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Research Report

Supplement to "Asymptotic distributions of the sample mean, autocovariances and autocorrelations of long-memory time series"

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Abstract. This report contains supplementary results, mostly details of proofs, related to those in the paper "Asymptotic distributions of the sample mean, autocovariances and autocorrelations of long-memory time series", Journal of Econometrics, vol. 73 (1996), pp. 261–284.

This report contains supplementary results related to those in the paper "Asymptotic distributions of the sample mean, autocovariances and autocorrelations of long-memory time series" (Hosking, 1996). In the order in which they occur in Hosking (1996), these results are: proof of an assertion at the end of Section 1; proofs of Theorems 1 and 2; explicit formulas for some of the low-order cumulants of the Rosenblatt and modified Rosenblatt distributions; a proof that these distributions approach the Normal distribution as $\alpha \to \frac{1}{2}$; and proofs of Theorems 8 and 9. All of the material in Hosking (1984b) is contained either in Hosking (1996) or in this report. Equation numbers (S.1), etc., are in this report; equation numbers (1), etc., are in Hosking (1996).

Proof that (2), (3) and (4a) together imply (1).

Suppose that the time series $\{y_t\}$ satisfies (2), (3), for some β with $\frac{1}{2} < \beta < 1$, and (4a). Then the autocovariance function $\{\gamma_k\}$ of $\{y_t\}$ satisfies (1) with $\lambda = \sigma^2 \delta^2 \Gamma(2\beta - 1)\,\Gamma(1-\beta)/\Gamma(\beta)$.

Let $\Psi_j = \delta\Gamma(j+1-\beta)/\Gamma(j+1)$, $j = 0, 1, 2, \ldots$. Then $\{\Psi_j\}$ is a bounded positive decreasing sequence and $\Psi_j \sim \delta j^{-\beta} \sim \psi_j$ as $j \to \infty$. Thus

1. there exists $C > 0$ such that $|\psi_j| \le C\Psi_j$, $j = 0, 1, 2, \ldots$;
2. for all $\epsilon > 0$ there exists J such that $j \ge J \implies (1-\epsilon)\Psi_j < \psi_j < (1+\epsilon)\Psi_j$.

From (2) and (4a) we have $\gamma_k = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k}$. Now

$$\begin{aligned}
\Gamma_k &\equiv \sigma^2 \sum_{j=0}^{\infty} \Psi_j \Psi_{j+k} \\
&= \sigma^2\delta^2 \frac{\Gamma(1-\beta)\Gamma(k+1-\beta)}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{\Gamma(j+1-\beta)\Gamma(j+k+1-\beta)\Gamma(k+1)}{\Gamma(j+1)\Gamma(1-\beta)\Gamma(k+1-\beta)\Gamma(j+k+1)} \\
&= \sigma^2\delta^2 \frac{\Gamma(1-\beta)\Gamma(k+1-\beta)}{\Gamma(k+1)}\, F(1-\beta, k+1-\beta; k+1; 1) \\
&= \sigma^2\delta^2 \frac{\Gamma(1-\beta)\Gamma(2\beta-1)\Gamma(k+1-\beta)}{\Gamma(\beta)\Gamma(k+\beta)} \\
&\sim \sigma^2\delta^2 \frac{\Gamma(1-\beta)\Gamma(2\beta-1)}{\Gamma(\beta)}\, k^{1-2\beta} \qquad \text{as } k \to \infty;
\end{aligned}$$

here $F(\alpha,\beta;\gamma;x)$ is the hypergeometric function. Thus for all $\epsilon > 0$ we have

$$\left|k^{2\beta-1}(\gamma_k - \Gamma_k)\right| \le \sigma^2 k^{2\beta-1}\left(\sum_{j=0}^{J-1}|\psi_j\psi_{j+k}| + \sum_{j=0}^{J-1}\Psi_j\Psi_{j+k} + \sum_{j=J}^{\infty}|\psi_j\psi_{j+k} - \Psi_j\Psi_{j+k}|\right)$$

$$\le \sigma^2 k^{2\beta-1}J(C^2+1)\Psi_0\Psi_k + \sigma^2 k^{2\beta-1}(2\epsilon+\epsilon^2)\sum_{j=J}^{\infty}\Psi_j\Psi_{j+k}\,.$$

This expression can be bounded by a constant multiple of ϵ for k sufficiently large, since the first term is $\mathrm{O}(k^{2\beta-1}\Psi_k) = \mathrm{O}(k^{\beta-1}) \to 0$ as $k \to \infty$ and the second term is bounded by $(2\epsilon+\epsilon^2)k^{2\beta-1}\Gamma_k$ with $k^{2\beta-1}\Gamma_k$ being bounded in k. Thus $\gamma_k \sim \Gamma_k$ as $k \to \infty$; the result follows.

Proof of Theorem 1.

Write the variance of y_t as

$$\operatorname{var}\bar{y} = n^{-2}\left\{2\sum_{t=1}^{n}(n-t)\,\gamma_t + n\gamma_0\right\}$$

$$= 2n^{-\alpha}\,.\,n^{-1}\sum_{t=1}^{n}(1-t/n)n^{\alpha}\gamma_t + n^{-1}\gamma_0\,.$$

As $n \to \infty$ the sum $n^{-1}\sum(1-t/n)n^{\alpha}\gamma_t$ converges to the integral

$$\int_0^1 (1-t)\,\lambda\,t^{-\alpha}\,dt = \lambda/\{(1-\alpha)\,(2-\alpha)\}\,.$$

Since $n^{-1}\gamma_0 = \mathrm{o}(n^{-\alpha})$ as $n \to \infty$, the result follows.

Proof of Theorem 2.

The proof is based on Theorem 18.6.5 of Ibragimov and Linnik (1971), which is given below as Lemma 1. It implies that if a time series $\{y_t\}$ satisfies (2), with $\sum_{j=0}^{\infty}\psi_j^2 < \infty$ and $\{a_t\}$ a series of independent and identically distributed random variables with zero mean and finite variance, and if $n^2\operatorname{var}\bar{y} \to \infty$ as $n \to \infty$, then $(\bar{y}-\mu)/(\operatorname{var}\bar{y})^{1/2} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)$ as $n \to \infty$. Under conditions (1)–(3) and (4a), we have $\psi_j = \mathrm{O}(j^{-\beta})$ with $\beta > \frac{1}{2}$, and so $\sum\psi_j^2 < \infty$; and, from Theorem 1, $n^2\operatorname{var}\bar{y} = \mathrm{O}(n^{2-\alpha}) \to \infty$ as $n \to \infty$. Theorem 2 now follows.

Lemma 1 (Ibragimov and Linnik, 1971).

Let $\{X_j\}$ be a sequence of independent and identically distributed random variables with $\mathrm{E}\,X_0 = 0$ and $\mathrm{E}\,X_0^2 = 1$, and let

$$Y_j = \sum_{k=-\infty}^{\infty} c_{k-j} X_k$$

where

$$\sum_{k=-\infty}^{\infty} c_k^2 < \infty.$$

If $\sigma_n^2 = \mathrm{E}(Y_1 + \ldots + Y_n)^2 \to \infty$ as $n \to \infty$, then $(Y_1 + \ldots + Y_n)/\sigma_n \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)$.

Proof of Lemma 1.

The proof of their Theorem 18.6.5 given by Ibragimov and Linnik (1971) is defective, but can easily be corrected, as we now show.

The first stage of Ibragimov and Linnik's proof is to let $c_{k,n} = c_{k-1} + \ldots + c_{k-n}$, so that

$$\sigma_n^2 = \sum_{k=-\infty}^{\infty} c_{k,n}^2\,,$$

and show that $c_{k,n}/\sigma_n$ tends to zero uniformly in k as $n \to \infty$. It is this part of the proof that is in error: Ibragimov and Linnik's equations (18.6.14)–(18.6.16) are all incorrect. A correct proof runs as follows.

We have $c_{j,n} - c_{j-1,n} = c_{j-1} - c_{j-n-1}$, so

$$c_{j,n}^2 = (c_{j-1} - c_{j-n-1})^2 + 2(c_{j-1} - c_{j-n-1})c_{j-1,n} + c_{j-1,n}^2\,. \tag{S.1}$$

Applying (S.1) k times we have

$$\begin{aligned}
c_{j,n}^2 &= \sum_{i=0}^{k-1}(c_{j-i-1} - c_{j-i-n-1})^2 + 2\sum_{i=0}^{k-1}(c_{j-i-1} - c_{j-i-n-1})c_{j-i-1,n} + c_{j-k,n}^2 \\
&\le \sum_{i=-\infty}^{\infty}(c_i - c_{i-n})^2 + 2\sum_{i=-\infty}^{\infty}|(c_i - c_{i-n})c_{i,n}| + c_{j-k,n}^2 \\
&\le 4\sum_i c_i^2 + 2\left\{\sum_i (c_i - c_{i-n})^2 \sum_i c_{i,n}^2\right\}^{1/2} + c_{j-k,n}^2 \\
&\le 4\sum_i c_i^2 + 8\sigma_n\left(\sum_i c_i^2\right)^{1/2} + c_{j-k,n}^2\,,
\end{aligned}$$

where the last two steps are both applications of Cauchy's inequality. Thus

$$c_{j,n}^2/\sigma_n^2 \le 4\sigma_n^{-2}\sum_i c_i^2 + 8\sigma_n^{-1}\left(\sum_i c_i^2\right)^{1/2} + c_{j-k,n}^2/\sigma_n^2 \,. \tag{S.2}$$

In (S.2) we may choose k so that $c_{j-k,n}^2/\sigma_n^2$ is arbitrarily small, and this yields

$$|c_{j,n}|/\sigma_n \le a_n \equiv \left[8\sigma_n^{-1}\left\{\left(\sum_{i=-\infty}^{\infty} c_i^2\right)^{1/2} + \tfrac{1}{2}\sigma_n^{-1}\sum_{i=-\infty}^{\infty} c_i^2\right\}\right]^{1/2} , \tag{S.3}$$

in which $a_n \to 0$ as $n \to \infty$. Equations (S.1), (S.2) and (S.3) are the corrected versions of Ibragimov and Linnik's (18.6.15), (18.6.16) and (18.6.14) respectively.

Now defining $a_{k,n} = c_{k,n}/\sigma_n$ (not $a_{k,n} = c_{k,n}$ as stated by Ibragimov and Linnik), the remainder of Ibragimov and Linnik's proof may be followed to obtain the final result.

Cumulants of the Rosenblatt and modified Rosenblatt distributions

Hosking (1984b) evaluated the integrals I_r and K_r for $r \le 4$. For convenience we reproduce those results here.

Let I_r be defined by (6) and K_r by (13)–(15). Let

$$J_r = \int_0^1 \dots \int_0^1 |x_1 - x_2|^{-\alpha}|x_2 - x_3|^{-\alpha} \dots |x_{r-1} - x_r|^{-\alpha}\, dx_1\, dx_2 \dots dx_r \,,$$

i.e., J_r is similar to I_r but with the term $|x_r - x_1|^{-\alpha}$ omitted from the integrand. Then

$$I_2 = 1/\{(1-\alpha)(1-2\alpha)\}\,, \tag{S.4}$$

$$I_3 = 4\{\Gamma(1-\alpha)\}^2/\{(2-3\alpha)\Gamma(3-2\alpha)\}\,, \tag{S.5}$$

$$\begin{aligned} I_4 = {} & \frac{6\{\Gamma(1-\alpha)\}^3}{(3-4\alpha)\Gamma(4-3\alpha)} + \frac{1}{(1-\alpha)^3(3-4\alpha)}\,{}_3F_2\left[\begin{matrix}1,\alpha,2-2\alpha\\ 2-\alpha,3-2\alpha\end{matrix}\right] \\ & + \frac{2\{\Gamma(2-2\alpha)\}^2}{(1-\alpha)^3\Gamma(4-4\alpha)}\,{}_3F_2\left[\begin{matrix}1,\alpha,\alpha\\ 2-\alpha,2-\alpha\end{matrix}\right]\,; \end{aligned} \tag{S.6}$$

$$K_2 = I_2 - 2J_2 + J_1^2\,, \tag{S.7}$$

$$K_3 = I_3 - 3J_3 + 3J_2J_1 - J_1^3\,, \tag{S.8}$$

$$K_4 = I_4 - 4J_4 + 4J_3J_1 + 2J_2^2 - 4J_2J_1^2 + J_1^4\,; \tag{S.9}$$

$$J_1 = 2/\{(1-\alpha)(2-\alpha)\}\,,$$

$$J_2 = 2\{\Gamma(1-\alpha)\}^2/\Gamma(4-2\alpha) + 2/\{(1-\alpha)^2(3-2\alpha)\}\,,$$

$$J_3 = \frac{2\{\Gamma(1-\alpha)\}^3}{\Gamma(5-3\alpha)} + \frac{4\{\Gamma(1-\alpha)\}^3}{(1-\alpha)(4-3\alpha)\Gamma(3-2\alpha)} + \frac{2}{(1-\alpha)^3(3-2\alpha)}\,{}_3F_2\left[\begin{matrix}1,\alpha-1,3-2\alpha\\ 2-\alpha,4-2\alpha\end{matrix}\right],$$

$$\begin{aligned}J_4 = {} & \frac{2\{\Gamma(1-\alpha)\}^4}{\Gamma(6-4\alpha)} + \frac{4\{\Gamma(1-\alpha)\}^3}{(1-\alpha)(5-4\alpha)\Gamma(4-3\alpha)} \\ & + \frac{2\{\Gamma(1-\alpha)\}^2}{(1-\alpha)^2(4-3\alpha)\Gamma(3-2\alpha)}\,{}_3F_2\left[\begin{matrix}1,\alpha-1,4-3\alpha\\ 3-2\alpha,5-3\alpha\end{matrix}\right] \\ & + \left\{\frac{4}{(1-\alpha)^3(3-2\alpha)(5-4\alpha)} + \frac{12\Gamma(1-\alpha)\Gamma(3-3\alpha)}{(1-\alpha)(3-2\alpha)\Gamma(6-4\alpha)}\right\}\,{}_3F_2\left[\begin{matrix}1,\alpha,3-2\alpha\\ 2-\alpha,4-2\alpha\end{matrix}\right] \\ & + \frac{4}{(1-\alpha)^3(4-3\alpha)}\int_0^1 t^{1-\alpha}F(1,\alpha;2-\alpha;t)\,F(\alpha-1,4-3\alpha;5-3\alpha;t)\,dt \\ & + \frac{8\Gamma(1-\alpha)\Gamma(2-2\alpha)}{(1-\alpha)\Gamma(5-3\alpha)}\int_0^1 t^{1-\alpha}F(1,\alpha;2-\alpha;t)\,F(\alpha-1,3-2\alpha;5-3\alpha;t)\,dt\,;\end{aligned}$$

here $F(\alpha,\beta;\gamma;x)$ is the hypergeometric function and

$${}_3F_2\left[\begin{matrix}1,\alpha,\beta\\ \gamma,\delta\end{matrix}\right] \equiv \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha)\Gamma(j+\beta)}{\Gamma(j+\gamma)\Gamma(j+\delta)}$$

is a generalized hypergeometric function of unit argument (Bailey, 1935).

Proofs of these results are tedious but not difficult, and are not given in detail. To obtain K_r we substitute $g(x,y)$ from (15). into (14); the resulting expressions simplify to (S.7)–(S.9). To evaluate the integrals for I_r and J_r we break the region of integration into sections of the form $0 < x_{i_1} < \ldots < x_{i_r} < 1$ for $i_1,\ldots,i_r$ a permutation of $1,\ldots,r$, and within each section integrate over $x_{i_1},\ldots,x_{i_r}$ in some convenient order. For each integration we transform the dummy variable so that the range of integration is $(0,1)$ and use the expressions

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)\,,$$

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-xt)^{-\gamma}dt = B(\alpha,\beta)F(\alpha,\gamma;\alpha+\beta;x)\,,$$

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F(\gamma,\delta;\epsilon;t)\,dt = B(\alpha,\beta)\,{}_3F_2\left[\begin{matrix}\alpha,\gamma,\delta\\ \alpha+\beta,\epsilon\end{matrix}\right]$$

(Gradshteyn and Ryzhik, 1980, pp. 284, 286, 849).

As an example of the method we derive (S.6). We have

$$I_4 = \int_0^1\int_0^1\int_0^1\int_0^1 |x_1-x_2|^{-\alpha}\,|x_2-x_3|^{-\alpha}\,|x_3-x_4|^{-\alpha}\,|x_4-x_1|^{-\alpha}\,dx_1\,dx_2\,dx_3\,dx_4\,.$$

Using the symmetry of the integrand under the cyclic transformation $x_1 \to x_2 \to x_3 \to x_4 \to x_1$ we can break up the integral as

$$\int_{[0,1]^4} = 8\int_{x_1>x_2>x_3>x_4} + 8\int_{x_1>x_2>x_4>x_3} + 8\int_{x_1>x_3>x_2>x_4}\,. \tag{S.10}$$

For the first term we perform the successive integrations

$$\int_{x_3}^{x_1}(x_1-x_2)^{-\alpha}(x_2-x_3)^{-\alpha}dx_2 = (x_1-x_3)^{1-2\alpha}B(1-\alpha,1-\alpha)\,,$$

$$\int_{x_4}^{x_1}(x_1-x_3)^{1-2\alpha}(x_3-x_4)^{-\alpha}dx_3 = (x_1-x_4)^{2-3\alpha}B(1-\alpha,2-2\alpha)\,,$$

$$\int_0^1\int_0^{x_1}(x_1-x_4)^{2-4\alpha}dx_4\,dx_1 = 1/\{(3-4\alpha)\,(4-4\alpha)\}\,;$$

thus the first term of (S.10) is $8B(1-\alpha,1-\alpha)B(1-\alpha,2-2\alpha)/\{(3-4\alpha)\,(4-4\alpha)\}$, which simplifies to the first term of (S.6). The second term of (S.10), after the integrations

$$\begin{aligned}
&\int_{x_4}^{x_1}(x_1-x_2)^{-\alpha}(x_2-x_3)^{-\alpha}dx_2\\
&\quad= (x_1-x_4)^{1-\alpha}(x_4-x_3)^{-\alpha}(1-\alpha)^{-1}F\Big(1,\alpha;2-\alpha;(x_4-x_1)/(x_4-x_3)\Big)\\
&\quad= (x_1-x_4)^{1-\alpha}(x_1-x_3)^{-\alpha}(1-\alpha)^{-1}F\Big(\alpha,1-\alpha;2-\alpha;(x_1-x_4)/(x_1-x_3)\Big)\,,
\end{aligned}$$

where we have used 9.131.1 of Gradshteyn and Ryzhik (1980, p. 1043),

$$\begin{aligned}
&\int_{x_3}^{x_1}(x_1-x_4)^{1-2\alpha}(x_4-x_3)^{-\alpha}F\Big(\alpha,1-\alpha;2-\alpha;(x_1-x_4)/(x_1-x_3)\Big)\,dx_4\\
&\quad= (x_1-x_3)^{2-3\alpha}\int_0^1 t^{-\alpha}(1-t)^{1-2\alpha}F(\alpha,1-\alpha;2-\alpha;1-t)\,dt\\
&\quad= (x_1-x_3)^{2-3\alpha}B(1-\alpha,2-2\alpha)\,{}_3F_2\left[\begin{matrix}\alpha,1-\alpha,2-2\alpha\\ 2-\alpha,3-3\alpha\end{matrix}\right]\,,
\end{aligned}$$

and

$$\int_0^1 \int_0^{x_1} (x_1 - x_3)^{2-4\alpha} dx_3\, dx_1 = 1/\{(3-4\alpha)(4-4\alpha)\},$$

yields

$$\frac{8B(1-\alpha, 2-2\alpha)}{(1-\alpha)(3-4\alpha)(4-4\alpha)} {}_3F_2 \left[\begin{matrix} \alpha, 1-\alpha, 2-2\alpha \\ 2-\alpha, 3-3\alpha \end{matrix} \right],$$

which reduces to the last term of (S.6) after the application of a transformation of the ${}_3F_2$ function given by Bailey (1935, p. 98). Similarly, the last term of (S.10), after integrations over x_2, x_3, x_4 and x_1. successively, yields the second term on the right side of (S.6). This completes the proof.

Limits of the Rosenblatt and modified Rosenblatt distributions as $\alpha \to \frac{1}{2}$

As $\alpha \to \frac{1}{2}$, the standardized cumulants $\kappa_r/\kappa_2^{r/2}$ and $\tilde{\kappa}_r/\tilde{\kappa}_2^{r/2}$, $r \geq 3$, tend to zero.

For the Rosenblatt distribution we shall show that as $\alpha \to \frac{1}{2}$, $I_2 \to \infty$ while I_r, $r \geq 3$, remain finite. It will then follow that the standardized cumulants

$$\tilde{\kappa}_r/\tilde{\kappa}_2^{r/2} = 2^{r/2-1}(r-1)!\, I_r/I_2^{r/2}$$

have the stated behavior.

It is clear from (S.4) and (S.5) that $I_2 \to \infty$ as $\alpha \to \frac{1}{2}$, while $I_3 \to 8\pi$ as $\alpha \to \frac{1}{2}$. Now consider the integral defining I_r, $r \geq 4$. If $x_1 \neq x_{r-1}$, we have

$$\frac{1}{(x_1 - x_r)(x_r - x_{r-1})} = \frac{1}{(x_1 - x_{r-1})} \left(\frac{1}{x_1 - x_r} + \frac{1}{x_r - x_{r-1}} \right).$$

Thus

$$\begin{aligned} |(x_1 - x_r)(x_r - x_{r-1})|^{-\alpha} &\leq |x_1 - x_{r-1}|^{-\alpha} \left(\frac{1}{|x_1 - x_r|} + \frac{1}{|x_r - x_{r-1}|} \right)^{\alpha} \\ &\leq |x_1 - x_{r-1}|^{-\alpha}\, 2^{\alpha} \left(|x_1 - x_r|^{-\alpha} + |x_r - x_{r-1}|^{-\alpha} \right), \end{aligned}$$

since for $x, y > 0$ and $\alpha > 0$ we have

$$(x+y)^{\alpha} = 2^{\alpha}\{(x+y)/2\}^{\alpha} \leq 2^{\alpha}\{\max(x,y)\}^{\alpha} \leq 2^{\alpha}(x^{\alpha} + y^{\alpha}).$$

Thus

$$\int_0^1 |x_{r-1}-x_r|^{-\alpha}|x_r-x_1|^{-\alpha}\,dx_r$$
$$\le |x_1-x_{r-1}|^{-\alpha}\,2^{\alpha}\left\{\int_0^1 |x_{r-1}-x_r|^{-\alpha}dx_r+\int_0^1 |x_r-x_1|^{-\alpha}dx_r\right\}$$
$$=|x_1-x_{r-1}|^{-\alpha}\,2^{\alpha}\left\{x_1^{1-\alpha}+(1-x_1)^{1-\alpha}+x_{r-1}^{1-\alpha}+(1-x_{r-1})^{1-\alpha}\right\}/(1-\alpha)$$
$$\le 8|x_1-x_{r-1}|^{-\alpha}\qquad \text{for } 0<x_1<1,\quad 0<x_{r-1}<1,\quad 0<\alpha<\tfrac{1}{2}$$

(the first of these inequalities is essentially the same as equation (19) of Rosenblatt (1979), but we have given more details of the derivation). Thus performing the integral over x_r in I_r, $r\ge 4$, yields

$$I_r \le 8I_{r-1},$$

and, by iteration,

$$I_r \le 8^{r-3}I_3 = 8^{r-2}\pi.$$

Thus each I_r, $r\ge 3$, remains finite as $\alpha\to\frac{1}{2}$.

For the modified Rosenblatt distribution, we observe that the difference $g(x,y)-|x-y|^{-\alpha}$ can be bounded by a constant independent of α. Hence we deduce the existence of constants m, M, $0<m<M<\infty$, such that

$$m|x-y|^{-\alpha} \le g(x,y) \le M|x-y|^{-\alpha}$$

for all x, y, α such that $0<x<1$, $0<y<1$, $x\ne y$, and $0<\alpha<\frac{1}{2}$. Thus

$$m^r I_r \le K_r \le M^r I_r,$$

and so as $r\to\frac{1}{2}$ we have $K_2\to\infty$ while K_r, $r\ge 3$, remains finite. As in the case of the Rosenblatt distribution, it follows that the standardized cumulants $\kappa_r/\kappa_2^{r/2}$, $r\ge 3$, tend to zero as $\alpha\to\frac{1}{2}$.

Proof of Theorem 8.

(i) If $d=0$ this is a standard result (Anderson, 1971, Theorem 7.7.8). If $0<d<\frac{1}{2}$ the result follows from Theorem 2. If $-\frac{1}{2}<d<0$, then $\{y_t\}$ satisfies (2) and (4a) and has an autocovariance function $\{\gamma_k\}$ with $\gamma_k\sim\lambda k^{-\alpha}$, $\lambda<0$, $1<\alpha<2$, as $k\to\infty$ and $\sum_{k=-\infty}^{\infty}\gamma_k=0$. Under these conditions $\operatorname{var}\bar{y}$ still satisfies

$$\operatorname{var}\bar{y}\sim 2\lambda n^{-\alpha}/\{(1-\alpha)(2-\alpha)\},$$

but the proof is not the same as that of Theorem 1. Write the variance of y_t as

$$\begin{aligned}
\operatorname{var}\bar{y} &= n^{-2} \sum_{t=-(n-1)}^{n-1} (n-|t|)\gamma_t \\
&= n^{-1} \sum_{t=-\infty}^{\infty} \gamma_t - 2n^{-1} \sum_{t=n}^{\infty} \gamma_t - 2n^{-2} \sum_{t=1}^{n-1} t\,\gamma_t\,. \\
&= T_1 + T_2 + T_3\,, \qquad \text{say.}
\end{aligned}$$

T_1 is zero and the other terms are of asymptotic form

$$T_2 \sim -2n^{-1} \sum_{t=n}^{\infty} \lambda\, t^{-\alpha} \sim -2n^{-1}\,\lambda\, n^{1-\alpha}/(\alpha-1) = 2\lambda\, n^{-\alpha}/(1-\alpha)\,,$$

$$T_3 \sim -2n^{-2} \sum_{t=1}^{n-1} \lambda\, t^{1-\alpha} \sim -2n^{-2}\,\lambda\, n^{2-\alpha}/(2-\alpha) = -2\lambda\, n^{-\alpha}/(2-\alpha)\,;$$

thus

$$\operatorname{var}\bar{y} \sim 2\lambda n^{-\alpha}/\{(1-\alpha)(2-\alpha)\}\,.$$

The result now follows from Lemma 1.

(ii) When $d = -\frac{1}{2}$, the proof runs as in the case $-\frac{1}{2} < d < 0$, with $T_1 = 0$ and $T_2 = \mathrm{O}(n^{-2})$, but now

$$T_3 \sim -2n^{-2} \sum_{t=1}^{n-1} \lambda\, t^{-1} \sim -2\lambda\, n^{-2} \log n$$

dominates T_2 asymptotically, and

$$\operatorname{var}\bar{y} \sim -2\lambda n^{-2} \log n\,.$$

The result now follows from Lemma 1.

Proof of Theorem 9.

Let γ_0^x be the lag-k autocovariance of an ARIMA$(0,d,0)$ process with white-noise variance 1. We have

$$\gamma_k^x = \frac{\Gamma(1-2d)\Gamma(k+d)}{\Gamma(d)\Gamma(1-d)\Gamma(k+1-d)}$$

(Hosking, 1981), and

$$\sum_{s=-\infty}^{\infty} \gamma_s^x \gamma_{s+k}^x = \left\{ \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \right\}^2 \sum_{s=-\infty}^{\infty} \frac{\Gamma(s+d)\Gamma(s+k+d)}{\Gamma(s+1-d)\Gamma(s+k+1-d)}$$

$$= \frac{\Gamma(1-4d)\Gamma(k+d)\Gamma(1-k-d)}{\Gamma(d)\Gamma(1-d)\Gamma(k+1-2d)\Gamma(1-k-2d)} = \gamma_k^+ \,, \tag{S.11}$$

where we have used Dougall's formula (Slater, 1966, p. 180) and the reflection formula for gamma functions, $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$. When $q=0$ we have $\gamma_k = \sigma^2 \gamma_k^x$ and substitution of (S.11) in (11) and (21) yields the expressions for $\mathrm{cov}(c_k, c_\ell)$ and $\mathrm{cov}(r_k, r_\ell)$ given in Theorem 9. When $q \neq 0$ we have

$$\gamma_k = \sigma^2 \sum_{j=-q}^{q} \gamma_j^u \gamma_{k-j}^x$$

and a similar substitution again yields the expressions for $\mathrm{cov}(c_k, c_\ell)$ and $\mathrm{cov}(r_k, r_\ell)$ given in Theorem 9.

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