all $x \in F$ yields an open set O which contains F, but is disjoint from $\overline{F} \setminus F$. If O' is the complement of $\overline{F}$, then O' is open, because $\overline{F}$ is closed. But now $\overline{F} \setminus F$ is the complement of the union of O and O', which is open, so $\overline{F} \setminus F$ is closed. ■

By Proposition 1 and the discussion following it imbedded submanifolds are common. Quadric surfaces, algebraic and analytic varieties, and even semialgebraic and semianalytic varieties provide examples. The sets and their boundary elements of geometrical modelling provide others.

Explicit knowledge of which of the connected Lie subgroups of R can appear as the identity component of the symmetries of a subset F of $\Re^3$ is often used in applications, as was illustrated in the introduction. Several of the Lie subgroups of Proposition 11 or Theorem 1 cannot be such a connected component. First, note that the action of Lie subgroup T on any point in $\Re^3$ yields all of $\Re^3$. Thus none of the subgroups listed in Proposition 11 that contain T can belong to $Aut(F)$ for any F, unless $F = \Re^3$, in which case $Aut(F) = R$. Second, note that the action of $T(2)$ on any point in $\Re^3$ yields the affine plane in $\Re^3$ passing through the point and parallel to the xy-coordinate plane. Thus, if $T(2)$ belongs to $Aut(F)$, so does $M(1;0)$. This implies that $T(2)$ cannot be the identity component of $Aut(F)$. But since $T(2)$ is contained in $M(3;\mu)$ for all μ, while $M(1;0)$ is not contained in $M(3;\mu)$, unless

$\mu = 0$, this also implies that $M(3;\mu)$ cannot be the identity component of $Aut(F)$, if $\mu \neq 0$. All the rest of the Lie subgroups appearing in Proposition 11 have a simple surface for which they are the identity component of the group of symmetries of the surface. The identity has a plane minus a triangle with three unequal sides. S has the sphere. $T(1)$ has the right elliptic, noncircular, cylinder. $M(1;0)$ has the right circular cone, while sweeping an open line segment along an appropriate helix serves for $M(1;\mu)$, when $\mu \neq 0$. $M(3;0)$ has the plane. In fact, $M(3;0)$ is the set of planar rigid motions. Clearly, if H is a connected Lie group, then whether H can or cannot be the identity component of the symmetry group of some surface is completely determined by the result for its isomorphic image in the list. Combining these observations with Propositions 2, 12, 13 and Theorem 1 yields

Theorem 2 *If F is a proper subset of $\Re^3$ with a closed limit set, then the identity component of Aut(F) is one and only one of the following Lie subgroups: the identity alone; the group of rotations about an arbitrary but fixed point; the group of translations in an arbitrary but fixed line; the screw group of pitch μ about an arbitrary but fixed line; the product of the rotation group about an arbitrary but fixed line and the group of translations along the same line; the product of the rotation group about an arbitrary but fixed line and the group of translations along the plane orthogonal to the line. The identity component of the symmetry group of an imbedded surface in $\Re^3$ is one of these Lie subgroups, and conversely, each of these subgroups is the identity component of the symmetry group of some imbedded surface in $\Re^3$.*

We recall that the screw group of pitch 0 about a line is the rotation group about the line and that all the subgroups are topologically closed.

Sometimes in applications a collection of rigid motions, which preserve more than one point set, is considered. For example, when primitive features of geometric part specification are combined to form a composite feature, they may be treated as a single entity for functional reasons, and the collection of rigid motions preserving the composite feature can be of interest. We could, of course, consider the symmetries of the union of the point sets, but this can contain many rigid motions that do not preserve any of the original point sets. For example, if F_1 is a triangle with three unequal sides, and F_2 is its complement, then $Aut(F_1 \cup F_2) = Aut(\Re^3) = R$, while $Aut(F_1)$ and $Aut(F_2)$ each contain only the identity. Thus it is often more useful to consider collection of rigid motions, which simultaneously preserve each of the point sets, or to be more precise, if $\mathcal{F}$ is a collection of subsets of $\Re^3$, to consider

$$Aut(\mathcal{F}) = \{r \in R : r(F) = F \text{ for all } F \in \mathcal{F}\} .$$

What can be said about $Aut(\mathcal{F})$? It is easy to see that $Aut(\mathcal{F}) = \cap_{F\in\mathcal{F}} Aut(F)$ and that $Aut(\mathcal{F})$ is a group. If $Aut(F)$ is closed for all $F \in \mathcal{F}$, then the intersection of them all is closed, so that $Aut(\mathcal{F})$ is a Lie subgroup of R by Proposition 3. However, in general not every $Aut(F)$ need be closed, so that even if we know that they are all Lie groups, we cannot conclude that $Aut(\mathcal{F})$ is a Lie group.

On the other hand, we can say more about the identity component of $Aut(\mathcal{F})$. To simplify the presentation, for any $K \subseteq R$ with $I \in K$ let K_I denote the identity component of K, that is, the connected component of K that contains I. Now note that if $\mathcal{G}$ is a collection of Lie subgroups of R, then for all $G \in \mathcal{G}$ we have that G_I is a Lie subgroup of R by Proposition 2 and is closed by Corollary 3. It follows that $\cap_{G\in\mathcal{G}} G_I$ is closed, and it is easy to see that $\cap_{G\in\mathcal{G}} G_I$ is a subgroup of R, so that it is a Lie subgroup of R by Proposition 3. Another application of Proposition 2 implies that $(\cap_{G\in\mathcal{G}} G_I)_I$ is a Lie subgroup of R, and since a simple argument shows that $(\cap_{G\in\mathcal{G}} G_I)_I = (\cap_{G\in\mathcal{G}} G)_I$, we have that $(\cap_{G\in\mathcal{G}} G)_I$ is a Lie subgroup of R. Thus, if $Aut(F)$ is Lie subgroup of R for all $F \in \mathcal{F}$, we can conclude that $Aut(\mathcal{F})_I$ is Lie subgroup of R. These considerations demonstrate the following proposition and its corollary.

Proposition 14 *If $\mathcal{G}$ is a collection of Lie subgroups of R, then the identity component of $\cap_{G\in\mathcal{G}} G$ is a Lie subgroup of R, and $(\cap_{G\in\mathcal{G}} G)_I = (\cap_{G\in\mathcal{G}} G_I)_I$.*

Corollary 4 *If $\mathcal{F}$ is a collection of subsets of $\Re^3$, such that $Aut(F)$ is a Lie subgroup of R for each $F \in \mathcal{F}$, then the identity component of $Aut(\mathcal{F})$ is a Lie subgroup of R, and $Aut(\mathcal{F})_I = (\cap_{F\in\mathcal{F}} Aut(F)_I)_I$.*

We have presented rich classes of examples of sets that satisfy the assumption of Proposition 12 that the set has a closed limit set. To close we show by example that this assumption is not vacuous, nor trivial. Let E be the xy-coordinate plane. Let $\mathbf{u}$ be the element of T with $\mathbf{u}_{14} = \mathbf{u}_{24} = 0$ and $\mathbf{u}_{34} = 1$. Let $\mathbf{Q}$ be the set containing all rational multiples of $\mathbf{u}$. Define F to be the union of all images of E by elements of $\mathbf{Q}$. F is the set that contains all parallel copies of E with rational z-coordinates. F is a surface, but clearly not an imbedded surface. The limit set of F is $\Re^3 \setminus F$, which is clearly not closed. $Aut(F)$ is the product of $\mathbf{Q}$ and $M(3,0)$, which again is clearly not closed, although it is a Lie subgroup of R. The identity component of $Aut(F)$ is $M(3,0)$.

7 Summary Remarks

This report began with an introduction to Lie groups and algebras that stressed the results in $\Re^n$ and $Gl(n,\Re)$ and assumed only a background in multivariable calculus and the small amount of linear algebra usually treated there. It then used the results of the introductory exposition to derive a classification of the connected Lie subgroups of the rigid motions. It accomplished this by appeal to explicit calculations in the Lie algebra of the rigid motions, which yielded an explicit catalog of all i-isomorphism classes of subalgebras, and then a passage from each subalgebra to its associated connected Lie group. The report closed by considering symmetry groups of subsets of $\Re^3$. The mildly restrictive assumption that a point set has a closed limit set was shown to imply that the symmetry group of the point set is a closed Lie subgroup of the rigid motions. This allowed the results from the previous sections to be applied and resulted in a classification of the identity components of the symmetry groups for the rich class of point sets that have a closed limit set. Imbedded submanifolds of $\Re^3$ were shown to satisfy this condition, and since the surfaces of kinematics and geometric part specification are imbedded submanifolds of $\Re^3$, it follows that the identity components of the symmetry groups of these surfaces are precisely the subgroups of Theorem 2. Finally, the identity component of the intersection of Lie symmetry groups was shown to be a Lie subgroup of R.

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