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Composing Distribution Function Zones For Statistical Tolerance Analysis

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Abstract

More and more often in industry practical exigencies force the designer to specify part tolerances statistically. Since most products are assemblies of parts, in order to assess the part-level designs he must consider their assembly-level implications. In particular, he must be able to combine specified part-level variabilities to obtain an assembly-level variability. In the simplest situation it is assumed that the populations of parts are statistically independent and that the relevant part-level values combine linearly to produce a linear gap function. Even this simplest case has wide applicability. In ISO two approaches for statistically specifying mechanical parts are being discussed. One of these defines acceptability of a population of parts by requiring that the distribution function of the relevant values of the parts be bounded by a pair of specified distribution functions. For this approach to be useful, then, a method to compose part-level specifications to yield assembly-level description is required. This paper supplies such a method for the case of statistically independent populations of parts and linear gap functions.

Keywords: statistics, distribution functions, tolerance analysis

1 Introduction

In design practice employing traditional worst case tolerancing the acceptability of an individual part is specified by requiring that the value of some dimension of the part falls between some upper and lower limits. In design practice employing statistical tolerancing the acceptability of an individual part is not specified; rather, the acceptability of a population of parts is specified by requiring that the values of some dimension of the parts satisfies some probabilistic criterion.

Although aspects of statistical tolerancing have been applied in industry for many years([1], [2], [3]), only recently has there been much effort to formalize and codify the practice. In 1994 the American Society of Mechanical Engineers made a modest beginning by devoting three out of the two hundred and thirty pages of its latest revision of the dimensioning and tolerancing standard ([4]) to statistical tolerancing. They defined a symbol to indicate that statistical tolerancing was being employed, but provided no further syntax or semantics. In [5] three possible interpretations of statistical tolerancing were provided. The first two of these are (population-) parameter based in that they define acceptability by constraining the mean μ and standard deviation σ of the population to a specified zone in μ-σ space. The last in contrast defines acceptability by constraining the distribution function of the population to being one of a specified collection of distribution functions, called a distribution function zone. As reported in [6] and [7] ISO is currently progressing towards promulgating standards in statistical tolerancing based on both the parameter based approach and the distribution function zone approach.

To this point we have focused on part-level considerations. However, since most products are assemblies of parts, this is clearly not sufficient. Ideally, a designer should start with an assembly "budget" for allowable variation, and distribute it to the constituent parts. In reality, this problem is attacked by several iterations of tolerance analysis, where part variations are composed to determine the assembly variation. In general, this can be extremely difficult, even in the case of traditional worst-case tolerancing, if the parts are linked in geometrically complex ways. However, if the parts are linked in a one dimensional linear way, leading to the common stackup analysis and a linear gap function, the analysis is much simpler. Often this is the case, and when it is not, it is often still possible to obtain useful results by linearizing the problem. In statistical tolerance analysis an added complexity arises, if the parts are statistically dependent. Again it is customary to assume away the problem, and in fact it is often reasonable to do so by assuming statistical independence. In [8] techniques for composition of statistically independent parts with a linear gap function were developed for the parameter based approach. This paper provides techniques for this composition problem for the distribution function zone approach

Section 2 describes distribution function zones and establishes some terminology and notations. In Section 3 the linear gap function and its relationship to convolutions are discussed. The actual task of composing the part specifications into assembly specification is then addressed by showing that bounds on distribution functions are preserved under convolution. Section 4 provides an example of a composition.

2 DFZones

In 1974 the West German standards organization made the first serious attempt in the direction of codifying statistical tolerancing by modifying the specification of a worst-case tolerance interval[9]. The designer could specify a central subinterval of the worst-case interval and a percentage. This was to be interpreted as requiring a population of parts such that the values of all of the parts would lie in the worst-case interval, with the percentage of values in the central subinterval being at least the specified percentage, and with the percentage of values in either of the remaining two subintervals being no more than half the remaining percentage. One of the two approaches for statistical tolerancing of mechanical parts currently under consideration in ISO [7] is semantically identical and syntactically almost identical to this German standard. Figure 1 illustrates such a specification.

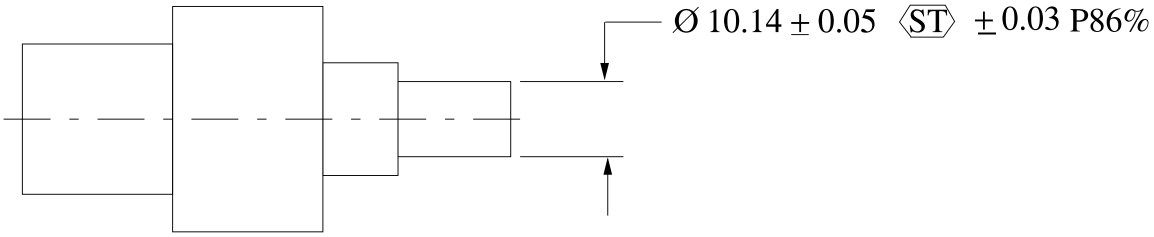

Figure 1: Proposed syntax for the DFZone–based specification of statistical tolerancing. The semantics is that in any acceptable population of parts, at least 86% of the diameters shall be within 10.14 ± 0.03, at most 7% shall be within $10.14^{-0.03}_{-0.05}$, and at most 7% shall be within $10.14^{+0.05}_{+0.03}$. In addition, these parts shall be produced under a state of statistical control.

The syntactic and semantic simplicity of the foregoing treatment is appealing, but, if this proposed standard is successful, one easily envisions that certain extensions would likely be useful. For example, in statistical tolerancing one is usually willing to accept some risk that some small fraction of a population is out of the worst-case limits, while the standard provides no means to describe this. If one central subinterval is useful, might not more subintervals be? A balanced distribution of probability is often appropriate, but not always. Each of these can easily be accommodated by adjustments to the standard, but at significant cost to the simplicity. Fortunately, there is a more powerful approach based on distribution functions that encompasses these and much more in a general framework. It has the added advantage that it is more convenient for further analysis and permits us to appeal directly to standard probabilistic and analytic tools. To begin its development we recall several definitions.

A distribution function, denoted DF, is a non-decreasing, right-continuous function F mapping $F : X \in I\!R \rightarrow F(X) \in [0, 1]$ with $\lim_{X\rightarrow-\infty} F(X) = 0$ and $\lim_{X\rightarrow\infty} F(X) = 1$. Distribution functions are of interest in probability theory, because for any real random variable x the function defined for all $X \in I\!R$ by

$$F_x(X) = Pr\{x \leq X\}$$

is a DF, which contains all probabilistic information about x. Conversely, any DF is the DF of some random variable.

If L and U are two DFs with $L(X) \leq U(X)$ for all X, then we write $L \leq U$. If $\mathcal{F}$ is a collection of DFs, then we denote by $Z(L, U; \mathcal{F})$ the collection of DFs bounded by L and U, that is,

$$Z(L, U; \mathcal{F}) = \{F \in \mathcal{F} : L \leq F \leq U\}.$$

We refer to $Z(L, U; \mathcal{F})$ as the DFZone in $\mathcal{F}$ bounded by L and U. The choice of $\mathcal{F}$ restricts attention to a relevant family of distributions. For example, by choosing $\mathcal{F}$ to be the DFs of all normal random variables and making appropriate definitions of L and U we can obtain the DFZones of [5]. When $\mathcal{F}$ is the collection of all DFs, we simplify the notation to $Z(L, U)$, and refer to $Z(L, U)$ as the DFZone bounded by L and U.

The tolerance specifications considered above are readily described in these terms. If the worst case tolerance interval is $[t_1, t_2]$, the central subinterval is $[s_1, s_2]$, and the percentage is p, then let

$$L(X) = \begin{cases} 0 & \text{for } X < s_2 \\ \frac{1+p}{2} & \text{for } s_2 \leq X < t_2 \\ 1 & \text{for } t_2 \leq X \end{cases} \quad \text{and} \quad U(X) = \begin{cases} 0 & \text{for } X < t_1 \\ \frac{1-p}{2} & \text{for } t_1 \leq X < s_1 \\ 1 & \text{for } s_1 \leq X \end{cases} \tag{1}$$

Now if x is the random variable of values of the part, and F_x its DF, then the tolerance specification is that $F_x \in Z(L, U)$. Alternatively, we can express this graphically by saying that the graph of F_x lies between the graph of L and that of U. Figure 2 shows this graphical interpretation for the example of Figure 1.

In the sequel we will encounter sums of scalar multiples of random variables, and it will be useful to know how the DFs of a random variables and bounds on DFs of random variables transform under multiplication of the random variables by a real scalar. Of course if x is a random variable, and a a real number, then ax is a random variable.

First consider how the DF of a random variable transforms under multiplication. If $a > 0$, then $Pr\{ax \leq X\} = Pr\{x \leq X/a\}$ for all real X, so that

$$F_{ax}(X) = F_x(X/a) \quad \text{for } a > 0 \tag{2}$$

for all real X.

For even the simplest negative multiple the situation is somewhat more complex:

$$\begin{aligned} F_{-x}(X) &= Pr\{-x \leq X\} = Pr\{x \geq -X\} = 1 - Pr\{x < -X\} \\ &= 1 - (Pr\{x \leq -X\} - Pr\{x = -X\}) = 1 - F_x(-X) + Pr\{x = -X\} \end{aligned} \tag{3}$$

Real complexity arises only when there are singleton sets with nonzero probability, or, equivalently, when F_x is not continuous, since F_x is continuous at X, if and only if

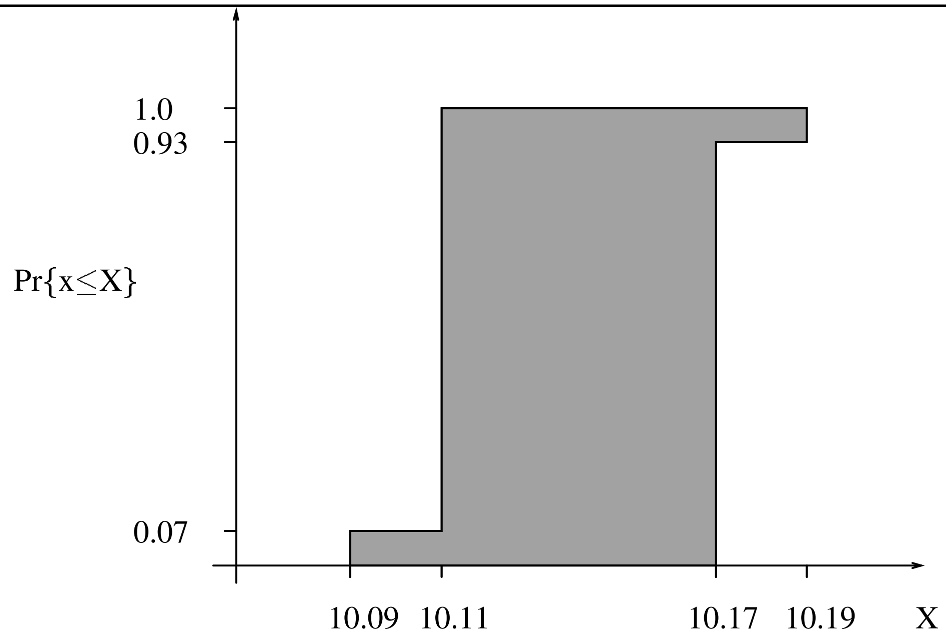

Figure 2: Semantics of the DFZone–based specification. The actual value of a diameter is denoted as x. Any population of diameters whose distribution function falls within the shaded zone is acceptable.

$Pr\{x = X\} = 0$. It might be argued that if x is the value of a part dimension, then singleton sets with nonzero probability are unrealistic, but since the bounding functions of DFZones may well be discontinuous, as are those of (1), we still need to address the issue.

Since F_{-x} is a DF, $F_x(-X) - Pr\{x = -X\}$ is continuous from the right by (3). Since $F_x(-X)$ is a decreasing function, there is a unique function $\overleftarrow{F_x}$, such that $\overleftarrow{F_x}(-X)$ is continuous from the right for all X, and $F_x(X) = \overleftarrow{F_x}(X)$ for all X, where F_x is continuous. It follows that

$$\overleftarrow{F_x}(-X) = F_x(-X) - Pr\{x = -X\} \tag{4}$$

for all X. Now it follows from (2), (3), and (4) that

$$F_{-ax}(X) = 1 - \overleftarrow{F_x}(-X/a) \quad \text{for } a > 0 \tag{5}$$

for all real X. In loose terms this all says that $F_{-ax}(X)$ equals $1 - F_x(-X/a)$ modulo the need to make $1 - F_x(-X/a)$ continuous from the right, which can be done in a unique way.

Now consider how bounds on DFs of random variables transform under multiplication of the random variables by some scalar. Let F_x and F_y be DFs with $F_x \leq F_y$, and let $a > 0$. The situation with positive multiples is easy. It follows directly from (2) that $F_{ax} \leq F_{ay}$. Negative multiples are a little more work. Since $F_x \leq F_y$, it follows that $1 - F_x(-X/a) \geq 1 - F_y(-X/a)$ for all X. Since $F_x(X) = \overleftarrow{F_x}(X)$ for all X where F_x is continuous, and $F_y(X) = \overleftarrow{F_y}(X)$ for all X where F_y is continuous, it follows that $F_{-ax}(X) \geq F_{-ay}(X)$ for all X such that F_x and F_y are simultaneously continuous at $-X/a$. F_x and F_y can each have only countably many points where they are discontinuous, so that there are only countably many points where either one is discontinuous. This implies that the set of X such that F_x and F_y are simultaneously continuous at $-X/a$ is dense. Since F_{-ax} and F_{-ay} are continuous from the right, it follows that $F_{-ay} \leq F_{-ax}$.

The proposition closing this section now follows immediately from (2), (5), and these results on bounds.

Proposition 1 *If $a > 0$ and $F_x \in Z(L, U)$, then it follows that $F_{ax} \in Z(L(X/a), U(X/a))$ and that $F_{-ax} \in Z(1 - \overleftarrow{U}(-X/a), 1 - \overleftarrow{L}(-X/a))$.*

3 Gaps, Convolutions, and DFZone Composition

From a theoretical perspective, the goal of this section is to determine the DF of an affine linear combination of independent random variables and then to propagate DFZone bounds on the random variables to a DFZone bound on the combinations. From a more practical perspective, the goal is to be able to express the variability of some dimension of an assembly described by a linear gap function of statistically independent values of certain dimensions of the constituent parts of the assembly. We will use the results of the last section on transformations of DFs to write the gap function in a simpler form. In this simpler form we can directly apply standard results of probability theory (see [10], for example) to express the DF of the gap function as a convolution. A simple inequality in convolutions will then lead to zone composition.

We begin by assuming that some critical dimension g of an assembly can be obtained as an affine linear combination of some finite set of values $\{x_i\}_{i\in\mathcal{I}}$ of its constituent parts, where $\mathcal{I}$ is some finite index set, that is,

$$g = a + \sum_{i\in\mathcal{I}^+} a_i x_i + \sum_{i\in\mathcal{I}^-} -a_i x_i \tag{6}$$

where a is a real, $\mathcal{I}^+$ and $\mathcal{I}^-$ are finite index sets, and $a_i > 0$ for all i in $\mathcal{I}^+$ or $\mathcal{I}^-$. The positive and negative contributions are separated, because in each case the DF is determined slightly differently.

If $\{x_i\}_{i\in\mathcal{I}}$ is a set of mutually independent random variables, then so is the set of random variables $\{a_i x_i\}_{i\in\mathcal{I}^+} \cup \{-a_i x_i\}_{i\in\mathcal{I}^-}$. Thus $g - a$ is a finite sum of mutually independent random variables.

Now turn to convolutions. If F and G are DFs, then their convolution is defined to be

$$F * G(X) = \int_{t\in\mathbb{R}} G(X-t)dF(t).$$

The convolution of two DFs is a DF, and acting on DFs the convolution operation is both commutative and associative. Hence, it is reasonable to speak of the product of a finite number of DFs without reference to their order or grouping. Thus if $\{F_i\}_{i\in\mathcal{I}}$ is a finite collection of DFs we will denote the convolution of the DFs by $\prod^*_{i\in\mathcal{I}} F_i$.

Convolutions are useful in this context, because if x and y are independent random variables, then $F_{x+y}(X) = F_x * F_y$. Thus by (6)

$$F_g \;=\; F_a \;*\; \textstyle\prod^*_{i\in\mathcal{I}^+} F_{a_i x_i} \;*\; \prod^*_{i\in\mathcal{I}^-} F_{-a_i x_i} \tag{7}$$

We will now show that the property of DFs being bounded by other DFs is preserved under convolution.

Proposition 2 *If F_1, F_2, G_1 and G_2 are DFs with $F_1 \leq F_2$ and $G_1 \leq G_2$, then it follows that $F_1 * G_1 \leq F_2 * G_2$.*

Proof :

$$\begin{aligned} F_1 * G_1(X) &= \int_{t\in\mathbb{R}} F_1(X-t)dG_1(t) \leq \int_{t\in\mathbb{R}} F_2(X-t)dG_1(t) = \int_{t\in\mathbb{R}} G_1(X-t)dF_2(t) \\ &\leq \int_{t\in\mathbb{R}} G_2(X-t)dF_2(t) = G_2 * F_2(X) = F_2 * G_2(X). \square \end{aligned}$$

As an immediate consequence, we have

Corollary 1 *$F_1 \in Z(L_1, U_1)$ and $F_2 \in Z(L_2, U_2)$ imply that $F_1 * F_2 \subset Z(L_1 * L_2, U_1 * U_2)$.*

We now have at hand all the tools needed to compose DFZones under the assumptions of independence of the random variables and a linear gap function, that is, we can now supply an essential tool for statistical tolerance analysis for distribution function based specification.

Theorem 1 *Let $\{x_i\}_{i \in \mathcal{I}}$ be a finite collection of random variables satisfying $x_i \in Z(L_i, U_i)$ for all $i \in \mathcal{I}$. Let g be the linear gap function given by*

$$g = a + \sum_{i \in \mathcal{I}^+} a_i x_i + \sum_{i \in \mathcal{I}^-} -a_i x_i$$

where a is a real, $\mathcal{I}^+$ and $\mathcal{I}^-$ are finite index sets, and $a_i > 0$ for all i in $\mathcal{I}^+$ or $\mathcal{I}^-$. If

$$L \quad = \quad F_a \ * \ \textstyle\prod^*_{i \in \mathcal{I}^+} L_i(X/a_i) \ * \ \prod^*_{i \in \mathcal{I}^-}(1 - \overleftarrow{U}_i(-X/a_i))$$

and

$$U \quad = \quad F_a \ * \ \textstyle\prod^*_{i \in \mathcal{I}^+} U_i(X/a_i) \ * \ \prod^*_{i \in \mathcal{I}^-}(1 - \overleftarrow{L}_i(-X/a_i))$$

then $g \in Z(L, U)$.

Proof : Proposition 1 implies that

$$F_{a_i x_i} \in Z(L_i(X/a_i), U_i(X/a_i))$$

for all i in $\mathcal{I}^+$ and that

$$F_{-a_i x_i} \in Z(1 - \overleftarrow{U}_i(-X/a_i), 1 - \overleftarrow{L}_i(-X/a_i))$$

for all i in $\mathcal{I}^-$. Trivially $F_a \in Z(F_a, F_a)$. $F_g = F_a \ * \ \prod^*_{i \in \mathcal{I}^+} F_{a_i x_i} \ * \ \prod^*_{i \in \mathcal{I}^-} F_{-a_i x_i}$ by (7). Proposition 8 now implies that

$$F_g \leq F_a \ * \ \textstyle\prod^*_{i \in \mathcal{I}^+} U_i(X/a_i) \ * \ \prod^*_{i \in \mathcal{I}^-}(1 - \overleftarrow{L}_i(-X/a_i))$$

and that

$$F_a \ * \ \textstyle\prod^*_{i \in \mathcal{I}^+} L_i(X/a_i) \ * \ \prod^*_{i \in \mathcal{I}^-}(1 - \overleftarrow{U}_i(-X/a_i)) \leq F_g$$

from which the claim follows. □

If each of the bounding DFs in each of the part-level DFZones in the hypotheses of the theorem is a step function with only finitely many jumps, the bounding DFs of the composed DFZone can can be described more explicitly. Since this added assumption is clearly met by the standard under consideration by ISO, it will be pursued here. First, if F_x is a step function with only finitely many jumps, and $a > 0$, then so are F_{ax} and F_{-ax} by (2) and (5). Next, if F is a DF, which is step function with only finitely many jumps, and we let the set of x-values where the jumps of F occur be denoted by $J(F)$, and for $X \in J(F)$ let $\delta(F, X)$ denote the value of the jump of F at X, then

$$F(X) = \sum_{\substack{Y \in J(F) \\ Y \leq X}} \delta(F, Y) \tag{8}$$

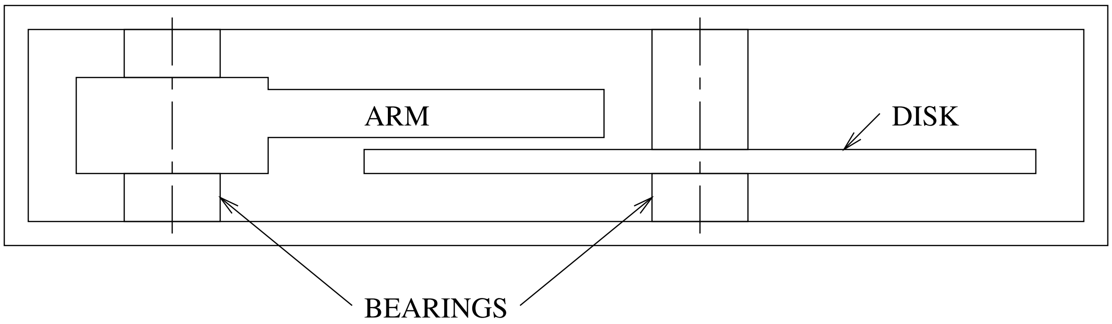

Figure 3: A disk-drive assembly indicating critical parts

On two such DFs, F and G, the convolution is trivial:

$$F * G(X) = \sum_{\substack{Y \in J(F) \\ Z \in J(G) \\ Y + Z \leq X}} \delta(F, Y)\delta(G, Z) \tag{9}$$

It is easy to see that $F * G$ is a step function with $J(F * G) = \{Y + Z : Y \in J(F), Z \in J(G)\}$, and $\delta(F * G, X) = \sum \delta(F, Y)\delta(G, Z)$ with the sum taken over all $Y \in J(F)$ and $Z \in J(G)$ with $Y + Z = X$. Thus, if each of the bounding DFs in each of the part-level DFZones in the hypotheses of the theorem are step functions with only finitely many jumps, then L and U of the conclusion of the theorem are also step functions with only finitely many jumps.

4 An Example

To illustrate the techniques described thus far, let us consider as a concrete example the statistical specifications of a magnetic storage product. Figure 3 shows a vertical section view of a disk-drive assembly. Four critical parts in this assembly are identified for our consideration: a magnetic disk which stores the data, an arm that swings in an angular movement to enable a magnetic head to read and write data onto the disk, and two bearings that support the disk and the arm. This figure is a grossly simplified version of an actual disk-drive cross-sectional view, but is sufficient for our purpose of illustration.

A critical assembly characteristic in such a disk-drive is the arm-to-disk clearance, shown in Figure 4 as the gap g.

For proper functioning of the disk-drive, this clearance g should be neither too large nor too small. It is, of course, controlled by the critical dimensions of the four parts we identified earlier. More precisely, we have an instance of the linear gap function (6) in the

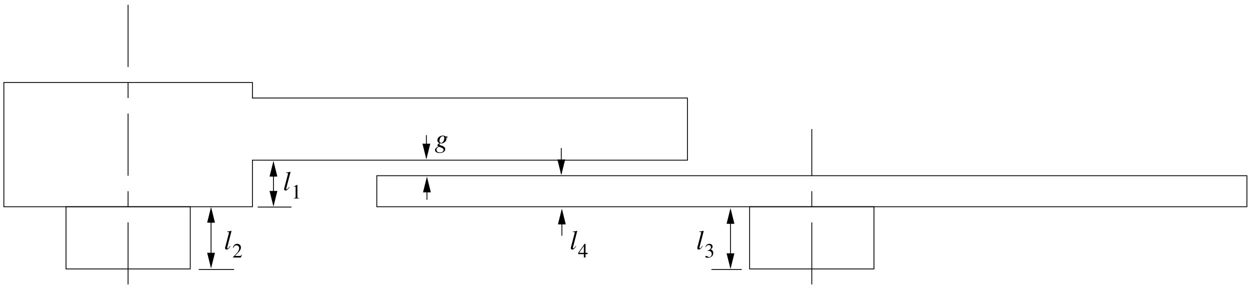

Figure 4: Critical part dimensions and an assembly characteristic in a disk-drive assembly. The critical assembly characteristic is the arm-to-disk spacing g which is related to the critical part dimensions l_1, l_2, l_3, and l_4 by the linear "gap" function $g = l_1 + l_2 - l_3 - l_4$.

Part	C.D.	Stat. Spec.
Arm	l_1	1.75 ± 0.05 ST $\pm 0.03P75\%$
Arm_Bearing	l_2	2.00 ± 0.07 ST $\pm 0.05P86\%$
Disk_Bearing	l_3	2.00 ± 0.07 ST $\pm 0.05P86\%$
Disk	l_4	1.00 ± 0.03 ST $\pm 0.02P90\%$

Table 1: Statistical specifications on critical dimensions of the disk-drive example. C.D. = Critical Dimension, N.D. = Nominal Dimension, Stat. Spec. = Statistical tolerance specification.

form

$$g = l_1 + l_2 - l_3 - l_4. \tag{10}$$

Each of the critical dimensions belongs to a different part in the disk-drive assembly, and can be justifiably assumed to be mutually independent random variables.

Part-level statistical specifications of these critical dimensions are summarized in Table 1. If we proceed as indicated in (1) and define

$$L_1(X) = \begin{cases} 0 & \text{for } X < 1.78 \\ .875 & \text{for } 1.78 \le X < 1.80 \\ 1 & \text{for } 1.80 \le X \end{cases} \quad \text{and} \quad U_1(X) = \begin{cases} 0 & \text{for } X < 1.70 \\ .125 & \text{for } 1.70 \le X < 1.72 \\ 1 & \text{for } 1.72 \le X \end{cases}$$

$$L_2(X) = \begin{cases} 0 & \text{for } X < 2.05 \\ .93 & \text{for } 2.05 \le X < 2.07 \\ 1 & \text{for } 2.07 \le X \end{cases} \quad \text{and} \quad U_2(X) = \begin{cases} 0 & \text{for } X < 1.93 \\ .07 & \text{for } 1.93 \le X < 1.95 \\ 1 & \text{for } 1.95 \le X \end{cases}$$

$$L_4(X) = \begin{cases} 0 & \text{for } X < 1.02 \\ .95 & \text{for } 1.02 \le X < 1.03 \\ 1 & \text{for } 1.03 \le X \end{cases} \quad \text{and} \quad U_4(X) = \begin{cases} 0 & \text{for } X < .97 \\ .05 & \text{for } .97 \le X < .98 \\ 1 & \text{for } .98 \le X \end{cases}$$

then the part specifications can be restated as $l_1 \in Z(L_1, U_1)$, $l_2 \in Z(L_2, U_2)$, $l_3 \in Z(L_2, U_2)$, and $l_4 \in Z(L_4, U_4)$. Now by applying Theorem 1 and using (2), (5), and (9) in the calculations we obtain

$$L(X) = \begin{cases} 0 & \text{for } X < .90 \\ .7189481 & \text{for } .90 \leq X < .91 \\ .7567875 & \text{for } .91 \leq X < .92 \\ .9677231 & \text{for } .92 \leq X < .93 \\ .978825 & \text{for } .93 \leq X < .94 \\ .9983594 & \text{for } .94 \leq X < .95 \\ .9993875 & \text{for } .95 \leq X < .96 \\ .9999694 & \text{for } .96 \leq X < .97 \\ 1 & \text{for } .97 \leq X \end{cases} \quad \text{and} \quad U(X) = \begin{cases} 0 & \text{for } X < .53 \\ .0000306 & \text{for } .53 \leq X < .54 \\ .0006125 & \text{for } .54 \leq X < .55 \\ .0016406 & \text{for } .55 \leq X < .56 \\ .021175 & \text{for } .56 \leq X < .57 \\ .0322769 & \text{for } .57 \leq X < .58 \\ .24332125 & \text{for } .58 \leq X < .59 \\ .2810519 & \text{for } .59 \leq X < .60 \\ 1 & \text{for } .60 \leq X \end{cases}$$

as the bounding DFs, that is, we find $Z(L, U)$ as a composition of the part-level specifications for l_1, l_2, l_3, and l_4. Figure 5 shows this DFZone.

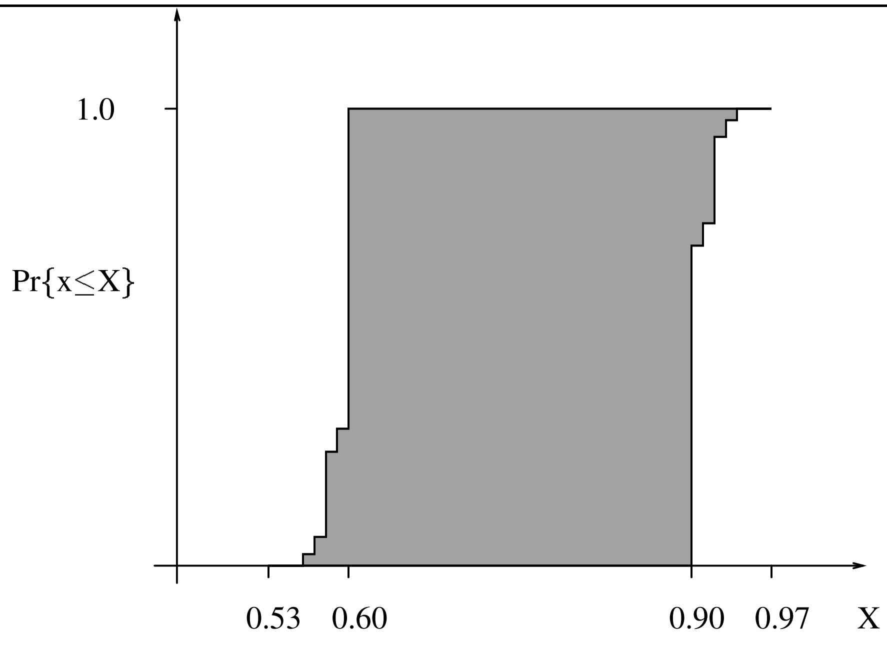

Figure 5: Composed DFZone for the arm-to-disk clearance g.

At this stage a one-sided risk assessment of all possible assemblies is no more complex than a table look-up. For example, let us assume that the assembly is required to have a gap between .535 and .947. Since $F(.535) \leq U(.535) = .0000306$ for any $F \in (Z(L, U)$, no more than .0000306 of the assemblies of any population of assemblies have values for g of less than .535, and in fact this value is attained by an assembly with part-level DFs of U_1, U_2,

L_2, and L_4 for l_1, l_2, l_3, l_4, respectively. Similarly, since $F(.947) \geq L(.947) = .9983594$, no more than $1 - .9983594 = .0016406$ of the assemblies of any population of assemblies have values for g of more than .947, and this is attained.

5 Some Concluding Remarks

This paper has discussed DFZones and their composition. It showed that these zones can be useful in description of part-level variation. The relationship between linear gap functions of independent variables and convolutions allowed the DF of a gap to be developed for specific part-level populations. An inequality in DFs was shown to be preserved under convolution, and this was used to compose part-level DFZone based specifications to form an assembly-level DFZone. It was noted that DFZones described by step functions, like those under consideration by ISO, lead to a composed zone also described by step functions, that is, DFZones of this type are closed under composition. A detailed consideration of an example closed the paper. In particular, one-sided risk assessment was seen to be direct in this framework.

Several interesting questions remain. First, is the restriction of the bounding functions to being DFs needed or useful? Examination of the developments here seem to indicate that weakening the restriction to increasing functions would function as well and would allow specification that some percent of a population is nonconforming without any further specification as to how. Next, the role of the collection of DFs, $\mathcal{F}$, in $Z(L, U; \mathcal{F})$ needs attention. For example, if $L^{'} \leq L$ and $L^{'}(X) < L(X)$ for some (X), it easy to see that $Z(L^{'}, U)$ never equals $Z(L, U)$, but for some such L and $L^{'}$ there are choices of $\mathcal{F}$ and U, such that $Z(L^{'}, U; \mathcal{F})$ and $Z(L, U; \mathcal{F})$ coincide. When can we say this is the case? Finally, is there anything meaningful and useful that can be said about two-sided risk assessment? It is not true that one DF can always attain both one-sided risks, and even if one does, it is not clear when we may claim that this DF corresponds to an assembly.

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