RC 20965(92550) 13JUN97
Computer Science/Mathematics

IBM Research Report

Graph Coloring via Synchronization of Coupled Oscillators

Chai Wah Wu

IBM Research Division
T.J. Watson Research Center
Yorktown Heights, New York 10598

IBM Research Division
Almaden · T.J. Watson · Tokyo · Zurich

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Chai Wah Wu

IBM Thomas J. Watson Research Center
P. O. Box 218
Yorktown Heights, NY 10598, U.S.A.

Abstract

In this paper, we study the possibility of coloring graphs by means of synchronized coupled oscillators. We consider an array of coupled oscillators as a graph by associating the oscillators to vertices and the coupling to edges. When the coupled array is synchronized, the phase of the oscillators can be considered as the color associated with the corresponding vertices. We prove that for connected 2-colorable graphs, we can construct a coupled array which generates the 2-coloring for that graph. For the general case, numerical simulation results with connected 3-colorable graphs suggest that the coupled array of oscillators can color graphs with a small number of colors in most cases. Some complexity issues of the system and comparisons with antivoter models of graph coloring will be discussed. We also conjecture that the system can be used to approximate the star chromatic number of the graph.

1 Introduction

In this paper, we utilize synchronized arrays of coupled oscillators to color graphs. We prove that for connected 2-colorable graphs, we can construct a coupled array which generates the corresponding 2-coloring. For the general case, numerical simulation results with connected 3-colorable graphs suggest that although it does not solve the NP-complete problem of coloring with the least number of color, for most of the time it can color graphs with a small number of colors. The problem formulation and the basic setup are presented in Section 2. In Section 3 we present the result that for each connected 2-colorable graph, a corresponding array of coupled oscillators can be constructed which generate the 2-coloring. In Section 4 we consider an array of oscillators for coloring arbitrary graphs and present numerical results. In Section 5 we discuss some complexity aspects of the proposed systems. In Section 6 we compare the proposed system in Section 4 to antivoter models of graph coloring. In Section 7 we present some concluding remarks including a conjecture that the system in Section 4 can be used to approximate the star chromatic number of the graph.

2 Problem Formulation

Given a graph $G = (V, E)$, a (vertex) coloring of G is an assignment of a color to each vertex, such that no two vertices with the same color are adjacent to each other. In other words, a coloring c with k colors can be described as a function $c : V \to \{1, 2, \cdots, k\}$ such that if v_i is adjacent to v_j, then $c(v_i) \neq c(v_j)$. Without loss of generality we assume that the vertex set is $V = \{1, 2, \cdots, N\}$. We define $v(i)$ to be the degree of the i-th vertex, i.e. the number of edges connected to vertex i. A graph is called k-colorable if it can be colored with k colors. The least number of colors a graph can be colored with is called the chromatic number of G, denoted $\chi(G)$. It is clear that no connected graphs are 1-colorable except the trivial graph of one vertex and no edges. We will only consider graphs with no self-loops since any graphs with self-loops cannot be colored. For 2-colorable graphs, a simple algorithm suffices to generate the 2-coloring. First assign an arbitrary vertex one color, then assign its neighbors the second color, then assign its neighbors' neighbors the first color, etc. until the entire graph is colored[1]. This algorithm can also be used to check whether the graph is 2-colorable, as the resulting coloring is not a valid coloring of the graph if the graph is not 2-colorable. For graphs where the chromatic number is higher than 2, things are more complicated. In fact, for this case the problem of coloring a graph with the minimal number of colors is NP-hard [1]. There has been recent work in constructing effective algorithms which can color a graph with the number of colors much less than the number of vertices [1]. The purpose of this paper is to present a continuous time dynamical system for coloring a graph with a small number of colors.

As in [2], we consider an array of coupled oscillators as a graph by associating vertices to the oscillators and edges to the coupling. The oscillators are coupled such that they are synchronized at steady state. At steady state, all oscillators generate a periodic waveform of the same period and a phase can be assigned to each oscillator. The relationship between the phases in synchronized arrays of oscillators have been studied before, e.g. [3, 4]. At steady state the phase of the oscillator can be considered as a color associated with the corresponding vertex. We want this procedure to generate a valid coloring of the graph. We use lowercase to denote scalars and boldface type to denote vectors and matrices.

[1] For disconnected graphs, this procedure has to be repeated for each connected component.

3 Coloring of 2-Colorable Graphs via Synchronized Oscillators

Consider an array of coupled relaxation oscillators and its associated graph G shown in Fig. 1.

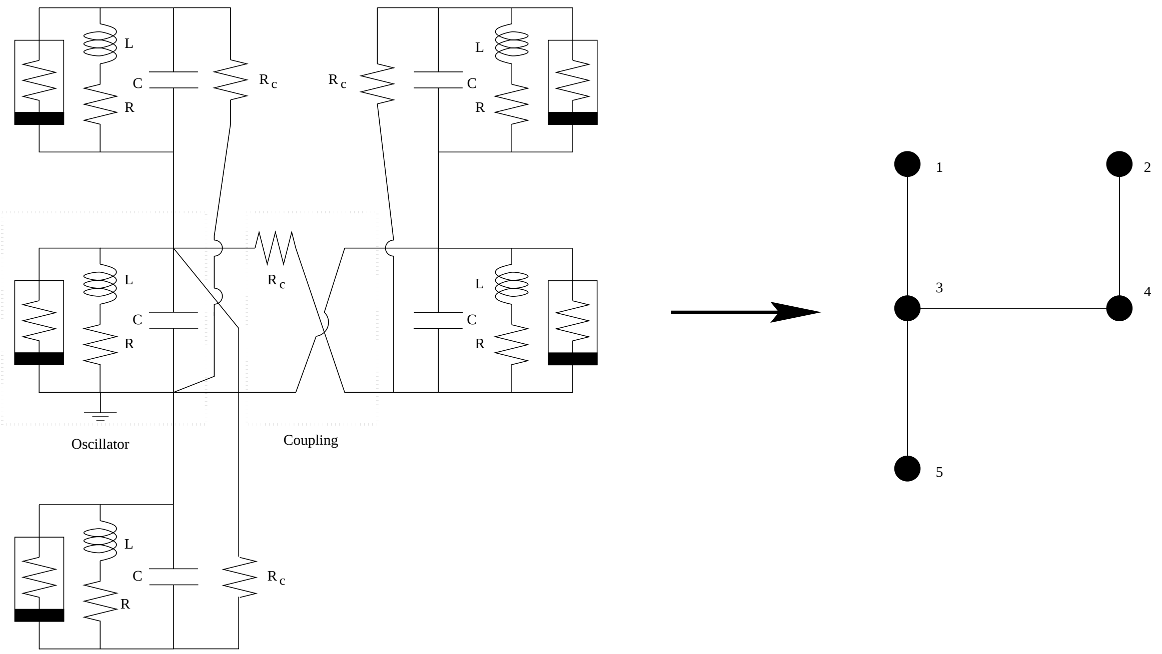

Figure 1: Arrays of coupled relaxation oscillators and its associated graph.

The state equations for this coupled system are given by:

$$\begin{array}{rcl} C\frac{dv_j}{dt} & = & -i_j - g(v_j) - \frac{1}{R_c}(\mathbf{G}\mathbf{v})_j \\ L\frac{di_j}{dt} & = & v_j - i_j/R \end{array} \tag{1}$$

where v_j is the voltage accross the capacitors, i_j is the current through the inductors, $\mathbf{v} = (v_1, v_2, \ldots, v_N)^T$, $N = 5$ and $g(v) = -\epsilon(v - \frac{v^3}{3})$ is the nonlinear v-i characteristic of the nonlinear resistor[2]. For the coupling configuration in Fig. 1, the matrix $\mathbf{G}$ is given by:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Note that $\mathbf{G} = \mathbf{A} + \mathrm{diag}(v(i))$, where $\mathbf{A}$ is the adjacency matrix of the associated graph and $\mathrm{diag}(v(i))$ is the diagonal matrix with the degree of the vertices on the diagonal. We want to prove that the array synchronizes to a steady state which gives

[2]Also called Chua's diode.

a 2-coloring of the graph whenever the graph is 2-colorable. Note that a 2-coloring is essentially unique (up to permutation of the two colors) when G is connected.

In particular, we prove the result for a more general class of oscillators with underlying graph G:

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \vdots \\ \dot{\mathbf{x}}_N \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1) \\ \mathbf{f}(\mathbf{x}_2) \\ \vdots \\ \mathbf{f}(\mathbf{x}_N) \end{pmatrix} - \alpha(\mathbf{G} \otimes \mathbf{D})\mathbf{x} \tag{2}$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N)^T$ and $\mathbf{G} = \mathbf{A} + \mathrm{diag}(v(i))$ and $\mathbf{A}$ is the adjacency matrix of the graph G and $\mathrm{diag}(v(i))$ is the diagonal matrix with the diagonal entries consisting of the degrees of the vertices. The Kronecker product is defined as:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \\ a_{m1}\mathbf{B} & & a_{mn}\mathbf{B} \end{pmatrix}$$

where $\mathbf{A} = \{a_{ij}\}$.

The matrix $\mathbf{D}$ is a coupling matrix which describes the coupling between two oscillators. Each $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$ is an oscillator and N is the number of oscillators in the array. We say that $\mathbf{x}_i$ and $\mathbf{x}_j$ are adjacent, if the corresponding vertices are adjacent in the underlying graph G, i.e. $a_{ij} \neq 0$.

Theorem 1 *Consider the array of coupled oscillators in Eq. (2) with underlying graph G where $\mathbf{f}$ is an odd function, i.e., $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$ for all $\mathbf{x}$. Suppose that a matrix $\mathbf{D}$ and a symmetric positive definite matrix $\mathbf{V}$ exist such that*

$$(\mathbf{z} - \mathbf{w})^T \mathbf{V}(\mathbf{f}(\mathbf{z}, t) - \mathbf{D}\mathbf{z} - \mathbf{f}(\mathbf{w}, t) + \mathbf{D}\mathbf{w}) \leq -c\|\mathbf{z} - \mathbf{w}\|^2 \tag{3}$$

for all $\mathbf{z}$ and $\mathbf{w}$ and some $c > 0$. Assume also that $\mathbf{VD}$ is positive semidefinite.

If G is connected and 2-colorable and $\frac{1}{\alpha}$ is less than the algebraic connectivity of G[3], then for two oscillators $\mathbf{x}_i$ and $\mathbf{x}_j$ which are adjacent, they are synchronized out of phase, i.e., $\mathbf{x}_j \to -\mathbf{x}_i$ as $t \to \infty$. In particular, by associating the phase of the oscillators to two colors, we get a 2-coloring of the graph.

Proof Consider a 2-coloring of G. Let $\mathbf{B}$ be the diagonal matrix such that $\mathbf{B}_{ii} = 1$ if the i-th vertex is of one color, and $\mathbf{B}_{ii} = -1$ if the i-th vertex is of the other color. If $\mathbf{A}_{ij} \neq 0$, then $\mathbf{B}_{ii} = -\mathbf{B}_{jj}$. Therefore $(\mathbf{BAB})_{ij} = \mathbf{B}_{ii}\mathbf{A}_{ij}\mathbf{B}_{jj} = -\mathbf{A}_{ij}$. Thus $\mathbf{BAB} = -\mathbf{A}$. Consider $\mathbf{C} = \mathbf{BGB}$. $\mathbf{C} = \mathbf{BAB} + \mathbf{B}\mathrm{diag}(v(i))\mathbf{B} = -\mathbf{A} + \mathrm{diag}(v(i))$. Therefore $\mathbf{C}$ is equal to the Laplacian matrix $\mathrm{diag}(v(i)) - \mathbf{A}$ of the graph G, and thus can be decomposed as $\mathbf{C} = \mathbf{E}\mathbf{E}^T$ where $\mathbf{E}$ is the vertex-edge incidence matrix of G.

[3]The algebraic connectivity of a graph G is defined as the smallest nonzero eigenvalue of $\mathrm{diag}(v(i)) - \mathbf{A}$.

Consider the state transformation $\tilde{\mathbf{x}} = (\mathbf{B} \otimes \mathbf{I})\mathbf{x}$ where $\mathbf{I}$ is the identity matrix. We write $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N)^T$. Then the state equations can be rewritten as:

$$\begin{pmatrix} \dot{\tilde{\mathbf{x}}}_1 \\ \dot{\tilde{\mathbf{x}}}_2 \\ \vdots \\ \dot{\tilde{\mathbf{x}}}_N \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\tilde{\mathbf{x}}_1) \\ \mathbf{f}(\tilde{\mathbf{x}}_2) \\ \vdots \\ \mathbf{f}(\tilde{\mathbf{x}}_N) \end{pmatrix} - \alpha(\mathbf{B} \otimes \mathbf{I})(\mathbf{G} \otimes \mathbf{D})(\mathbf{B} \otimes \mathbf{I})\tilde{\mathbf{x}} \tag{4}$$

where we have used the fact that $\mathbf{f}$ is an odd function. The second term can be written as:

$$\alpha(\mathbf{B} \otimes \mathbf{I})(\mathbf{G} \otimes \mathbf{D})(\mathbf{B} \otimes \mathbf{I})\tilde{\mathbf{x}} = \alpha(\mathbf{BGB} \otimes \mathbf{D})\tilde{\mathbf{x}} = \alpha(\mathbf{C} \otimes \mathbf{D})\tilde{\mathbf{x}}$$

By applying the results in [5], this system is synchronized, i.e., $\tilde{\mathbf{x}}_i \to \tilde{\mathbf{x}}_j$ as $t \to \infty$ for all i, j. The conclusion then follows from the definition of $\mathbf{B}$. ∎

Since for a graph of N vertices the algebraic connectivity is always larger than or equal to $2\left(1 - \cos\left(\frac{\pi}{N}\right)\right)$, we can choose $\alpha > \frac{1}{2\left(1-\cos\left(\frac{\pi}{N}\right)\right)}$. Note that the coupling in Fig. 1 is a resistive 2-port where the leads are "twisted" in the output port. An intuitive interpretation of the above theorem is that if certain oscillators can be turned upside down as to "untwist" the coupling, then we would result in an array of oscillators with diffusive coupling and the results in [6] can then be applied. The oscillator can be turned upside down and its dynamics remain the same since all the components are bilateral[4]. For 2-colorable graphs, this untwisting can be done exactly when the oscillators of one color are turned upside down.

We can apply Theorem 1 to the coupled oscillators in Eq. (1) when we choose

$$\mathbf{D} = \begin{pmatrix} \frac{\epsilon+\delta}{C} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \quad R_c = \frac{1}{(\epsilon+\delta)\alpha}, \quad C, L > 0$$

It is easy to see that Eq. (3) is satisfied for any $\delta > 0$. It then follows that the system in Eq. (1) synchronizes and generates a 2-coloring if $R_c < \frac{a}{\epsilon}$ and the graph is connected and 2-colorable. The number a is the algebraic connectivity of the graph which in the case of Fig. 1 is equal to 0.5858. Since the graph in Fig. 1 is 2-colorable and connected, the system in Fig. 1 synchronizes and generates a 2-coloring if $R_c < \frac{0.5858}{\epsilon}$.

4 Coloring of Arbitrary Connected Graphs

In this section we present an array of coupled oscillators for coloring arbitrary graphs. As synchronization between all the oscillators can only occur if the graph is connected, we will only consider connected graphs. Given a graph G with adjacency matrix $\mathbf{A}$,

[4]This results in the vector field being symmetric with respect to the origin.

we generate an array with state equations as in Eq. (2) and

$$\mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \mathbf{f}(\mathbf{x}_i) = \begin{pmatrix} -y_i + 5(x_i - \frac{x_i^3}{3}) \\ x_i \end{pmatrix},$$
$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \alpha = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_{NN} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

Whereas before α is a scalar, here α is a matrix with $\alpha_{ii} = 0.21/v(i)$. The initial conditions are chosen at random from the interval $[-1, 1]$. This system can be implemented by substituting the oscillator shown in Fig. 2 for the oscillators in Fig. 1. For regular graphs, all the α_{ii} are equal, and by choosing $R_c = \frac{1}{\alpha_{ii}}$ we can eliminate the controlled current source in Fig. 2.

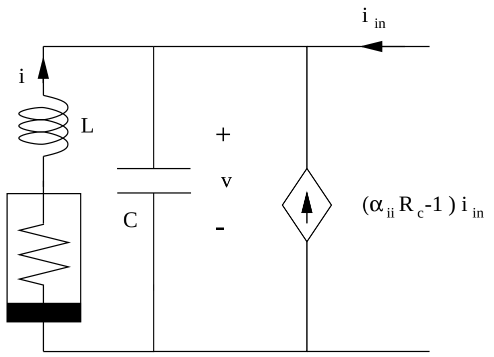

Figure 2: Substituting this oscillator for the oscillators in Fig. 1 results in dimensionless state equations of the form Eq. (2) and Eq. (5).

Although Theorem 1 cannot be applied to this case when G is 2-colorable, computer simulations indicate that the array synchronizes to the proper 2-coloring. For more general graphs, a more complicated algorithm is needed to map a synchronized state of the oscillators onto a coloring scheme. Let $0 \leq \phi(i) < 2\pi$ be the phase of the i-th oscillator at the synchronized state. For each $0 \leq d \leq 2\pi$, we associate a mapping which maps ϕ onto a coloring c of the graph G. We denote this map as the decoding map μ_d. The map μ_d is defined such that vertices v_i and v_j are colored the same color if and only if the difference between the phase of the i-th oscillator and the j-th oscillator is less than or equal to d and v_i and v_j are not adjacent in the graph. The phase difference is computed modulo 2π. More precisely

$$\mu_d(v_i) = \mu_d(v_j) \Leftrightarrow (h(\phi(i), \phi(j)) \leq d \quad \text{AND} \quad v_i \quad \text{is not adjacent to} \quad v_j)$$

where the function h calculates the difference between the phases of oscillators i and j (modulo 2π) and is defined as $h(a, b) = \min(|a - b|, |a - b + 2\pi|, |a - b - 2\pi|)$. This in general does not uniquely determine μ_d even up to a permutation of the colors. Note that μ_0 is the map used to determine the 2-coloring in Theorem 1.

It's clear that each μ_d generates a valid coloring of the graph. For graphs which are not 2-colorable, most of the time at the synchronized state all the oscillators will have different phases, so that μ_0 will generate the trivial coloring of assigning each vertex a different color. In our graph coloring algorithm, for each synchronized state we increase d from 0 or decrease d from 2π, until a valid coloring is found with the minimal number of colors. The pseudocode for this algorithm is shown in Fig. 3. Thus the steady state form a relaxed coloring of the graph, where we assign to each vertex the phase of the corresponding oscillator. What is desired is that adjacent vertices have phases far apart from each other. The property of "far apart" is captured by $h(\phi(i), \phi(j)) > d$. The algorithm tries to find the d which gives a valid coloring, minimal among all d. An alternative algorithm can be obtained by replacing `Simulate system for T seconds` with `Simulate system until steady state` and removing the `WHILE` and `ENDWHILE` statements.

```
BEGIN
      totaltime = 0
      WHILE totaltime < maxtime DO
            Simulate system for T seconds.
            Get state vector at end of simulation and calculate φ(i).
            d = 0
            DO
               d = d + δ
               use μ_d to generate coloring.
               IF coloring generated by μ_d is satisfactory
                  STOP and RETURN coloring generated by μ_d.
               ENDIF
            UNTIL d = 2π
            totaltime = totaltime + T
      ENDWHILE
      RETURN coloring generated by μ_d.
END
```

Figure 3: Pseudocode of algorithm to generate coloring from phase information of the oscillators. An alternative algorithm can be obtained by replacing "Simulate for T seconds" with "Simulate until steady state" and removing the WHILE and ENDWHILE statements.

The constant δ should be taken to be inversely proportional to the number of vertices. We have simulated Eq. (2) with the oscillators defined as Eq. (5) with $\mathbf{G}$ corresponding to various small graphs with chromatic number 3. Most of the time, the system generates a 3-coloring. In particular, we have conducted experiments with 300 graphs with the number of vertices ranging from 4 to 16. The graphs are generated randomly using the algorithm in [7] where we make the edge density p a uniform random variable in the range $[2/N, 1]$. The generated graphs have either chromatic number 2 or 3. The maximum time simulated (`maxtime` in Fig. 3) is 2000, which corresponds to approximately 160 cycles of oscillations. The system colored all

2-colorable graphs (except for one) correctly, and of the remaining graphs which have chromatic number 3, it colored more than 79% with three colors, and more than 94% with less than or equal to 6 colors. Of the graphs which were minimally colored with 3 colors, the correct coloring was found in less than 100 cycles.

As an example, consider the following 3-colorable graph with 12 vertices shown in Fig. 4. The waveforms of x_i are shown in Fig. 5. The waveforms are translated in the vertical axis to avoid clutter, with oscillator 1 at the bottom and oscillator 12 at the top. At $t = 0$, the random initial conditions result in random phases for the 12 oscillators. Near the end of the simulation ($t = 100$), we can clearly see that the phases can be grouped into 3 groups, with oscillators 1-4 in one group, oscillators 5-8 in a second group, and oscillators 9-12 in the third group. Assigning oscillators in each group to one color results in a valid 3-coloring of the graph.

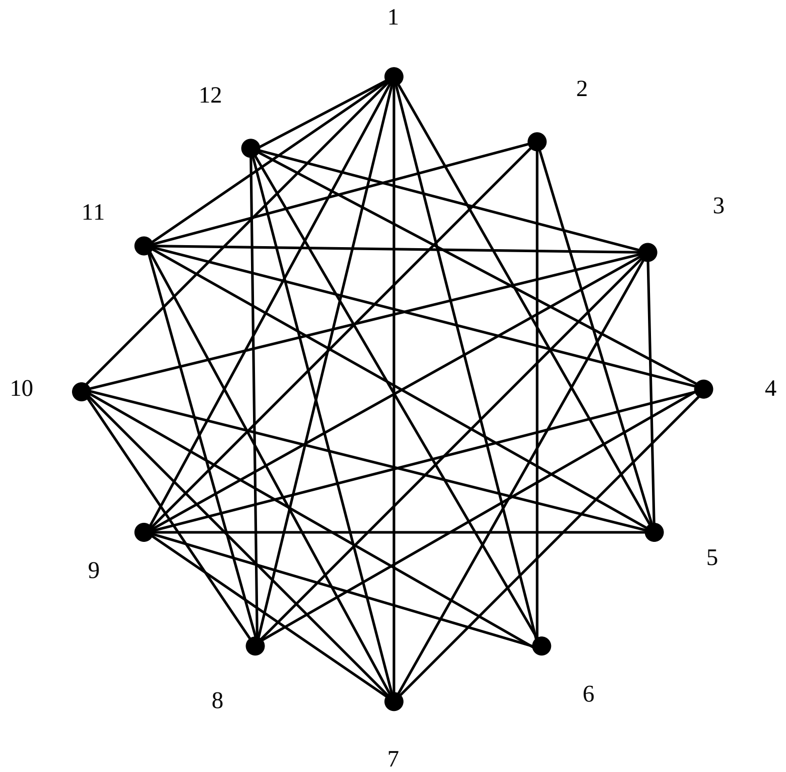

Figure 4: Graph with 12 vertices and chromatic number 3.

As a second example, consider the connected bipartite graph with 16 vertices shown in Fig. 6. The waveforms of x_i are shown in Fig. 7. It is clear that after a few cycles the first 8 oscillators are synchronized in phase, while the other 8 oscillators are synchronized at opposite phase, creating a valid 2-coloring for the graph.

5 Complexity Issues

The space complexity of the proposed system is on the order of $O(N^2)$, since the number of edges of a fully connected graph is on the order of $O(N^2)$. The proof of

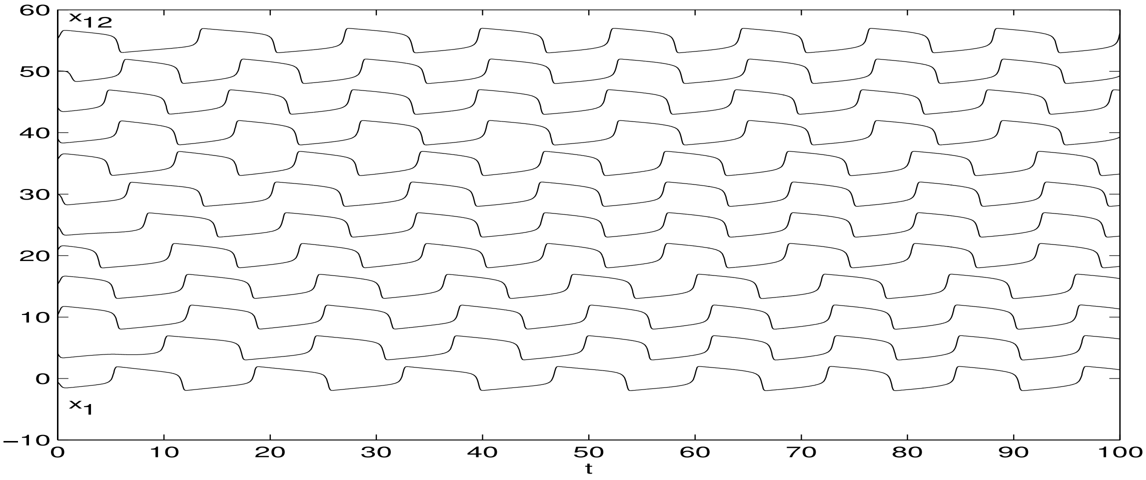

Figure 5: Waveforms corresponding to x_i for coloring the graph in Fig. 4. The curves are translated in the vertical axis to avoid clutter. x_1 is the lowest curve and x_{12} is the highest curve.

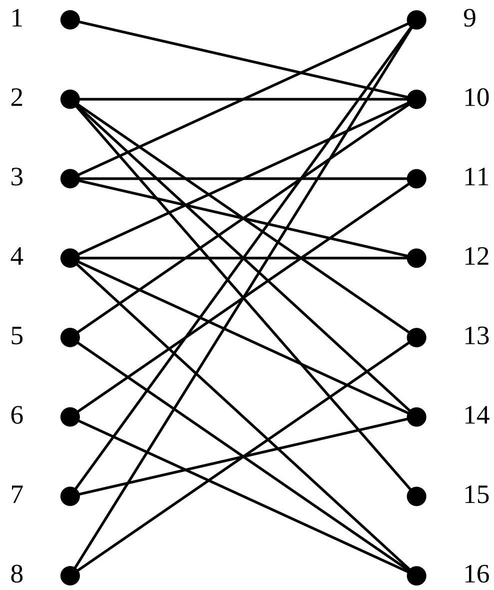

Figure 6: Connected bipartite graph with 16 vertices.

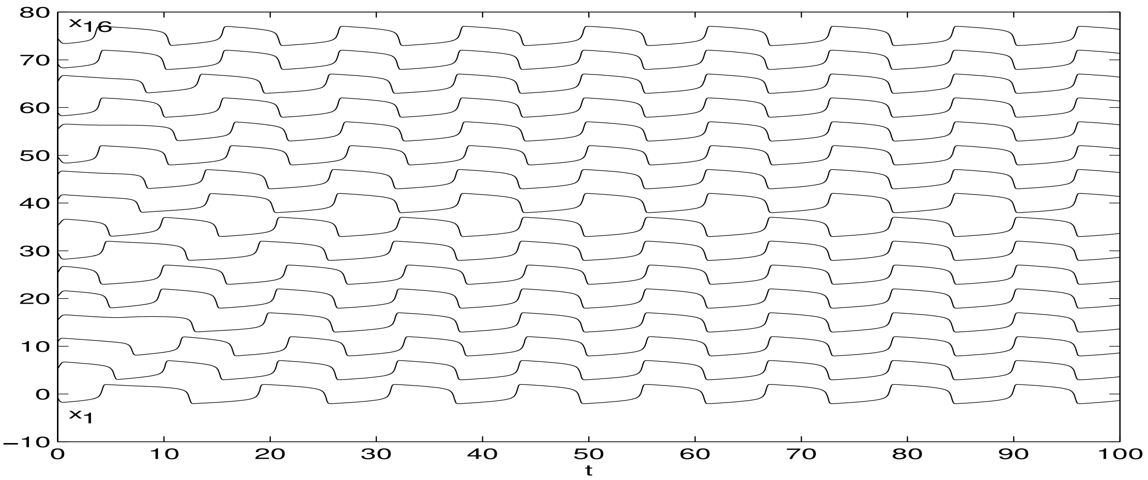

Figure 7: Waveforms corresponding to x_i for coloring the graph in Fig. 6. The curves are translated in the vertical axis to avoid clutter. x_1 is the lowest curve and x_{16} is the highest curve.

Theorem 1 is based on results in [6] which is proved by using a Lyapunov energy function. As this Lyapunov function decreases exponentially, it can be shown that it takes $O(1)$ time for a system satisfying Theorem 1 to converge to the synchronized state, when the initial conditions are chosen to be small in proportion to N. Simulating this system on a digital computer would then require a time on the order of $O(N^2)$ per unit time step of the differential equations. For the more general case, the algorithm in Fig. 3 requires a time on the order of $O(N^3)$ if simulated on a digital computer. However, it is not clear how long it takes for the system described by Eq. (2) and Eq. (5) to reach steady state (if at all) although numerical simulations indicate this time to be $O(1)$. If that is the case, an analog continuous-time implementation of the system in Section 4 along with the alternative algorithm (see caption of Fig. 3) would run in $O(1)$ time.

6 Antivoter Model for Graph Coloring

The antivoter model proposed in [8] for graph coloring shares some similarities and differences with the present approach of coupled oscillators. In the antivoter model, adjacent vertices repel each other at random intervals. We believe that in the system of coupled oscillators, adjacent oscillators also "repel" each other so that they will not share the same phase. However, the repulsion here is differential and occur in continuous time. In both the antivoter model and the coupled oscillators model, there are theoretical results available for 2-colorable graphs, but none are available for more general graphs. One difference between the two approaches is that the antivoter model apparently always generates a 3-coloring if one waits long enough while the present model seemingly doesn't do that. One the other hand, the antivoter model will not return a valid coloring if the algorithm is stopped prematurely or the graph is not 3-colorable, while in the present model, after a fixed initial period it seems the algorithm almost always generates a valid coloring when stopped at any time. Furthermore, the antivoter model for 3-coloring will not return a 2-coloring for 2-colorable graphs, while simulation indicates that the present system almost always returns a 2-coloring for 2-colorable graphs.

7 Concluding Remarks

We have proposed a system of synchronized coupled oscillators for coloring graphs. The system as described does not operate perfectly for graphs which are not 2-colorable. Once in a while, the system would not generate a valid coloring or would converge to a stable equilibrium point, in which case the system has to be simulated using a different set of initial conditions and different α. This behavior requires further investigation. Another future research direction is studying what the optimal α is.

In [9], Vince defined the concept of a *star chromatic number* which is a generalization of the standard chromatic number. The star chromatic number χ^* is defined as the minimum (or limit) of the k-chromatic numbers. Defining $\mathbb{Z}_k$ as $\mathbb{R}/k\mathbb{Z}$, the k-chromatic number χ_k corresponds to a $\mathbb{Z}_k$-coloring of the graph where the colors are elements of the ring $\mathbb{Z}_k$ with the colors of adjacent vertices as far apart as possible. The metric used in $\mathbb{Z}_k$ is the circular norm: $|x|_k$ is the distance of x to the nearest multiple of k. χ_k is defined as

$$\chi_k = \min_c \frac{k}{\min_{(i,j)\in E} |c(i) - c(j)|_k}$$

It can be shown that $\chi - 1 < \chi^* \leq \chi$ [9].

We can think of the phase of the oscillator as elements of $\mathbb{Z}_k$ with k normalized to 2π. The function h defined earlier then corresponds to the circular norm $|\cdot|_k$ and the procedure of defining μ_d corresponds to a mapping from a $\mathbb{Z}_k$-coloring to a standard graph coloring. Therefore we conjecture that at a synchronized state of the system which generates the minimal (standard) coloring of the graph, this state also corresponds to a $\mathbb{Z}_k$-coloring of the graph which generates χ^*. Therefore the conjecture implies that we can calculate χ^* as:

$$\frac{2\pi}{\min_{(i,j)\in E} h(\phi(i), \phi(j))} \tag{6}$$

For bipartite graphs (i.e. graphs with chromatic number 2), the oscillators corresponding to the two colors are at opposite phase at synchronization, and thus Eq. (6) evaluates to 2. Thus the conjecture is true for bipartite graphs since $\chi^* = 2$ for bipartite graphs. Initial simulations with graphs of the form $G_{n,m}$ (as defined in [9]) support this conjecture.

One of the conclusions of this paper is that continuous-time analog coupled systems can perform complex operations in parallel which were solely in the domain of digital algorithms before. The success of Hopfield networks in solving combinatorial problems and Cellular Neural Networks [10] in solving image processing applications are other examples.

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