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Variational Problems Arising from Balancing Several Error Criteria

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Abstract: The purpose of this paper is to describe various results which pertain to a class of variational problems in which two or more error criteria are to be controlled simultaneously in some optimal fashion. This type of problem has wide application in optimal control, statistical estimation and approximation theory. We make no pretense that our treatment is exhaustive either in the applications we cover or in the analysis we use. Rather, we wish to point out some general properties of these problems that are useful, survey some interesting examples which arise in applications, and give an analysis of certain classes of these variational problems where total positivity plays a central role.

1. Introduction

The purpose of this paper is to describe various results which pertain to a class of variational problems in which two or more error criteria are to be controlled simultaneously in some optimal fashion. This type of problem has wide application in optimal control, statistical estimation and approximation theory. We make no pretense that our treatment is exhaustive either in the applications we cover or in the analysis we use. Rather, we wish to point out some general properties of these problems that are useful, survey some interesting examples which arise in applications, and give an analysis of certain classes of these variational problems where total positivity plays a central role.

The problems that we refer to have a general presentation. We suppose that two functionals $G_1(x)$ and $G_2(x)$ are given which both measure the "desirability" of x. Thus we are led to control both simultaneously.

Perhaps, the simplest example of what we have in mind is smoothing splines. Here we are given data $y_1, \dots, y_m$ which is assumed to be inaccurate and wish to find a smooth function that passes near these values at $t_1 < \cdots < t_m$ in some interval $[a,b]$. Thus we desire to balance the fidelity of our function values $f(t_1), \dots, f(t_m)$ to the noisy data $y_1, \dots, y_m$ with the smoothness of f. Here a standard choice of functionals are

(1.1) $$G_1(f) = \sum_{j=1}^{m} |f(t_j) - y_j|^2$$

and

(1.2) $$G_2(f) = \int_a^b |f^{(n)}(t)|^2 dt.$$

A smoothing spline is the unique function which minimizes a positive combination of both. That is, which solves

$$(1.3) \qquad \min \{G_1(f) + \sigma G_2(f): f \in W_2^n[a,b]\}$$

where $W_2^n [a,b]$ is the Sobolev space

$$W_2^n [a, b] = \{f: f, \dots, f^{(n-1)} \text{abs. cont.}, f^{(n)} \in L_2[a,b]\},$$

and $\sigma > 0$ is some prescribed smoothing parameter. Much is known about this problem which we will not dwell upon here. The only point we wish to raise here concerns the choice of σ. There are several strategies. A useful one is based on the technique of cross-validation. That is, a good σ is determined by how well the smoothing spline to a subset of the data fits the remaining data, Wahba [25]. Another possibility is to choose σ interactively. Thus, if you like the graph of f corresponding to a given σ, fine, if not adjust σ and proceed if necessary. This hit and miss approach although not to be seriously recommended in general may have merit in specific problems. A more reasonable approach can be designed if one had a preferred error tolerance for the data error, say some $\varepsilon > 0$. In this case one might adjust σ so that the solution to (1.3) satisfies $G_1(f) \leq \varepsilon$. Even better, we can abandon (1.3) altogether and consider

$$(1.4) \qquad \begin{aligned} &\text{minimize } G_2(f). \\ &G_1(f) \leq \varepsilon, \quad f \in W_2^n[a,b] \end{aligned}$$

The fact that these extremal problems (1.3), (1.4) are the same is of no surprise and we will highlight their precise relationship in a general context later in Section 2. As a final remark about smoothing splines we mention that some general facts about smoothing noisy data in a Hilbert when f is constrained to lie in a convex set appear in Micchelli and Utreras [17].

A problem of a different type is studied in Forsythe, Golub [9], Spjotvoll [23], Walsh [26], whose motivation comes from both nonlinear parameter estimation and optimal control. Here A is some given $n \times n$ matrix and b some fixed vector in $\mathbb{C}^n$. It is then desired to minimize $\|Ax - b\|$ subject to $\|x\| \leq 1$ (and also $\|x\| = 1$) where $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$ is the usual Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Thus, in this case, $G_2(x) = \|Ax - b\|$ and $G_1(x) = \|x\|$. A characterization of the solution and various other facts about this problem are given in the above references.

Another example comes from optimal filter design. Here the problem is to find an optimal impulse response $u(t)$ of a filter so that the response to a given input signal s of finite support lies within specified pointwise bounds and at the same time minimizes the effect of input noise. When the noise is assumed to be zero - mean white noise the output noise power is proportional to the squared L^2 - norm of u. Thus the problem takes the form

$$\text{minimize} \int_{-\infty}^{\infty} |u(t)|^2 dt$$

subject to

$$\varepsilon^-(t) \leq \psi(t) := (s * u)(t) = \int_{-\infty}^{\infty} u(\tau)s(t-\tau)d\tau \leq \varepsilon^+(t).$$

If we let $d := ½(\varepsilon^+ + \varepsilon^-)$ and $\varepsilon := ½(\varepsilon^+ - \varepsilon^-)$ then we get the equivalent problem

$$\min \{\|u\|_2^2 : \|Ku - d\|_\infty \leq 1\} \tag{1.6}$$

where $Ku: = s * u$, $\|u\|_2^2 : = \int_{-\infty}^{\infty} |u(t)|^2 dt$ and $\|f\|_\infty : = \sup_{-\infty < t < \infty} |f(t)|/\varepsilon(t)$. Thus here we have $G_2(u) = \|u\|_2^2$ and $G_1(u) = \|Ku - d\|_\infty$. Further details and computational algorithms for solutions of this problem appear in Evans, Fortmann, Cantoni [7] and Evans, Cantoni, Fortmann [8].

In contrast, in the series of papers [3-5], a detailed analysis of a class of optimal filter design problems were given. These problems are surveyed in Berkowitz [3]. We wish to mention only one such problem of this type considered in Berkowitz and Pollard [4]. Specifically, they consider the minimum of

$$\left(\int_0^\infty |y(t)|\,dt\right)^2 + \int_0^\infty |y''(t)|^2\,dt \tag{1.7}$$

where $y \in L_1(0, \infty)$, y' is absolutely continuous, $y'' \in L_2(0, \infty)$ and $y(0) = a, y'(0) = b$ are specified. Equivalently, if we let $f(t): = a + bt$, and $(Kh)(t): = \int_0^x (x - t)h(t)dt$, then (1.7) becomes

$$\|f - Kh\|_{L_1}^2 + \|h\|_{L_2}^2 \tag{1.8}$$

where $$\|f\|_{L_1} = \int_0^\infty |f(t)|\,dt, \quad \|f\|_{L_2}^2 = \int_0^\infty |f(t)|^2\,dt$$

and we minimize over all h such that $f - Kh \in L^1(0, \infty)$ and $h \in L^2(0, \infty)$. Thus our functionals are in this case $G_1(h) = \|f - Kh\|_{L_1}^2$, and $G_2(h) = \|h\|_{L_2}^2$.

Next, we describe a problem in control theory, Glashoff [11] and Glashoff and Weck [12]. The problem here is to choose a temperature $u(t, \xi)$, $t \in [0,T]$, $\xi \in \partial\Omega \subseteq \mathbb{R}^n$, of a medium surrounding a body Ω to satisfy prescribed pointwise constraints and so that the temperature distribution y(T,x) at time T of the body is as close as possible to some desired temperature $z(x)$, $x \in \Omega$. Under some regularity hypothesis on Ω we have

$$y(u;\, t,\, x) = \int_{\Gamma_t} g(t,x,\, \tau,\, \xi)u(\tau,\, \xi)d\tau d\xi$$

for some Green's function g and $\Gamma_t := (0,t)x\partial\Omega$. Thus the problem reduces to minimizing

$$\text{(1.9)} \qquad \|Su - z\|_{\infty}(\overline{\Omega}) := \operatorname*{ess\ sup}_{x \in \Omega} |(Su)(x) - z(x)|$$

where

$$(Su)(x) := y(u;\, T,\, x)$$

subject to $u \in L_{\infty}(\Gamma)$ where $\Gamma := (0,T)x\partial\Omega$ and

$$\text{(1.10)} \qquad \|u\|_{\infty}(\Gamma) \leq 1$$

which again gives us $G_2(u) = \|Su - z\|_{\infty}$, $G_1(u) = \|u\|_{\infty}$, but now we have two L^{∞} - norms.

In the totally different context of functional analysis, the Peetre K-functional leads us to similar problems Butzer and Berens [6], Bergh and Löfström [2]. The K-functional is an essential tool in various problems in functional analysis and approximation theory, especially in problems concerned with the Bernstein inverse theorem on order of approximation. The K-functional is defined relative to two Banach spaces X_1, X_2 which are assumed to be continuously imbedded in some linear Hausdorff space

X. Then their algebraic sum $X_1 + X_2 = \{x: x = x_1 + x_2, x_i \in X_i, i = 1,2\}$ is a Banach space relative to the norm (K-functional)

$$K(t,x) = \|x\|_{X_1 + X_2} := \inf_{x = x_1 + x_2} (\|x_1\|_1 + t\|x_2\|_2).$$

In this case $G_1(x_2) = \|x - x_2\|_1$ and $G_2(x_2) = \|x\|_2$. It is only in rare cases that the K-functional of a pair of spaces can be found explicitly and so one does not generally study the variational problem directly. Instead, concrete bounds from above and below for K are sought in terms of some useful measure of x. This may take the form of the modulus of continuity of higher order differences for Sobolev spaces.

Approximation theory provides several further instances of the type of variational problem which interests us here. We mention the problem of best operator approximation with a fixed bound on its norm, to a given operator Stechkin [24]. The general description of the problem is as follows: Let X, Y be Banach spaces and U an unbounded (generally) linear operator of X into Y with domain $D_U \subseteq X$. Let V be another linear operator with domain $D_V \subseteq D_U$ and range in a normed linear space Z. The problem is to study the variational problem

$$\inf_{\|S\| \leq N} \sup_{\|Vx\|_Z \leq 1} \|Ux - Sx\|_Y$$

where the infinum is taken over all bounded linear operators S from X into Y, and N is a prescribed positive constant. Here our functionals are $G_1(S) = \|S\|$ and $G_2(S) = \sup \{\|Ux - Sx\|: \|Vx\|_Z \leq 1\}$.

Vectorial approximation, Gearhart [10] is also very much in the spirit of the problems in which we are interested. This problem is similar to the rest and has the manifestation of simultaneously best approximating a function f and its derivative f' by polynomials. In its general form, it concerns the following notion. We suppose X is some set on which G_1, and G_2 are defined. Given a subset C of X, we say that $u_0 \in C$ is a best vectorial approximation to f from C if there does not exist a $u \in C$ such that both

$$G_1(f-u) < G_1(f-u_0) \text{ and } G_2(f-u) < G_2(f-u_0).$$

We will discuss this problem a bit more in Section 2.

Finally, as our last example we describe the point of view of optimal recovery which gives a model of computation that provides a general framework for the study of estimation problems under limited information. We suppose U is a bounded linear operator from a linear space X into a normed linear space Z, K a convex subset of X, and I another bounded linear operator from X into another normed linear space Y. We wish to estimate Ux given $x \in K$ and any observation $y = Ix + w$ where $\|w\|_Y \leq \varepsilon$, $\varepsilon > 0$, fixed. The intrinsic error in the worst case is given by

$$E = \inf_A \sup_{\substack{x \in K \\ \|w\|_Y \leq \varepsilon}} \|Ux - A(Ix + w)\|_Z \tag{1.11}$$

where we minimize over all mappings A from $\{Ix + w, x \in K, \|w\| \leq \varepsilon\}$ into Z, Micchelli [14], Micchelli and Rivlin [16]. We can bound the intrinsic error from above by

$$\inf_B \{ \sup_{x \in K} \|Ux - BIx\|_Z + \varepsilon \|B\| \} \tag{1.12}$$

where here we minimize over all *bounded linear* operators B from Y to Z. This bound is sometimes sharp, Micchelli [14], Micchelli and Rivlin [16]. Here again as in the operator approximation problem, we have

$$G_2(B) = \sup_{x \in K} \|Ux - BIx\|_Z$$

and

$$G_1(B) = \|B\|.$$

2. Basic Facts

The first result we wish to present concerns simultaneously controlling a family of real-valued functions $G(t,x)$ where $t \in T$ and $x \in K$, a given convex subset of a linear space X. We make the following hypothesis:

(*i*) $$\sup_{t \in T} |G(t,x)| < \infty, \quad x \in K,$$

(*ii*) $$x \to G_t(x) := G(t,x) \text{ is convex on } K,$$

(*iii*) Given any $x,\ y \in K$ such that $G(t,x) < G(t,y)$ for all $t \in T$ there exists a positive constant $c > 0$ such that

$$0 < c \leq G(t,y) - G(t,x), \quad t \in T.$$

Clearly, (i) and (iii) are satisfied if T is a compact Hausdorff space, K a convex subset of a topological vector space X and $G(t, x)$ is continuous on $T \times K$.

To state the result we have in mind we let $B(T)$ be the space of real-valued bounded functions on T and let $\leq$ be the natural pointwise ordering on $B(T)$, i.e. $f \leq g, f, g \in B(T)$ means $f(t) \leq g(t)$, for all $t \in T$. A linear function L in the algebraic dual of $B(T)$ is said to be nonnegative if $f \geq 0$ implies $Lf \geq 0$.

We say that $x_0 \in K$ is a best G-vectorial approximation if there does not exist an $x \in K$ such that $G(t, x) < G(t, x_0)$ for all $t \in T$.

Theorem 2.1. *An element $x_0 \in K$ is a best G-vectorial approximation if and only if there exists a nonnegative nontrivial linear functional L on B(T) such that*

$$L(G(\cdot\,, x_0)) = \min_{x \in K} L(G(\cdot\,, x)). \tag{2.1}$$

Remark 2.1. The functional $x \to L(G(\cdot\,, x))$ is convex and so this theorem reduces the problem of finding a vectorial approximation to that of minimizing some convex function. In particular, for a finite set of convex functionals $\{G_1, \dots, G_m\}$ this says that vectorial approximations come from minimizing

$$\sigma_1 G_1(x) + \cdots + \sigma_m G_m(x), \quad x \in K \tag{2.2}$$

for some choice of $\sigma_1 \geq 0, \dots, \sigma_m \geq 0, \ \sum_{i=1}^{m} \sigma_i > 0$.

Proof: Suppose L and $x_0 \in K$ satisfy (2.1) and x_0 is not a best G-vectorial approximation. Then there is an $\bar{x} \in K$ such that $G(t, \bar{x}) < G(t, x_0)$ for all $t \in T$. Let e be the function in $B(T)$ which is identically one. Since L is a nonnegative nontrivial linear functional on $B(T)$ it follows that $Le > 0$. From property (iii) there is a constant c such that

$$G(t, \bar{x}) \leq G(t, x_0) - c, \quad t \in T. \tag{2.3}$$

Hence by (2.1), (2.3) and the nonnegativity of L we get

$$L(G(\cdot\,, x_0)) \leq L(G(\cdot\,, \bar{x})) \leq L(G(\cdot\,, x_0)) - cLe,$$

which is a contradiction.

Conversely, let x_0 be a best G-vectorial approximation. Define

$$C_1 = \{f: f \in B(T), \quad f(t) < G(t, x_0), t \in T\}$$

and

$$C_2 = \text{convexhull } \{G(\cdot , x): x \in K\}.$$

From property (i), C_1 and C_2 are convex subsets of $B(T)$. They are disjoint, since otherwise for some $\lambda_1 \geq 0, \dots , \lambda_n \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad x_i \in K$, we would have by property (ii)

$$G(t, \sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i G(t, x_i) < G(t, x_0).$$

contradicting the choice of x_0 as a best G-vectorial approximation. From property (i) there is a $d \in \mathbb{R}$ such that $d \leq G(t, x_0)$. It then follows that $f_0(t) := d - \varepsilon$ is an internal point of C_1, for any $\varepsilon > 0$.

Thus by the basic separation theorem for convex sets, see Royden [22, p. 176], there is a nontrivial linear functional L and an $\alpha \in \mathbb{R}$ such that

(2.4) $$Lf \leq \alpha, \quad f \in C_1$$

(2.5) $$Lf \geq \alpha, \quad f \in C_2.$$

For any $f \in B(T)$, $f(t) \leq 0$, $t \in T$; and any $d^1 < d$ we have $d^1 + \lambda f \in C_1$ for all $\lambda \geq 0$. Hence by (2.4), $L(\lambda f + d^1) \leq \alpha$ and so we get $Lf \leq 0$. That is, L is a nonnegative linear functional. Similarly, because $G(\cdot\, , x_0) + \lambda \in C_1$, for $\lambda < 0$ we get by (2.4), $L(G(\cdot\, , x_0)) \leq \alpha$ while (2.5) gives $L(G(\cdot\, , x)) \geq \alpha$ for all $x \in K$. This completes the proof. □

This result shows us that in the special case of two convex functionals, vectorial approximation is equivalent to the minimum problem

$$(2.6) \qquad \inf_{x \in K} \sigma_1 G_1(x) + \sigma_2 G_2(x), \qquad \sigma_1, \sigma_2 \geq 0, \qquad \sigma_1 + \sigma_2 = 1.$$

It will be more convenient for us to discuss the function

$$(2.7) \qquad (G_1 + G_2)(\sigma) : = \inf_{x \in K} \{G_1(x) + \sigma G_2(x)\}, \quad \sigma > 0.$$

It is evident that (2.6) and (2.7) are the same problem if $\sigma_1 \in (0, 1)$ in (2.6). (Simply divide (2.6) by σ_1.)

We discuss the relationship of (2.7) to the two problems

$$(2.8) \qquad (G_1/G_2)(t) : = \inf\{G_1(x) \colon G_2(x) \leq t\},$$

where $t \geq \mu(G_2) \colon = \inf_{x \in K} G_2(x)$ and

$$(2.9) \qquad (G_2/G_1)(t) : = \inf\{G_2(x) \colon G_1(x) \leq t\},$$

where $t \geq \mu(G_1)$.

In what follows, we also assume that G_1 and G_2 are bounded below on K. Thus, without loss of generality we will require that G_1 and G_2 are nonnegative. We start with some general properties of $(G_1/G_2)(t)$ and $(G_2/G_1)(t)$.

Let

$$\Gamma = \{y = (y_1, y_2) : G_i(x) \leq y_i, \ \ i = 1, 2, \ \text{for some } x \in K\}.$$

Γ is a generalization of Gagliardo diagrams studied in connection with the K-functional (see e.g. Bergh, Lofstrom [2, Chap. 3 and 7]). Since each G_i is convex and nonnegative, it easily follows that Γ is a convex unbounded subset in the first quadrant of $\mathbb{R}^2$. To understand quite explicitly the usefulness of Γ, set

$$\begin{aligned} \mu [G_1; G_2] &= \lim_{t \to \mu (G_2)^+} (G_1/G_2)(t) \\ &= \sup_{t > \mu (G_2)} (G_1/G_2)(t). \end{aligned}$$

(Note that (G_1/G_2) is obviously nonincreasing), and similarly, we define $\mu[G_2; G_1]$. The graph of Γ is given by

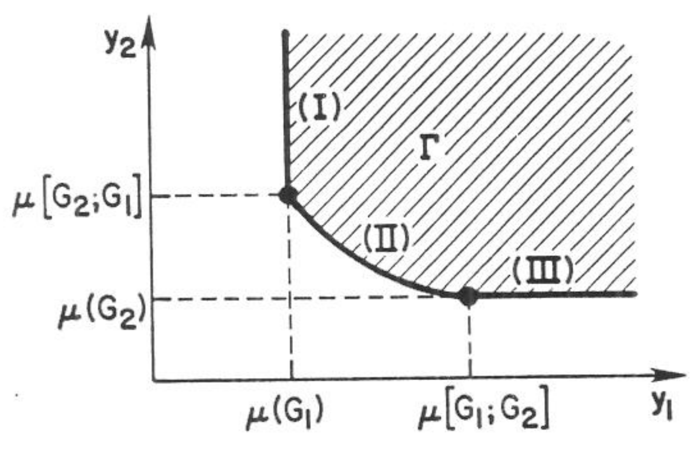

Figure 1

We remark that each of these segments *(I)*, *(II)*, and *(III)* of $\partial\Gamma$ may be empty. The boundary sections *(II)* and *(III)* (without the left-endpoint) are the "essential" graph of $(G_2/G_1)(t)$ (where $t = y_1$). At the left-endpoint of *(II)* , i.e., at $y_1 = t = \mu(G_1)$,, it may be that $(G_2/G_1)(\mu(G_1))$ is not defined. This will occur if $\inf\{G_1(x) : x \in K\}$ is not attained. Moreover, even if $\min\{G_1(x) : x \in K\}$ does exist, then $(G_2/G_1)(\mu(G_1)) \geq \mu[G_2; G_1]$ and equality need not hold. The graph of $(G_1/G_2)(t)$ is obtained by interchanging the axes. To see that this is indeed the case, we list and prove the following facts.

Theorem 2.2. *Let G_1, G_2 be convex nonnegative functions on a convex subset K of a given linear space. Then*

(1) $(G_1/G_2)(t)$ *is nonincreasing and convex on its domain of definition, and continuous on the interior thereof.*

(2) $(G_1/G_2)(t) = \mu(G_1)$ *for* $t \geq \mu[G_2; G_1]$.

(3) $(G_1/G_2)(t)$ *is strictly decreasing on* $(\mu(G_2), \mu[G_2; G_1])$.

(4) $(G_2/G_1)\,((G_1/G_2)(t)) = t$ *for* $t \in (\mu(G_2), \mu[G_2; G_1])$.

Proof. (1). It is obvious that $(G_1/G_2)(t)$ in nonincreasing. The convexity is a simple consequence of the convexity of Γ. The continuity follows from general considerations regarding convex nonnegative functions.

(2). Since $G_1(x) \geq \mu(G_1)$ for all x, we have $(G_1/G_2)(t) \geq \mu(G_1)$ for all t. $(G_2/G_1)(t)$ is nonincreasing. Thus from the definition of $\mu[G_2; G_1]$, we have that since $t > \mu[G_2; G_1]$, there exists a sequence $\{x_n\}$ in K such that $G_2(x_n) \leq t$ while $G_1(x_n) < \mu(G_1) + \frac{1}{n}$. But then $(G_1/G_2)(t) < \mu(G_1) + \frac{1}{n}$, implying that $(G_1/G_2)(t) \leq \mu(G_1)$.

(3). Suppose that there are two values $t_1, t_2 \in (\mu(G_2), \mu[G_2; G_1])$ such that $(G_1/G_2)(t_1) = (G_1/G_2)(t_2)$, $t_1 < t_2$. Since (G_1/G_2) is convex and nonincreasing it easily follows that $(G_1/G_2)(t) =$ constant for $t \geq t_1$. Thus (2) implies that $(G_1/G_2)(t) = \mu(G_1)$, for all $t \geq t_1$, for some $t_1 \in (\mu(G_2), \mu[G_2; G_1])$. Consequently, given any $\delta > 0$ there is a $x_\delta \in K$ with

$$G_1(x_\delta) \le \mu(G_1) + \delta, \quad G_2(x_\delta) \le t.$$

Hence $(G_2/G_1)(\mu(G_1) + \delta) \le t$ for all $\delta > 0$ and therefore by the definition of $\mu[G_2; G_1]$ we get

$$\mu[G_2; G_1] \le t, \quad \text{for any } t \ge t_1.$$

But $t_1 < \mu[G_2; G_1]$, which is a contradiction.

(4). We first note for $t \in (\mu(G_2), \mu[G_2; G_1])$, $(G_1/G_2)(t)$ is in the domain of (G_2/G_1) because

$$\mu(G_1) < (G_1/G_2)(t) < \mu[G_1; G_2].$$

Let $\bar{t} := (G_2/G_1)((G_1/G_2)(t))$ and choose any $\delta > 0$. From the definition of $(G_1/G_2)(t)$ there exists $x_\delta \in K$ with

$$G_1(x_\delta) \le (G_1/G_2)(t) + \delta$$
$$G_2(x_\delta) \le t.$$

Hence

$$(G_2/G_1)((G_1/G_2)(t) + \delta) \le t$$

which by the continuity of G_2/G_1 in the open interval $(\mu(G_1), \mu[G_1; G_2])$ gives, as $\delta \to 0^+$ that $\bar{t} \le t$. For the reverse inequality, let $\delta > 0$. Then there is an $x_\delta \in K$ such that

$$G_2(x_\delta) \le \bar{t} + \delta, \quad G_1(x_\delta) \le (G_1/G_2)(t).$$

Consequently, $(G_1/G_2)(\bar{t} + \delta) \le (G_1/G_2)(t)$ and so $\bar{t} + \delta \ge t$, for otherwise we would contradict the strict decrease of (G_1/G_2). Letting $\delta \to 0^+$ proves the result. □

Before studying the function $(G_1 + G_2)(\sigma), \sigma > 0$, we consider the relationship between it and the functions $(G_1/G_2)(t)$ and $(G_2/G_1)(t)$. This relationship is geometrically given as follows. For $\sigma > 0$, consider the tangent line to Γ with slope $-1/\sigma$. The y_1-intercept (see Figure 1) of this tangent line is the value $(G_1 + G_2)(\sigma)$. To obtain $(G_2/G_1)(t)$ from $(G_2 + G_1)(\sigma)$ for $t > \mu(G_1)$, we do the reverse. That is, consider the lines with slope $-1/\sigma$ and y_1-intercept $(G_1 + G_2)(\sigma)$. The supremum of the y_2-values of this line at t is $(G_2/G_1)(t)$. This is all equivalent to the following analytic statements.

Proposition 2.3. *Assume G_1 and G_2 are convex nonnegative functions on a convex subset K of a given linear space. Then*

(1)
$$(G_1 + G_2)(\sigma) = \inf_{t > \mu(G_1)} (t + \sigma(G_2/G_1)(t))$$
$$= \inf_{t > \mu(G_1)} ((G_1/G_2)(t) + \sigma t)$$

(2) For $t > \mu(G_1)$

$$(G_2/G_1)(t) = \sup_{\sigma > 0} \left(\frac{(G_1 + G_2)(\sigma) - t}{\sigma} \right).$$

Proof: We prove only the first part of (1). Pick any $\delta > 0$. Then there is an $x_\delta \in K$ with $G_2(x_\delta) \le \delta + (G_2/G_1)(t)$ and $G_1(x_\delta) \le t$. Thus

$$(G_1 + G_2)(\sigma) \le G_1(x_\delta) + \sigma G_2(x_\delta)$$
$$\le t + \sigma(\delta + (G_2/G_1)(t))$$

which gives $(G_1 + G_2)(\sigma) \le \inf_t (t + \sigma(G_2/G_1)(t))$ by letting $\delta \to 0^+$. For the other inequality choose $y^\delta \in K$ with

$$G_1(y^\delta) + \sigma G_2(y^\delta) \le (G_1 + G_2)(\sigma) + \delta .$$

Set $t_0 := G_1(y^\delta)$ so that

$$\begin{aligned} &\inf_{t > \mu(G_1)} (t + \sigma(G_2/G_1)(t)) \\ &\quad \le t_0 + \sigma(G_2/G_1)(t_0) \\ &\quad \le G_1(y^\delta) + \sigma G_2(y^\delta) \le (G_1 + G_2)(\sigma) + \delta. \end{aligned}$$

The other statements of the proposition follow similarly. □

We now consider some general properties of the function $(G_1 + G_2)(\sigma)$ for $\sigma \ge 0$. We introduce for $\sigma \ge 0$ the set

$$X_\sigma = \{x: (G_1 + G_2)(\sigma) = G_1(x) + \sigma G_2(x)\}$$

which may be empty for any particular σ. If $X_\sigma \neq \phi$, set

$$G_i^+(\sigma) = \sup_{x \in X_\sigma} G_i(x), \quad i = 1, 2$$

and

$$G_i^-(\sigma) = \inf_{x \in X_\sigma} G_i(x), \quad i = 1, 2.$$

Proposition 2.4. *Let G_1 and G_2 be nonnegative convex functions on K. Then*

(a) $(G_1 + G_2)(\sigma)$ is a nondecreasing continuous concave function of σ on $(0, \infty)$, and $(G_1 + G_2)(\sigma)/\sigma$ is nonincreasing in σ. Furthermore, if there exists an $\hat{x} \in K$ such that $G_2(\hat{x}) = 0$, then $(G_1 + G_2)(\sigma)$ is bounded.

(b) $\lim_{\sigma \to 0^+} (G_1 + G_2)(\sigma) = \mu(G_1)$ and $\lim_{\sigma \to \infty} (1/\sigma)(G_1 + G_2)(\sigma) = \mu(G_2)$.

(c) If there exists an $x^* \in K$ for which $G_1(x^*) = \mu(G_1)$, then for $0 < \sigma_1 < \sigma_2 < \infty$,

$$\frac{(G_1 + G_2)(\sigma_2) - (G_1 + G_2)(\sigma_1)}{\sigma_2 - \sigma_1} \le G_2(x^*).$$

(d) If $0 < \sigma_1 < \sigma_2 < \infty$, and $X_{\sigma_1}, X_{\sigma_2}$ are both non-empty, then

(i) $$G_2^+(\sigma_2) \le G_2^-(\sigma_1)$$

(ii) $$G_1^+(\sigma_1) \le G_1^-(\sigma_2).$$

(e) Assume K is closed in X, and G_1, G_2 are lower semi-continuous on K. If there exists $x_\sigma \in X_\sigma$ and x_∞ such that

$$\lim_{\sigma \to \infty} x_\sigma = x_\infty,$$

then

(i) $$G_1(x_\infty) = (G_1/G_2)(\mu(G_2))$$

(ii) $$G_2(x_\infty) = \mu(G_2).$$

If there exist $x_\sigma \in X_\sigma$ and x_0 such that

$$\lim_{\sigma \to 0^+} x_\sigma = x_0.$$

then

(iii) $$G_2(x_0) = (G_2/G_1)(\mu(G_1))$$

(iv) $$G_1(x_0) = \mu(G_1).$$

Proof: (a). By definition, $(G_1 + G_2)(\sigma)$ is a nondecreasing function of σ. Consequently, since $(G_1 + G_2)(\sigma)/\sigma = (G_2 + G_1)(1/\sigma)$ this function is a nonincreasing function of σ. If there exists an $\hat{x} \in K$ for which $G_2(\hat{x}) = 0$, then

$$0 \leq (G_1 + G_2)(\sigma) \leq G_1(\hat{x}) + \sigma G_2(\hat{x}) = G_1(\hat{x})$$

for all σ. Thus $(G_1 + G_2)(\sigma)$ is bounded. To prove the concavity, let $\sigma_1, \sigma_2 \in (0, \infty)$, $\lambda \in [0, 1]$, and $\sigma = \lambda\sigma_1 + (1 - \lambda)\sigma_2$. Then

$$\begin{aligned}(G_1 + G_2)(\sigma) &= \inf_{x \in K} (G_1(x) + \sigma G_2(x)) \\ &= \inf_{x \in K} [\lambda(G_1(x) + \sigma_1 G_2(x)) + (1 - \lambda)(G_1(x) + \sigma_2 G_2(x))] \\ &\geq \inf_{x \in K} \lambda(G_1(x) + \sigma_1 G_2(x)) + \inf_{x \in K} (1 - \lambda)(G_1(x) + \sigma_2 G_2(x)) \\ &= \lambda(G_1 + G_2)(\sigma_1) + (1 - \lambda)(G_1 + G_2)(\sigma_2).\end{aligned}$$

Thus $(G_1 + G_2)(\sigma)$ is concave and nondecreasing on $(0, \infty)$. Every such function is necessarily continuous on $(0, \infty)$.

(b) Since $(G_1 + G_2)(\sigma)$ is nondecreasing and bounded below (by zero) on $(0, \infty)$, the limit

$$\lim_{\sigma \to 0^+} (G_1 + G_2)(\sigma)$$

necessarily exists. Furthermore,

$$\begin{aligned}\lim_{\sigma \to 0^+} (G_1 + G_2)(\sigma) &= \inf_{\sigma > 0} (G_1 + G_2)(\sigma) \\ &= \inf_{x \in K} \inf_{\sigma > 0} (G_1(x) + \sigma G_2(x)) \\ &= \inf_{x \in K} G_1(x) \\ &= \mu(G_1).\end{aligned}$$

Since $\frac{1}{\sigma}(G_1 + G_2)(\sigma) = (G_2 + G_1)(\frac{1}{\sigma})$, the second result of (b) is verified.

(c) Let $G_1(x^*) = \mu(G_1)$. Since $(G_1 + G_2)(\sigma)$ is concave on $(0, \infty)$, we have for any $0 < \sigma_0 < \sigma_1 < \sigma_2 < \infty$,

$$\frac{(G_1 + G_2)(\sigma_2) - (G_1 + G_2)(\sigma_1)}{\sigma_2 - \sigma_1} \leq \frac{(G_1 + G_2)(\sigma_1) - (G_1 + G_2)(\sigma_0)}{\sigma_1 - \sigma_0}.$$

Now,

$$(G_1 + G_2)(\sigma_1) \leq G_1(x^*) + \sigma_1 G_2(x^*) = \mu(G_1) + \sigma_1 G_2(x^*).$$

From (b), $\lim_{\sigma_0 \to 0^+} (G_1 + G_2)(\sigma_0) = \mu(G_1)$. Substituting these two facts on the right hand side of the initial inequality, we obtain

$$\frac{(G_1 + G_2)(\sigma_2) - (G_1 + G_2)(\sigma_1)}{\sigma_2 - \sigma_1} \leq \frac{\mu(G_1) + \sigma_1 G_2(x^*) - \mu(G_1)}{\sigma_1} = G_2(x^*).$$

(d) Let $x_i \in X_{\sigma_i}$, $i = 1, 2$. Then by definition,

$$\begin{aligned} G_1(x_1) + \sigma_1 G_2(x_1) &\leq G_1(x_2) + \sigma_1 G_2(x_2) \\ &= G_1(x_2) + \sigma_2 G_2(x_2) - (\sigma_2 - \sigma_1) G_2(x_2) \\ &\leq G_1(x_1) + \sigma_2 G_2(x_1) - (\sigma_2 - \sigma_1) G_2(x_2). \end{aligned}$$

Since $\sigma_2 - \sigma_1 > 0$, this implies that $G_2(x_2) \leq G_2(x_1)$ which proves (i). Because

$$G_1(x_1) + \sigma_1 G_2(x_1) \leq G_1(x_2) + \sigma_1 G_2(x_2)$$

and $G_2(x_2) \leq G_2(x_1)$, we have $G_1(x_1) \leq G_1(x_2)$, which proves (ii).

(e) We prove (i) and (ii). The proofs of (iii) and (iv) are totally analogous. To prove (ii), note that

$$\frac{1}{\sigma} G_1(x_\sigma) + G_2(x_\sigma) \leq \frac{1}{\sigma} G_1(x) + G_2(x)$$

for any $x \in K$. Let $\sigma \to \infty$. Then in the limit,

$$G_2(x_\infty) \leq G_2(x)$$

for all $x \in K$. Thus $G_2(x_\infty) = \mu(G_2)$. (This also follows from (b).) To prove (i), let $\hat{x} \in K$ be such that $G_2(\hat{x}) = \mu(G_2)$. Then

$$G_1(x_\sigma) + \sigma G_2(x_\sigma) \le G_1(\hat{x}) + \sigma G_2(\hat{x}) \le G_1(\hat{x}) + \sigma G_2(x_\sigma).$$

Thus $G_1(x_\sigma) \le G_1(\hat{x})$ for all $\sigma > 0$, implying that $G_1(x_\infty) \le G_1(\hat{x})$ for all $\hat{x} \in K$ satisfying $G_2(\hat{x}) = \mu(G_2)$. Thus

$$G_1(x_\infty) \le (G_1/G_2)(\mu(G_2)).$$

Now

$$(G_1/G_2)(\mu(G_2)) = \inf_{G_2(x) \le \mu(G_2)} G_1(x) = \inf_{G_2(x) = \mu(G_2)} G_1(x) \le G_1(x_\infty)$$

since $G_2(x_\infty) = \mu(G_2)$. Thus $G_1(x_\infty) = (G_1/G_2)(\mu(G_2))$. □

3. Operators on Banach Spaces

We shall discuss in some detail the function $(G_1 + G_2)(\sigma)$ for a class of G_1 and G_2. To this end, we introduce the following notation. Let X and Y be normed linear spaces, and T a bounded (continuous) linear operator from X to Y. Associated with T is its adjoint T^*, a bounded linear operator from Y^* to X^*, where X^* (Y^*) is the continuous dual of X (Y). For a fixed $f \in X^*$, we set

(3.1) $$G_1(g) = \|f - T^*g\|_{X^*}$$

and

(3.2) $$G_2(g) = \|g\|_{Y^*}$$

for any $g \in Y^*$. Thus for $\sigma \geq 0$,

$$(G_1 + G_2)(\sigma) = \inf_{g \in Y^*} \|f - T^*g\|_{X^*} + \sigma \|g\|_{Y^*}. \tag{3.3}$$

We work with the dual spaces X^* and Y^*, and the adjoint operator T^* because of this next result which both implies that the above infimum is attained (for $\sigma > 0$), and also identifies the dual problem.

Theorem 3.1. *Let X^*, Y^* and T^* be as above. For $f \in X^*, f \neq 0$, and $\sigma > 0$,*

$$\sup_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f,h) = \min_{g \in Y^*} \|f - T^*g\|_{X^*} + \sigma \|g\|_{Y^*}. \tag{3.4}$$

Remark 3.1. By (f, h) we mean $f(h)$ since $f \in X^*$ and $h \in X$.

Proof: Rather than prove the theorem directly, we will use a result in Micchelli [14]. Namely,

$$\sup_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f,h) = \min_{g \in Y^*} \{ \sup_{\|h\|_X \leq 1} |(f, h) - (g, Th)| + \sigma \|g\|_{Y^*} \},$$

which actually holds in a more general setting, and is used in the theory of optimal recovery. Now,

$$\sup_{\|h\|_X \leq 1} |(f,h) - (g, Th)| = \sup_{\|h\|_X \leq 1} |(f - T^*g, h)| = \|f - T^*g\|_{X^*}$$

since $f - T^*g \in X^*$. Thus (3.4) holds. $\square$

It would be especially useful if the supremum on the left hand side of (3.4) was attained. That this is *not* so in general is easily seen, since for given $f \in X^*$ there need not exist an $h \in X$ attaining the supremum in

$$\sup_{\|h\|_X \le 1} (f, h). \tag{3.5}$$

To ensure that the supremum in (3.5) is attained, we assume that X has a pre-dual, and that f is in the pre-dual (and dual) of X. The Hahn-Banach theorem implies that the supremum in (3.5) is attained. However this assumption alone is insufficient to insure that the supremum is attained in

$$\sup_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f, h). \tag{3.6}$$

As an example, let $X = \ell_1 = c_0{}^*$, $Y = \mathbb{R}$, $\mathbf{f} = (1, 1/2, 1/3, 1/4, \dots) \in c_0$, and $T\mathbf{h} = \sum_{n=1}^{\infty} h_n$. Then it is not difficult to verify that

$$\sup_{\substack{\|\mathbf{h}\|_{\ell_1} \le 1 \\ |T\mathbf{h}| \le 1/2}} (\mathbf{f}, \mathbf{h}) = 3/4,$$

but the supremum is not attained. The reason that the supremum is not attained is that if

$$\mathbf{h}^n = (3/4, 0, \dots, 0, -1/4, 0, \dots, 0),$$

then $(\mathbf{f}, \mathbf{h}^n) = 3/4 - 1/4n \to 3/4$, $\|\mathbf{h}^n\|_{\ell_1} = 1$, $T\mathbf{h}^n = 1/2$. But while $(\mathbf{f}, \mathbf{h}^n) \to (\mathbf{f}, \mathbf{h}^0)$, where $\mathbf{h}^0 = (3/4, 0, 0, \dots)$ and $\|\mathbf{h}^0\|_{\ell_1} \le 1$, we have $T\mathbf{h}^0 > 1/2$. (Note that T is also compact.)

An additional assumption is necessary. To explain, let

$$K_\sigma = \{h \colon \|h\|_X \leq 1,\ \|Th\|_Y \leq \sigma\},$$

and $S_X = \{h \colon \|h\|_X \leq 1\}$. From the Banach-Alaoglu Theorem, S_X is weak*-compact. Since f is in pre-dual of X, this is exactly what we need. But we really need that K_σ be weak*-compact. Since $K_\sigma \subseteq S_X$, K_σ is weak*-compact if and only if K_σ is weak*-closed. Thus we must impose a condition (on T) which implies that K_σ is weak*-closed. For example, it suffices to assume that if h_n converges weak* to h_0 in S_X, then Th_n converges to Th_0 in norm (or in fact $(g,\ Th_n)$ converges to $(g,\ Th_0)$ for all $g \in Y^*$). In our examples, we will check that K_σ is weak*-closed for all $\sigma > 0$. Throughout this section, we always make this assumption. To recapitulate

Assumption I. T *is a bounded linear operator from X to Y. The element f is in the pre-dual of X, and*

$$K_\sigma = \{h \colon ||h||_X \leq 1,\ ||Th||_Y \leq \sigma\}$$

is weak-closed in X for all $\sigma > 0$.*

Assumption I implies that the supremum is attained in (3.6) and that (3.4) holds. Assumption I holds when Y has a pre-dual (recall that X is assumed to have a pre-dual) and there exists a bounded linear operator S from the pre-dual of Y to the pre-dual of X such that T is its adjoint. In this case T is continuous in the weak *-topologies. Thus if h_n converges weak * to h_0 in S_x, and $\|T h_n\|_Y \leq \sigma$ for all n, and $\|T h_n\|_Y \leq \sigma$ for all n, then $\|T h_0\|_Y \leq \sigma$. If, for example, X and Y are Hilbert spaces then this is necessarily satisfied by every bounded linear operator T from X to Y.

For ease of exposition, rather than using the notation of the previous section, we set

(3.7) $$E(\sigma) = \inf_{g \in Y^*} \|f - T^*g\|_{X^*} + \sigma\|g\|_{Y^*}$$

for $\sigma \in [0, \infty)$. For $\sigma > 0$ this infimum is attained (Theorem 3.1) and $g_\sigma \in Y^*$ will denote any function attaining this infimum. Note that the G_1 and G_2 of (3.1) and (3.2) satisfy the assumptions of the previous section. Thus, as a consequence of Proposition 2.4, we can list:

Proposition 3.2. *If Assumption I holds, then*

(i) *$E(\sigma)$ is a bounded, nondecreasing, concave, continuous function of σ on $(0, \infty)$ and $E(\sigma)/\sigma$ is nonincreasing.*

(ii) $\lim_{\sigma \to 0^+} E(\sigma) = \inf_{g \in Y^*} \|f - T^*g\|_{X^*}$. *Thus $E(\cdot)$ is continuous on $[0, \infty)$.*

(iii) *If there exists a $g_0 \in Y^*$ such that*

$$\|f - T^*g_0\|_{X^*} = \inf_{g \in Y^*} \|f - T^*g\|_{X^*},$$

then for all $0 < \sigma_1 < \sigma_2 < \infty$,

$$\frac{E(\sigma_2) - E(\sigma_1)}{\sigma_2 - \sigma_1} \leq \|g_0\|_{Y^*}.$$

(iv) *For $0 < \sigma_1 < \sigma_2 < \infty$,*

a) $$\|g_{\sigma_1}\|_{Y^*} \geq \|g_{\sigma_2}\|_{Y^*}$$

b) $$\|f - T^* g_{\sigma_1}\|_{X^*} \le \|f - T^* g_{\sigma_2}\|_{X^*}.$$

(v) For $\sigma > 0$,

$$\|g_\sigma\|_{Y^*} = \min\{\|g_\sigma + g\|_{Y^*} : g \in Y^*,\ T^* g = 0\}.$$

Remark 3.2. Statement (v) does not follow from Proposition 2.4, but is an immediate consequence of the definition of g_σ.

In the study of $E(\sigma)$ there is one very important value of σ which we denote by $\tilde{\sigma}$. The constant $\tilde{\sigma}$ is the smallest value in $[0, \infty)$ for which $E(\sigma)$ is a constant on $[\tilde{\sigma}, \infty)$. In fact $E(\sigma) = \|f\|_{X^*}$ thereon, and $\tilde{\sigma}$ always exists. We identify $\tilde{\sigma}$ as follows:

Proposition 3.3. *Assumption I holds. Then $E(\sigma) = \|f\|_{X^*}$ if and only if $\sigma \ge \tilde{\sigma}$, where*

(3.8) $$\tilde{\sigma} = \inf\left\{ \frac{\|Th\|_Y}{\|h\|_X} : h \in X,\ h \ne 0,\ (f, h) = \|f\|_{X^*} \|h\|_X \right\}.$$

Furthermore, if $\tilde{\sigma} > 0$ then the above infimum is attained.

Remark 3.3. Since, by Assumption I, f is in the pre-dual of X, there exists an $h \in X$, $h \ne 0$, satisfying $(f, h) = \|f\|_{X^*} \|h\|_X$. Thus the infimum in (3.8) is taken over a non-empty set

Proof: By Assumption I and (3.4), $E(\sigma) = \|f\|_{X^*}$ for $\sigma > 0$ if and only if

(3.9) $$\|f\|_{X^*} = \max_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f, h).$$

($\Rightarrow$). Assume $E(\sigma) = \|f\|_{X^*}$, $\sigma > 0$. Then there exists an $h \in X, h \neq 0$, which is admissible in (3.9), i.e., $\|h\|_X \leq 1$, $\|Th\|_Y \leq \sigma$, and $(f, h) = \|f\|_{X^*}$. Since

$$\|f\|_{X^*} = (f, h) \leq \|f\|_{X^*}\|h\|_X \leq \|f\|_{X^*},$$

we have $\|h\|_X = 1$. Thus

$$\frac{\|Th\|_Y}{\|h\|_X} \leq \sigma$$

which implies that $\widetilde{\sigma} \leq \sigma$.

($\Leftarrow$). Assume $\sigma > \widetilde{\sigma}$. Thus there exists an $\widetilde{h} \in X, \widetilde{h} \neq 0$, such that $(f, \widetilde{h}) = \|f\|_{X^*}\|\widetilde{h}\|_X$, and

$$\frac{\|T\widetilde{h}\|_Y}{\|\widetilde{h}\|_X} < \sigma.$$

Normalize $\widetilde{h}$ so that $\|\widetilde{h}\|_X = 1$. Thus $\|T\widetilde{h}\|_Y < \sigma$ and so we also have

$$\|f\|_{X^*} = (f, \widetilde{h}) \leq \max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f, h) = E(\sigma).$$

But $E(\sigma) \leq \|f\|_{X^*}$ for every $\sigma \geq 0$. Thus

$$E(\sigma) = \|f\|_{X^*}.$$

If $\widetilde{\sigma} > 0$, then $E(\widetilde{\sigma}) = \|f\|_{X^*}$ and from the first part of the proof of this proposition, there exists an $h \in X, h \neq 0$, satisfying $\|h\|_X = 1$, $\|Th\|_Y \leq \widetilde{\sigma}$ and $(f, h) = \|f\|_{X^*}\|h\|_X$. If $\|Th\|_Y < \widetilde{\sigma}$, we contradict our definition of $\widetilde{\sigma}$. The infimum in (3.8) is attained. □

Remark 3.4. It may happen that $\tilde{\sigma} = 0$, i.e., $E(\sigma) = ||f||_{X^*}$ for all $\sigma \geq 0$. This will occur if and only if

$$\inf_{g \in Y^*} ||f - T^*g||_{X^*} = ||f||_{X^*}.$$

Remark 3.5. An upper bound for $\tilde{\sigma}$ is $||T^*||$. For if $\sigma \geq ||T^*||$, then for any $g \in Y^*$

$$||f - T^*g||_{X^*} \geq ||f||_{X^*} - ||T^*g||_{X^*} \geq ||f||_{X^*} - ||T^*||\,||g||_{Y^*} \geq ||f||_{X^*} - \sigma||g||_{Y^*},$$

and thus $||f - T^*g||_{X^*} + \sigma||g||_{Y^*} \geq ||f||_{X^*}$ for all $g \in Y^*$.

If $f = T^*g^*$ for some $g^* \in Y^*(g^* \neq 0)$, then $E(0) = 0$. It is then also possible that for some $\sigma^* > 0$,

$$E(\sigma^*) = \sigma^*||g^*||_{Y^*}. \tag{3.10}$$

If (3.10) holds, then necessarily

$$E(\sigma) = \sigma||g^*||_{Y^*}$$

for all $0 \leq \sigma \leq \sigma^*$. This is an immediate consequence of either (iii) or (iv) (b) of Proposition 3.2. It is therefore of interest to characterize the largest possible value of σ^* for which (3.10) holds. This largest possible value we will denote by $\hat{\sigma}$. If $f \neq T^*g$ for any $g \in Y^*$, then $\hat{\sigma}$ does not exist. If $f = T^*g^*$ for some $g^* \in Y^*$, then $\hat{\sigma}$ does exist, but may equal 0 (since then $E(0) = 0$). Prior to characterizing $\hat{\sigma}$, we present a general lemma which will be used in subsequent results.

Lemma 3.4. *Assume* $\sigma > 0$.

i) If $g_\sigma \neq 0$, then there exists an $h_\sigma \in X$ satisfying $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y = \sigma$, and

$$(g_\sigma,\ Th_\sigma) = \sigma\|g_\sigma\|_{Y^*}. \tag{3.11}$$

ii) If $f \neq T^*g_\sigma$, then there exists an $h_\sigma \in X$ satisfying $\|h_\sigma\|_X = 1$, $\|Th_\sigma\|_Y \leq \sigma$, and

$$(f - T^*g_\sigma, h_\sigma) = \|f - T^*g_\sigma\|_{X^*}. \tag{3.12}$$

iii) If $g_\sigma \neq 0$ and $f \neq T^*g_\sigma$, then there exists an $h_\sigma \in X$ satisfying $\|h_\sigma\|_X = 1$, $\|Th_\sigma\|_Y = \sigma$, (3.11) and (3.12). Thus

$$\sigma = \frac{\|Th_\sigma\|_Y}{\|h_\sigma\|_X}.$$

Proof: Statement (iii) is a simple consequence of (i) and (ii), and is stated for convenience. Since $\sigma > 0$, from Assumption I and Theorem 3.1,

$$\max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f, h) = \|f - T^*g_\sigma\|_{X^*} + \sigma\|g_\sigma\|_{Y^*}.$$

Let h_σ attain the maximum on the left hand side. Then $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y \leq \sigma$, and

$$\begin{aligned}
&\|f - T^*g_\sigma\|_{X^*} + \sigma\|g_\sigma\|_{Y^*} = (f, h_\sigma) = (f - T^*g_\sigma, h_\sigma) + (T^*g_\sigma, h_\sigma) \\
&= (f - T^*g_\sigma, h_\sigma) + (g_\sigma, Th_\sigma) \leq \|f - T^*g_\sigma\|_{X^*}\|h_\sigma\|_X + \|g_\sigma\|_{Y^*}\|Th_\sigma\|_Y \\
&\leq \|f - T^*g_\sigma\|_{X^*} + \sigma\|g_\sigma\|_{Y^*}.
\end{aligned}$$

If $g_\sigma \neq 0$, i.e., $\|g_\sigma\|_{Y^*} \neq 0$, we must have

$$\sigma\|g_\sigma\|_{Y^*} = (g_\sigma, Th_\sigma) = \|g_\sigma\|_{Y^*}\|Th_\sigma\|_{Y}.$$

Thus $\|Th_\sigma\|_Y = \sigma$ and (3.11) holds.

If $f \neq T^*g_\sigma$, i.e., $\|f - T^*g_\sigma\|_{X^*} \neq 0$ we must have

$$\|f - T^*g_\sigma\|_{X^*} = (f - T^*g_\sigma, h_\sigma) = \|f - T^*g_\sigma\|_{X^*}\|h_\sigma\|_X.$$

Thus $\|h_\sigma\|_X = 1$ and (3.12) holds. □

We now present a characterization of $\hat{\sigma}$.

Proposition 3.5. $E(\sigma) = \sigma\|g^*\|_{Y^*}$ *for some* $\sigma > 0$ *and* $f = T^*g^*$ *if and only if*

$$\|g^*\|_{Y^*} = \min\{\|g^* + g\|_{Y^*}: g \in Y^*,\ T^*g = 0\} \tag{3.13}$$

and $\sigma \leq \hat{\sigma}$, *where*

$$\hat{\sigma} = \max\left\{ \frac{\|Th\|_Y}{\|h\|_X} : (g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y,\ h \neq 0 \right\}. \tag{3.14}$$

Remark 3.6. If $f = T^*g^*$ for some $g^* \in Y^*$ satisfying (3.13), but there exists no $h \in X$, $h \neq 0$, satisfying $(g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y$, then $\hat{\sigma} = 0$.

Proof: ($\Rightarrow$). Assume $E(\sigma) = \sigma\|g^*\|_{Y^*}$ for some $\sigma > 0$ where $f = T^*g^*$. That is, $g_\sigma = g^*$. From Proposition 3.2 (v), we obtain (3.13). Since $g_\sigma \neq 0$, we have from Lemma 3.4 (i), the existence of $h_\sigma \in X$ satisfying $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y = \sigma$, and

$$(g^*, Th_\sigma) = \|g^*\|_{Y^*}\|Th_\sigma\|_{Y}.$$

Thus

$$\sigma \le \frac{||Th_\sigma||_Y}{||h_\sigma||_X} \le \hat{\sigma}.$$

(By continuity, $\hat{\sigma}$ (the largest σ for which $E(\sigma) = \sigma||g^*||_{Y^*}$) exists.)

($\Leftarrow$). Assume (3.13) and (3.14) hold with $\hat{\sigma} > 0$. Let $\hat{h} \in X$ be such that $(g^*, T\hat{h}) = ||g^*||_{Y^*}||T\hat{h}||_Y$, and

$$\hat{\sigma} = \frac{||T\hat{h}||_Y}{||\hat{h}||_X}.$$

Normalize $\hat{h}$ so that $||\hat{h}||_X = 1$. Thus $||T\hat{h}||_Y = \hat{\sigma}$. Therefore

$$\hat{\sigma}||g^*||_{Y^*} \ge E(\hat{\sigma}) = \max_{\substack{||h||_X \le 1 \\ ||Th||_Y \le \hat{\sigma}}} (f, h) = \max_{\substack{||h||_X \le 1 \\ ||Th||_Y \le \hat{\sigma}}} (T^*g^*, h)$$
$$\ge (T^*g^*, \hat{h}) = (g^*, T\hat{h}) = \hat{\sigma}||g^*||_{Y^*}.$$

Thus $E(\hat{\sigma}) = \hat{\sigma}||g^*||_{Y^*}$. From Proposition 3.2, (iii) or (iv) (b), we get $E(\sigma) = \sigma||g^*||_{Y^*}$ for all $\sigma \in [0, \hat{\sigma}]$. □

We have identified $E(\sigma)$ for $\sigma \ge \tilde{\sigma}$ and for $\sigma \le \hat{\sigma}$, if $\hat{\sigma} > 0$ exists. There is, in general, no formula which explicitly gives $E(\sigma)$ for $\sigma \in (\hat{\sigma}, \tilde{\sigma})$ (or $\sigma \in (0, \tilde{\sigma})$ if $\hat{\sigma}$ does not exist). However, we do have:

Proposition 3.6. *Assume $\sigma < \tilde{\sigma}$ and $\sigma > \hat{\sigma}$, if $\hat{\sigma}$ exists, $\sigma > 0$. Then $g_\sigma \in Y^*$ is a solution to $E(\sigma)$ if and only if $g_\sigma \ne 0$, $f \ne T^*g_\sigma$, and there exists an $h_\sigma \in X$ with $Th_\sigma \ne 0$ satisfying*

(3.15) $$(f - T^*g_\sigma, h_\sigma) = ||f - T^*g_\sigma||_{X^*}||h_\sigma||_X$$

(3.16) $$(g_\sigma, Th_\sigma) = ||g_\sigma||_{Y*}||Th_\sigma||_Y.$$

Furthermore,

(3.17) $$\sigma = \frac{||Th_\sigma||_Y}{||h_\sigma||_X}.$$

Proof: ($\Rightarrow$). Since $\sigma < \tilde{\sigma}$, $g_\sigma \neq 0$. Since $\sigma > \hat{\sigma}$ (if $\hat{\sigma}$ exists) and $\sigma > 0$, $f \neq T^* g_\sigma$. Thus this direction is simply a restatement of Lemma 3.4, (iii).

($\Leftarrow$). Assume h_σ and g_σ are as above satisfying (3.15), (3.16) and (3.17). Normalize h_σ so that $||h_\sigma||_X = 1$. Thus $||Th_\sigma||_Y = \sigma$. Now

$$E(\sigma) = \max_{\substack{||h||_X \leq 1 \\ ||Th||_Y \leq \sigma}} (f, h) \geq (f, h_\sigma) = (f - T^* g_\sigma, h_\sigma) + (g_\sigma, Th_\sigma)$$
$$= ||f - T^* g_\sigma||_{X*} + \sigma ||g_\sigma||_{Y*}.$$

Thus g_σ is a solution to $E(\sigma)$. □

Remark 3.7. The converse direction does not imply that $\hat{\sigma} < \sigma < \tilde{\sigma}$. However, from Proposition 3.2, (iv), (a) and (b), it follows that if (3.15), (3.16) and (3.17) hold for $g_\sigma \neq 0$ and $f \neq T^* g_\sigma$, then $\hat{\sigma} \leq \sigma \leq \tilde{\sigma}$.

Proposition 3.4 delineates the infinite interval on which $E(\sigma)$ is a constant. Proposition 3.5. (if $\hat{\sigma} > 0$ exists) gives us an interval on which $E(\sigma)$ is linear. Is it true that $E(\sigma)$ is strictly concave on the complement of these two intervals? Not necessarily. However, we do give one condition implying the strict concavity thereon.

Proposition 3.5. *Let $I = (\hat{\sigma}, \tilde{\sigma})$ if $\hat{\sigma}$ exists, and $I = (0, \tilde{\sigma})$ otherwise. If X is strictly convex, then $E(\sigma)$ is strictly concave on I.*

Proof: If $E(\sigma)$ is not strictly concave on I, then $E(\sigma)$ is linear on some subinterval $[\sigma_1, \sigma_2]$ of I ($\sigma_1 < \sigma_2$). For $\lambda \in [0, 1]$, set $\sigma_\lambda = \lambda\sigma_1 + (1-\lambda)\sigma_2$. Then

$$E(\sigma_\lambda) = \lambda E(\sigma_1) + (1-\lambda)E(\sigma_2)$$

for $\lambda \in [0, 1]$. From the definition of $E(\sigma)$, it follows that if $g_{\sigma_\lambda} \in Y^*$ is a solution to $E(\sigma_\lambda)$ for some $\lambda \in (0,1)$, then it is also a solution to $E(\sigma_1)$ and $E(\sigma_2)$ and thus to $E(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_2]$. Set $g_{\sigma_\lambda} = \hat{g}$. Thus

$$E(\sigma) = ||f - T^*\hat{g}||_{X^*} + \sigma||\hat{g}||_{Y^*}$$

for all $\sigma \in [\sigma_1, \sigma_2]$.

Now $\hat{g} \neq 0$ and $f \neq T^*\hat{g}$ since $\hat{g} = g_\sigma$ for $\sigma \in [\sigma_1, \sigma_2]$. From Proposition 3.6 there exists for each $\sigma \in [\sigma_1, \sigma_2]$ an $h_\sigma \in X$, $Th_\sigma \neq 0$, satisfying (3.15), (3.16) and (3.17). Normalize h_σ so that $||h_\sigma||_X = 1$. Since we also have $||Th_\sigma||_Y = \sigma$, this implies that $h_{\sigma_1} \neq \alpha h_{\sigma_2}$ for any $\alpha \in \mathbb{R}$. Let $\lambda \in (0, 1)$, and

$$\hat{h}_{\sigma_\lambda} = \lambda h_{\sigma_1} + (1-\lambda)h_{\sigma_2}.$$

Then $||T\hat{h}_{\sigma_\lambda}||_Y \leq \sigma_\lambda$, and since X is strictly convex $||\hat{h}_{\sigma_\lambda}||_X < 1$.

Thus

$$E(\sigma_\lambda) = \max_{\substack{||h||_X \leq 1 \\ ||Th||_Y \leq \sigma}} (f, h) \geq (f, \hat{h}_{\sigma_\lambda}) = \lambda(f, h_{\sigma_1}) + (1-\lambda)(f, h_{\sigma_2})$$
$$= \lambda E(\sigma_1) + (1-\lambda)E(\sigma_2) = E(\sigma_\lambda).$$

Since equality holds,

$$E(\sigma_\lambda) = (f, \hat{h}_{\sigma_\lambda}).$$

But

$$(f, \hat{h}_{\sigma_\lambda}) = (f - T^*\hat{g}, \hat{h}_{\sigma_\lambda}) + (\hat{g}, T\hat{h}_{\sigma_\lambda}) \le ||f - T^*\hat{g}||_{X^*}||\hat{h}_{\sigma_\lambda}||_X + ||\hat{g}||_{Y^*}||T\hat{h}_{\sigma_\lambda}||_Y$$
$$< ||f - T^*\hat{g}||_{X^*} + \sigma_\lambda||\hat{g}||_{Y^*} = E(\sigma_\lambda).$$

This contradiction implies the proposition. □

Remark 3.8. If T is one-to-one and Y is strictly convex, then this same result also holds.

Let us now return to the case where $f = T^*g^*$ for some $g^* \in Y^*$ satisfying (3.13). In this case we have the following inequalities.

Lemma 3.8. *If $f = T^*g^*$, $g^* \in Y^*$ satisfying (3.13), $f \neq 0$, then*

$$\hat{\sigma} \le \frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}} \le \tilde{\sigma}.$$

Proof: By (3.8),

$$\tilde{\sigma} = \inf\left\{ \frac{||Th||_Y}{||h||_X} : h \in X,\ h \neq 0,\ (f, h) = ||f||_{X^*}||h||_X \right\}.$$

Let $h \in X$, $h \neq 0$, satisfy

$$(f, h) = ||f||_{X^*}||h||_X.$$

Such h exist since f is in the pre-dual of X. By definition, $f = T^*g^*$. Thus

$$(T^*g^*, h) = ||T^*g^*||_{X^*}||h||_X.$$

Furthermore,

$$(T^*g^*, h) = (g^*, Th) \leq ||g^*||_{Y^*}||Th||_Y.$$

Therefore

$$\frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}} \leq \frac{||Th||_Y}{||h||_X}$$

implying that

$$\frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}} \leq \tilde{\sigma}.$$

If $\hat{\sigma} = 0$, there is nothing to prove. Assume $\hat{\sigma} > 0$. Then from (3.14),

$$\hat{\sigma} = \max\left\{ \frac{||Th||_Y}{||h||_X} : (g^*, Th) = ||g^*||_{Y^*}||Th||_Y,\ h \neq 0 \right\}.$$

If $h \in X$, $h \neq 0$, and

$$(g^*, Th) = ||g^*||_{Y^*}||Th||_Y,$$

then since

$$(g^*, Th) = (T^*g^*, h) \leq ||T^*g^*||_{X^*}||h||_{X},$$

we have

$$\frac{||Th||_Y}{||h||_X} \leq \frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}}.$$

Thus,

$$\hat{\sigma} \leq \frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}}. \qquad \square$$

As a simple consequence of Lemma 3.8, and the analysis thereof, we can exactly delineate when $\tilde{\sigma} = \hat{\sigma}$. Note that from Lemma 3.8, when $f = T^*g^*$, $f \neq 0$, we must have $\tilde{\sigma} > 0$.

Proposition 3.9. *Let $f = T^*g^*$, $g^* \in Y^*$ satisfying (3.13), $f \neq 0$. Then $\tilde{\sigma} = \hat{\sigma}\ (> 0)$ if and only if there exists an $h^* \in X$, $h^* \neq 0$, satisfying*

$$||g^*||_{Y^*}||Th^*||_Y = (g^*, Th^*) = (T^*g^*, h^*) = ||T^*g^*||_{X^*}||h^*||_X. \tag{3.18}$$

In this case

$$\tilde{\sigma} = \hat{\sigma} = \frac{||T^*g^*||_{X^*}}{||g^*||_{Y^*}}. \tag{3.19}$$

Proof: $(\Rightarrow)$. If $\tilde{\sigma} = \hat{\sigma}$, then from Lemma 3.8, (3.19) holds. Let $h^* \in X$, $h^* \neq 0$, satisfy (3.8) for $\tilde{\sigma}(> 0)$. Then

(3.20) $$\frac{\|Th^*\|_Y}{\|h^*\|_X} = \widetilde{\sigma} = \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}}$$

and $(T^*g^*, h^*) = \|T^*g^*\|_{X^*}\|h^*\|_X$. Since $(T^*g^*, h^*) = (g^*, Th^*)$, and $\|T^*g^*\|_{X^*}\|h^*\|_X = \|Th^*\|_Y\|g^*\|_{Y^*}$ by (3.20), we obtain (3.18).

($\Leftarrow$). Assume $h^* \in X$, $h^* \neq 0$, satisfies (3.18). Then from Propositions 3.3 and 3.5 (with the h^* of (3.18)),

$$\widetilde{\sigma} \leq \frac{\|Th^*\|_Y}{\|h^*\|_X} \leq \hat{\sigma}.$$

However $\hat{\sigma} \leq \widetilde{\sigma}$ by definition. Thus $\widetilde{\sigma} = \hat{\sigma}$, and (3.19) is a consequence of Lemma 3.8. □

Remark 3.9. The content of Proposition 3.9 is not vacuous. There do exist nontrivial T, X, Y and f for which $\widetilde{\sigma} = \hat{\sigma}$. A general example of such a situation is given in the next section.

What about the uniqueness of the solution g_σ to $E(\sigma)$? In general we would not expect uniqueness unless we put restrictions on the norms. However, even in the best of cases we do not always have uniqueness. In the next section we give an example in the Hilbert space setting where uniqueness does not hold for $\sigma = \widetilde{\sigma}$.

Proposition 3.10. *If X^* and Y^* are strictly convex, then for $\sigma \neq \widetilde{\sigma}$, the solution g_σ for $E(\sigma)$ is unique.*

Proof: *Case 1.* $\sigma > \widetilde{\sigma}$.

From Proposition 3.2, (iv) (a), if $0 < \sigma_1 < \sigma_2 < \infty$ and g_{σ_i} is any solution for $E(\sigma_i)$, $i = 1, 2$, then $\|g_{\sigma_1}\|_{Y^*} \geq \|g_{\sigma_2}\|_{Y^*}$. Since $g_{\widetilde{\sigma}} = 0$ is a solution for $E(\widetilde{\sigma})$, it follows that $g_\sigma = 0$ is the only solution for $E(\sigma)$ for $\sigma > \widetilde{\sigma}$.

Case 2. $0 < \sigma < \widetilde{\sigma}$.

Assume $0 < \sigma < \tilde{\sigma}$ and g_1, g_2 are two distinct solutions for $E(\sigma)$ with $g_1 \neq g_2$. Since $\sigma < \tilde{\sigma}$, we have $g_1 \neq 0$ and $g_2 \neq 0$. It is easy to see that the solution set of $E(\sigma)$ is convex. In fact, if $\lambda \in [0, 1]$, and $g_\lambda = \lambda g_1 + (1 - \lambda)g_2$, then

$$\|f - T^*g_\lambda\|_{X^*} = \lambda\|f - T^*g_1\|_{X^*} + (1 - \lambda)\|f - T^*g_2\|_{X^*} \tag{3.21}$$

$$\|g_\lambda\|_{Y^*} = \lambda\|g_1\|_{Y^*} + (1 - \lambda)\|g_2\|_{Y^*}. \tag{3.22}$$

Since Y^* is strictly convex, we have from (3.22) that

$$g_1 = ag_2, \qquad a > 0.$$

If $a = 1$ we are finished. Thus

$$g_1 = ag_2, \qquad a > 0,\ a \neq 1. \tag{3.23}$$

If $T^*g_1 = T^*g_2$, then a contradiction ensues from (3.23) (since $g_1 \neq 0$, $g_2 \neq 0$) unless $T^*g_1 = T^*g_2 = 0$. But in this latter case we must have $E(\sigma) = \|f\|_{X^*}$ and thus $\sigma \geq \tilde{\sigma}$, a contradiction. Since $T^*g_1 \neq T^*g_2$, by a judicious choice of $\lambda \in (0, 1)$ we can assume that $f \neq T^*g_1$ and $f \neq T^*g_2$. Since X^* is strictly convex, we have from (3.22) that

$$f - T^*g_1 = b(f - T^*g_2), \quad b > 0.$$

If $b = 1$, then again $T^*g_1 = T^*g_2$. Thus

$$f - T^*g_1 = b(f - T^*g_2), \quad b > 0,\ b \neq 1. \tag{3.24}$$

Now, from (3.23) and (3.24)

$$||f - T^*g_2||_{X^*} + \sigma||g_2||_{Y^*} = ||f - T^*g_1||_{X^*} + \sigma||g_1||_{Y^*} = b||f - T^*g_2||_{X^*} + \sigma a||g_2||_{Y^*}.$$

Thus

$$(1 - b)\ ||f - T^*g_2||_{X^*} = \sigma(a - 1)\ ||g_2||_{Y^*}$$

with $a, b > 0,\ a \neq 1, b \neq 1$. Furthermore, from (3.24)

$$(1 - b)f = T^*g_1 - bT^*g_2 = (a - b)T^*g_2.$$

Thus,

$$\sigma\frac{(a-1)}{(1-b)}\ ||g_2||_{Y^*} = ||f - T^*g_2||_{X^*} = ||\left(\frac{a-b}{1-b}\right)T^*g_2 - T^*g_2||_{X^*} = ||\left(\frac{a-1}{1-b}\right)T^*g_2||_{X^*},$$

which implies that

$$\sigma||g_2||_{Y^*} = ||T^*g_2||_{X^*}.$$

Since $\sigma < \widetilde{\sigma}$, we have $E(\sigma) < ||f||_{X^*}$. Therefore,

$$||f||_{X^*} > E(\sigma) = ||f - T^*g_2||_{X^*} + \sigma||g_2||_{Y^*} \geq ||f||_{X^*} - ||T^*g_2||_{X^*} + \sigma||g_1||_{Y^*} = ||f||_{X^*}.$$

This contradiction proves the proposition. □

Remark 3.10. For $\sigma > \widetilde{\sigma}$, no assumptions on X^* or Y^* are needed to prove that $g_\sigma = 0$ is the unique solution for $E(\sigma)$. If $f = T^*g^*,\ g^* \in Y^*$ satisfies (3.13), $f \neq 0$, and $\hat{\sigma} > 0$, then using Proposition 3.2, (iv) (b), it follows that $g_\sigma = g^*$ is the unique solution for $E(\sigma)$ for $\sigma < \hat{\sigma}$ if T^* is $1-1$ or Y^* is strictly convex.

A different method of finding a solution g_σ to $E(\sigma)$ is via the more classical variational or perturbation approach. We recall that for f, g in a normed linear space W,

$$\tau_+^W(f, g) = \lim_{t \to 0^+} \frac{\|f + tg\|_W - \|f\|_W}{t}$$

always exists. This is a consequence of the fact that the quantity

$$\frac{\|f + tg\|_W - \|f\|_W}{t}$$

is a nondecreasing function of t bounded below on $(0, \infty)$.

Thus it easily follows that g_σ is a solution to $E(\sigma)$ if and only if

$$\tau_+^{X^*}(f - T^* g_\sigma, -T^* g) + \sigma\, \tau_+^{Y^*}(g_\sigma, g) \geq 0$$

for all $g \in Y^*$.

Furthermore $\sigma > \tilde{\sigma}$ if and only if $g_\sigma = 0$ is a solution (and the only solution) to $E(\sigma)$. Thus $\tilde{\sigma}$ is the infimum of σ for which

$$\tau_+^{X^*}(f, -T^* g) + \sigma \|g\|_{Y^*} \geq 0$$

for all $g \in Y^*$ (since $\tau_+^{Y^*}(0, g) = \|g\|_{Y^*}$). A similar analysis gives us a characterization of $\hat{\sigma}$. Stating formally, we have:

Proposition 3.11. *In the above notation,*

$$\tilde{\sigma} = \sup \left\{ \frac{-\tau_+^{X^*}(f, -T^*g)}{\|g\|_{Y^*}} : g \in Y^*, \quad g \neq 0 \right\},$$

and if $f = T^* g^*$ for some g^* satisfying (3.13), *then*

$$\tilde{\sigma} = \inf \left\{ \frac{-\|T^* g\|_X^*}{\tau_+^{Y^*}(g^*, g)} : g \in Y^*, \ \tau_+^{Y^*}(g^*, g) < 0 \right\}.$$

4. Hilbert Space: An Example

Let X and Y be Hilbert spaces, and T a compact operator from X to Y. Note that T automatically satisfies Assumption I. From Proposition 3.3, it immediately follows that $\tilde{\sigma} = \|Tf\|_Y / \|f\|_X$. The identification of $\hat{\sigma}$ is not as simple. However, it is possible to identify exactly when $\tilde{\sigma} = \hat{\sigma}$. To this end, we recall that the operator TT^* from $Y(= Y^*)$ into itself is compact, self-adjoint, and nonnegative. It has eigenvectors and eigenvalues (which are nonnegative and whose square roots are called the singular values or s-numbers of TT^*).

Proposition 4.1. *Let $f \in X$, $f \neq 0$. Then under the above assumptions, $\tilde{\sigma} = \hat{\sigma}(> 0)$ if and only if $f = T^*g$ where*

$$TT^* g^* = \lambda g^*$$

for some $\lambda > 0$. In this case $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$.

Remark 4.1. Note that in this case both the functions 0 and g^* are solutions to $E(\sigma)$ for $\sigma = \tilde{\sigma} = \hat{\sigma}$. The solution set is convex and thus αg^* is a solution for all $\alpha \in [0, 1]$ (see Proposition 3.10).

Proof. ($\Rightarrow$). Assume $\tilde{\sigma} = \hat{\sigma}$ and $f = T^* g^*$, $g^* \in Y$ satisfying (3.13). From Proposition 3.9 there exists and $h^* \in X$, $h^* \neq 0$, satisfying (3.18). The equalities in (3.18) imply that $Th^* = \alpha g^*$ and $T^* g^* = \beta h^*$ for some $\alpha, \beta > 0$. Thus

$$TT^* g^* = \lambda g^*$$

where $\lambda = \alpha\beta > 0$. From (3.19) it easily follows that $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$.

($\Leftarrow$). Assume $f = T^* g^*$, and $TT^* g^* = \lambda g^*$ for some $\lambda > 0$. Set $h^* = T^* g^* = f$. Then,

$$(h^*, h^*) = (T^* g^*,\ h^*) = (g^*, Th^*) = (g^*,\ TT^* g^*) = \lambda(g^*, g^*).$$

Furthermore,

$$\|g^*\|_Y \|Th^*\|_Y = \|g^*\|_Y \|TT^* g^*\|_Y = \lambda \|g^*\|_Y^2 = \lambda\,(g^*, g^*),$$

while

$$\|T^* g^*\|_X \|h^*\|_X = \|h^*\|_X \|h^*\|_X = \|h^*\|_X^2 = (h^*, h).$$

Thus (3.18) holds, and from (3.19) $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$. □

Let us now assume that $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, where we define $\hat{\sigma} = 0$ if $\hat{\sigma}$ does not exist. Let us now attempt to determine g_σ.

From Proposition 3.6, there exists an $h_\sigma \in X$, $Th_\sigma \neq 0$, satisfying (3.15), (3.16) and (3.17). From (3.15)

$$\alpha\, h_\sigma = f - T^* g_\sigma$$

for some $\alpha > 0$ (since $f \neq T^* g_\sigma$), and from (3.16)

$$\beta\, g_\sigma = T h_\sigma$$

for some $\beta > 0$ (since $g_\sigma \neq 0$). Thus it follows that

$$T(f - T^* g_\sigma) = \lambda(\sigma) g_\sigma, \tag{4.1}$$

which we can rewrite as

$$Tf = (TT^* + \lambda(\sigma) I) g_\sigma. \tag{4.2}$$

From (4.2) and (3.17), it may be shown that

$$\lambda(\sigma) = \sigma \frac{\|f - T^* g_\sigma\|_X}{\|g_\sigma\|_Y}. \tag{4.3}$$

If (4.1) and (4.3) hold, then so do (3.15), (3.16) and (3.17) for $h_\sigma = f - T^* g_\sigma$. From Proposition 3.2, *(iv)(a)*, $\|g_\sigma\|_Y$ is a nonincreasing function of σ, and from Proposition 3.2, *(iv)(b)*, $\|f - T^* g_\sigma\|_X$ is a nondecreasing function of σ. Thus for $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, the function $\lambda(\sigma)$ is a strictly increasing function of σ. Furthermore, it is not difficult to ascertain that $\lambda(\sigma)$ is also a continuous function of σ in $(\hat{\sigma}, \tilde{\sigma})$, and

$$\lim_{\sigma \uparrow \tilde{\sigma}} \lambda(\sigma) = \infty$$

(since $\lim_{\sigma \uparrow \tilde{\sigma}} \|g_\sigma\|_Y = 0$,) while

$$\lim_{\sigma \downarrow \hat{\sigma}} \lambda(\sigma) = 0.$$

Thus for all $\lambda \in (0, \infty)$ there is a one-to-one correspondence between solutions of (4.2), (4.3) and g_σ solving $E(\sigma)$.

This may also be seen from the following. Since TT^* is compact, self-adjoint and nonnegative, the operator $TT^* + \lambda I$ is invertible for all $\lambda > 0$. From (4.2), for $\lambda > 0$, set

$$\hat{g}_\lambda = (TT^* + \lambda I)^{-1}\, Tf.$$

Then $\hat{g}_\lambda$ is a solution to $E(\sigma)$, i.e., $\hat{g}_\lambda = g_\sigma$, for

$$\sigma = \frac{\lambda \|\hat{g}_\lambda\|_Y}{\|f - T^* \hat{g}_\lambda\|_X}.$$

A more explicit representation for g_σ can be given in the case where $f = T^* g^*, g^* \in Y$, $g^* \neq 0$. Assume $TT^* g^* \neq \lambda g^*$ for any $\lambda > 0$. Thus $\hat{\sigma} < \tilde{\sigma}$. Let $\{\lambda_i\}_{i=1}$ denote the nonzero eigenvalues of TT^* listed to their algebraic multiplicity. Let $\{g_i\}_{i=1}$ denote a corresponding set of orthonormal eigenvectors. That is,

$$TT^* g_i = \lambda_i g_i.$$

Since the $\{g_i\}_{i=1}$ are complete in the range of TT^*, and since we may assume that g^* satisfies (3.13), it then follows that

$$g^* = \sum_{i=1} \alpha_i g_i$$

for some $\{\alpha_i\}_{i=1}$ in ℓ_2.

For $\hat{\sigma} < \sigma < \tilde{\sigma}$, it follows from (4.2), or using the results of the previous section, that

$$g_\sigma = \sum_{i=1} \left(\frac{\lambda_i \alpha_i}{\lambda_i + \lambda} \right) g_i,$$

where

$$\sigma = \frac{\left(\sum_{i=1} \frac{\lambda_i^2 \alpha_i^2}{(\lambda_i + \lambda)^2} \right)^{1/2}}{\left(\sum_{i=1} \frac{\lambda_i \alpha_i^2}{(\lambda_i + \lambda)^2} \right)^{1/2}}.$$

Note that as $\lambda \uparrow \infty$, the above quantity tends to

$$\tilde{\sigma} = \frac{(\Sigma_{i=1} \lambda_i^2 \alpha_i^2)^{1/2}}{(\Sigma_{i=1} \lambda_i \alpha_i^2)^{1/2}}$$

while as $\lambda \downarrow 0$, this quantity tends to

$$\hat{\sigma} = \frac{(\Sigma_{i=1} \alpha_i^2)^{1/2}}{(\Sigma_{i=1} \alpha_i^2/\lambda_i)^{1/2}},$$

where if the denominator does not exist (diverges), then we understand it to mean that $\hat{\sigma} = 0$.

As examples, we consider two concrete cases. We will return in later sections to similar types of examples.

For simplicity of presentation, we let L^2 denote the usual Lebesgue space $L^2[0, 1]$, and ℓ_2^n the euclidean space in $\mathbb{R}^n$.

A.

$$E(\sigma) = \min_{g \in \ell_2^n} \|f - T^*(g)\|_{L^2}^2 + \sigma\|g\|_{\ell_2^n}.$$

Since T^* is, by assumption, a bounded linear operator from ℓ_2^n to L^2, it is necessarily of the form

$$T^*(g) = \sum_{i=1}^{n} g_i u_i$$

for some $u_1, \ldots, u_n \in L^2$, and

$$T(f) = \begin{pmatrix} \int f u_1 \\ \vdots \\ \int f u_n \end{pmatrix}.$$

Thus

$$TT^*(g) = C\,g$$

where C is the $n \times n$ Grammian matrix given by $C = (c_{ij})$, $c_{ij} = \int u_i u_j$, $i, j = 1, \ldots, n$. A simple calculation shows that for $\lambda > 0$,

$$\hat{g}_\lambda = (TT^* + \lambda I)^{-1}\, T(f) = (C + \lambda I)^{-1} \begin{pmatrix} \int f u_1 \\ \vdots \\ \int f u_n \end{pmatrix}.$$

B. In this case we are given $\mathbf{f} \in \ell_2^n$,

$$E(\sigma) = \min_{\mathbf{g} \in L^2} \|\mathbf{f} - T^*(\mathbf{g})\|_{\ell_2^n} + \sigma \|\mathbf{g}\|_{L^2},$$

and here

$$T^*(\mathbf{g}) = \begin{pmatrix} \int g\, u_1 \\ \vdots \\ \int g\, u_n \end{pmatrix}$$

where the $u_1, \ldots, u_n$ are in L^2. Thus

$$T(\mathbf{f}) = \sum_{i=1}^{n} f_i\, u_i,$$

and

$$TT^*(g) = \sum_{i=1}^{n} \left(\int g\, u_i \right) u_i.$$

For $\lambda > 0$

$$\hat{g}_\lambda = (TT^* + \lambda I)^{-1} T(\mathbf{f}) = \sum_{i=1}^{n} b_i u_i$$

where

$$\mathbf{b} = (C + \lambda I)^{-1} \mathbf{f}$$

and C is as in (Λ).

If we assume that the $u_1, \ldots, u_n$ are linearly independent, then $f = T^* g^*$ for some g^* satisfying (3.13). This g^* is necessarily of the form $g^* = \Sigma_{i=1}^{n} g^*{}_i u_i$ where $\mathbf{g}^* = (g^*{}_1, \ldots, g^*{}_n)$ is given by $g^* = C^{-1} \mathbf{f}$. C^{-1} exists since the $u_1, \ldots, u_n$ are linearly independent. In this case

$$\hat{\sigma} = \frac{\|g^*\|_{L^2}}{\|\mathbf{g}^*\|_{\ell_2^n}} .$$

If the $u_1, \ldots, u_n$ are the orthonormal, then $\tilde{\sigma} = \hat{\sigma}$ for all $\mathbf{f}$.

5. Diagonal Matrices: An Example

In this section we will explicitly determine solutions in one of the easier problems, namely

$$E_{pq}(\sigma) = \min_{\mathbf{g}} \|\mathbf{f} - D\mathbf{g}\|_p + \sigma \|\mathbf{g}\|_q \tag{5.1}$$

where $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$, $p, q \in [1, \infty]$, $\sigma \geq 0$, and D is an $n \times n$ diagonal matrix given by $D = \text{diag}\{d_1, \ldots, d_n\}$. By $\| \bullet \|_p$ and $\| \bullet \|_q$ we mean the usual ℓ_p^n and ℓ_q^n norms, respectively.

For any $\sigma \geq 0$ the minimum in $E_{pq}(\sigma)$ is easily seen to be attained by a $\mathbf{g}^\sigma$ such that

$$0 \leq | d_i g_i^{\sigma} | \leq |f_i|$$

and

$$f_i(d_i g_i^{\sigma}) \geq 0.$$

Furthermore, if $d_i = 0$ then we can always take $\mathbf{g}^{\sigma}$ such that $g_i^{\sigma} = 0$. As such we may assume, with no loss of generality, that

a) $$f_i > 0, \quad i = 1, \ldots, n$$

b) $$d_i > 0, \quad i = 1, \ldots, n.$$

We will use the dual problem to (5.1) in our construction of the solutions. The dual problem is:

(5.2) $$\max_{\substack{||\mathbf{h}||_{p'} \leq 1 \\ ||D\mathbf{h}||_{q'} \leq \sigma}} (\mathbf{f}, \mathbf{h})$$

where $1/p + 1/p' = 1/q + 1/q' = 1$, and $(\mathbf{f}, \mathbf{h}) = \sum_{l=1}^{n} f_l h_l$.

Remark 5.1. Before embarking on the analysis we note that essentially the same analysis will hold for the problem

$$\min_{g \in L^q} ||f - dg||_p + \sigma ||g||_q$$

where $f,\ d \in C(\Omega)$, Ω is a set of finite measure, d is strictly positive, and $\|\bullet\|_p$, $\|\bullet\|_q$ are the usual L^p and L^q norms on Ω. Such an analysis in the case $d \equiv 1$ was essentially carried out in Nilsson, Peetre [ ].

Determination of $\tilde{\sigma}$.

I. (i). $1 < p < \infty,\ 1 \le q \le \infty$

$$\tilde{\sigma} = \min \left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_{p'}} \ :\ (\mathbf{f}, \mathbf{h}) = ||\mathbf{f}||_p ||\mathbf{h}||_{p'} \right\}.$$

Since $1 < p < \infty$, $h_i = \alpha f_i^{p-1}$, $i = 1, \dots, n$. Thus

$$\tilde{\sigma} = \frac{||D\mathbf{f}^{p-1}||_{q'}}{||\mathbf{f}||_p^{p-1}}$$

where $\mathbf{f}^{p-1} = (f_1^{p-1}, \dots, f_n^{p-1})$.

I. (ii). $p = 1,\ 1 \le q \le \infty$.

$$\tilde{\sigma} = \min \left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_{\infty}} \ :\ (\mathbf{f}, \mathbf{h}) = ||\mathbf{f}||_1\ ||\mathbf{h}||_{\infty} \right\}.$$

Since

$$(\mathbf{f}, \mathbf{h}) = ||\mathbf{f}||_1\ ||\mathbf{h}||_{\infty}$$

(and $f_i > 0,\ i = 1, \dots, n$), $h_i = \alpha > 0,\ i = 1, \dots, n$. Thus with $\mathbf{e} = (1,\ 1, \dots, 1)$

$$\tilde{\sigma} = ||D\mathbf{e}||_{q'} = \left(\sum_{i=1}^{n} d_i^{q'} \right)^{1/q'},$$

independent of $\mathbf{f}$. If $q = 1$, we mean by the above that $\tilde{\sigma} = \max d_i$.

I. (iii). $p = \infty, \ 1 \le q \le \infty$.

$$\tilde{\sigma} = \min \left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_1} : (\mathbf{f}, \mathbf{h}) = ||\mathbf{f}||_\infty ||\mathbf{h}||_1 \right\}.$$

Let $J = \{j: f_j = ||\mathbf{f}||_\infty\}$. Thus

$$\tilde{\sigma} = \min \left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_1} : h_i \ge 0 \text{ for all } i, \text{ and } h_i = 0, \ i \notin J \right\}$$
$$= \frac{1}{\left(\sum_{j \in J} \left(\frac{1}{d_j} \right)^q \right)^{1/q}}.$$

If $q = \infty$, then $\tilde{\sigma} = \min_{j \in J} d_j$.

Determination of $\hat{\sigma}$.

II. (i). $1 \le p \le \infty, \ 1 < q < \infty$.

$$\hat{\sigma} = \max \left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_{p'}} : (\mathbf{g}^*, D\mathbf{h}) = ||\mathbf{g}^*||_q ||D\mathbf{h}||_{q'} \right\}$$

where $D\mathbf{g}^* = \mathbf{f}$, i.e. $g_i^* = f_i / d_i$. This may be rewritten as

$$\hat{\sigma} = \max\left\{ \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_{p'}} : (\mathbf{f}, \mathbf{h}) = ||D^{-1}\mathbf{f}||_q ||D\mathbf{h}||_{q'} \right\}.$$

Since $1 < q < \infty$,

$$d_i h_i = \alpha \left(\frac{f_i}{d_i} \right)^{q-1}, \quad i = 1, \dots, n, \ \alpha > 0.$$

Thus

$$\hat{\sigma} = \frac{\left(\sum_{i=1}^{n} \left(\frac{f_i}{d_i} \right)^q \right)^{1/q'}}{\left(\sum_{i=1}^{n} \left(\frac{1}{d_i} \left(\frac{f_i}{d_i} \right)^{q-1} \right)^{p'} \right)^{1/p'}}$$

II. (ii). $1 \le p \le \infty, \ q = 1.$

$$\hat{\sigma} = \max\left\{ \frac{||D\mathbf{h}||_{\infty}}{||\mathbf{h}||_{p'}} : (\mathbf{f}, \mathbf{h}) = ||D^{-1}\mathbf{f}||_1 ||D\mathbf{h}||_{\infty} \right\}.$$

Since $f_i/d_i > 0, \ i = 1, \dots, n$, we must have $d_i h_i = \alpha, \ i = 1, \dots, n$. Thus

$$\hat{\sigma} = \frac{1}{\left(\sum_{i=1}^{n} \left(\frac{1}{d_i} \right)^{p'} \right)^{1/p'}} .$$

If $p = 1$, then $\hat{\alpha} = \min_{i=1,\dots,n} d_i$.

II. (iii). $1 \leq p \leq \infty,\ q = \infty$.

$$\hat{\sigma} = \max\left\{ \frac{||D\mathbf{h}||_1}{||\mathbf{h}||_{p'}} : (\mathbf{f}, \mathbf{h}) = ||D^{-1}\mathbf{f}||_\infty ||D\mathbf{h}||_1 \right\}.$$

Let

$$\tilde{J} = \left\{ j: \left(\frac{f_j}{d_j} \right) = \max_{i=1\dots n} \left(\frac{f_i}{d_i} \right) \right\}.$$

Since

$$(\mathbf{f}, \mathbf{h}) = ||D^{-1}\mathbf{f}||_\infty \, ||D\mathbf{h}||_1,$$

we must have $h_j = 0$ if $j \notin \tilde{J}$ (and $h_i \geq 0$ for all i). Thus

$$\hat{\sigma} = \max\left\{ \frac{||D\mathbf{h}||_1}{||\mathbf{h}||_{p'}} : h_i \geq 0 \text{ for all } i, \text{ and } h_i = 0, \quad i \notin \tilde{J} \right\}$$

$$= \left(\sum_{j \in \tilde{J}} d_j^p \right)^{1/p}.$$

(where for $p = \infty$, $\hat{\sigma} = \max_{j \in \tilde{J}} d_j$.)

It is now easy to calculate when $\tilde{\sigma} = \hat{\sigma}$. From the above we obtain the following necessary and sufficient conditions for $\tilde{\sigma} = \hat{\sigma}$.

(a) $1 \le p < \infty,\ 1 \le q < \infty,$

$$d_i^q f_i^{p-q} = c > 0, \quad i = 1, \dots, n$$

(b) $1 \le p < \infty,\ q = \infty$

$$\frac{f_i}{d_i} = c > 0, \quad i = 1, \dots, n$$

(c) $1 \le q < \infty,\ p = \infty$

$$f_i = c > 0, \quad i = 1, \dots, n$$

(d) $p = \infty,\ q = \infty$

$$J \cap \widetilde{J} \neq \phi$$

where $J = \{j : f_j = \max_{i=1,\dots,n} f_i\}$ and $\widetilde{J} = \{j : \frac{f_j}{d_j} = \max_{i=1,\dots,n} f_i d_i\}$.

We wish to calculate $\mathbf{g}^\sigma$ (a solution to (5.1)) for each σ. For $\sigma > \widetilde{\sigma}$, $\mathbf{g}^\sigma = \mathbf{0}$ is a solution, and for $\sigma \le \hat{\sigma}$, $\mathbf{g}^\sigma = D^{-1}\mathbf{f}$ is a solution. Thus we consider

$$\hat{\sigma} \le \sigma \le \widetilde{\sigma}.$$

It is not convenient to calculate $\mathbf{g}^\sigma$ in terms of σ. Instead we calculate both σ and $\mathbf{g}^\sigma$ in terms of a different parameter.

We use the dual problem and recall from Proposition 3.6 that if $\widetilde{\mathbf{g}} \neq \mathbf{0}$, $\mathbf{f} \neq D\widetilde{\mathbf{g}}$ and $\widetilde{\mathbf{h}} \neq \mathbf{0}$ satisfy

$$(5.3) \qquad (\mathbf{f} - D\widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) = ||\mathbf{f} - D\widetilde{\mathbf{g}}||_p ||\widetilde{\mathbf{h}}||_{p'}$$

$$(5.4) \qquad (\widetilde{\mathbf{g}}, D\widetilde{\mathbf{h}}) = ||\mathbf{g}||_q ||D\widetilde{\mathbf{h}}||_{q'},$$

then $\widetilde{\mathbf{g}}$ is a solution to (5.1) for

$$\sigma = \frac{||D\widetilde{\mathbf{h}}||_{q'}}{||\widetilde{\mathbf{h}}||_{p'}}.$$

Thus we construct solutions to (5.1) by looking for $\widetilde{\mathbf{g}}$ and $\widetilde{\mathbf{h}}$ as above satisfying (5.3) and (5.4).

Case I. $1 < p < \infty$, $1 < q < \infty$.

Given $\lambda > 0$. For each $i = 1, \ldots, n$, let g_i solve

$$(5.5) \qquad d_i g_i + \lambda \left(\frac{g_i^{q-1}}{d_i} \right)^{\frac{1}{p-1}} = f_i, \; i = 1, \ldots n.$$

Set

$$(5.6) \qquad h_i = \frac{\dfrac{g_i^{q-1}}{d_i}}{\left(\displaystyle\sum_{j=1}^{n} \left(\frac{g_j^{q-1}}{d_j} \right)^{p'} \right)^{1/p'}}, \; i = 1, \ldots, n.$$

Thus $||\mathbf{h}||_{p'} = 1$, and from (5.6),

$$(\mathbf{g}, D\mathbf{h}) = ||\mathbf{g}||_q ||D\mathbf{h}||_{q'},$$

i.e., (5.4) holds.

From (5.5) and (5.6)

$$h_i = \alpha(f_i - d_i g_i)^{p-1}, \quad i = 1, \dots, n$$

for some $\alpha > 0$. Thus

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_p ||\mathbf{h}||_{p'}.$$

Thus (5.3) holds. Now

(5.7)
$$\sigma = \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_{p'}} = \frac{\left(\sum_{i=1}^{n} g_i^q\right)^{1/q'}}{\left(\sum_{i=1}^{n}\left(\frac{g_i^{q-1}}{d_i}\right)^{p'}\right)^{1/p'}}.$$

The $\mathbf{g}$ obtained from (5.5) solves (5.1) for the above σ. Both $\mathbf{g}$ and σ are continuous functions of λ.

Note that for $\lambda = 0$ in (5.5), $d_i g_i \equiv f_i$, $i = 1, \dots, n$, and

$$\sigma = \frac{\left(\sum_{i=1}^{n}\left(\frac{f_i}{d_i}\right)^q\right)^{1/q'}}{\left(\sum_{i=1}^{n}\left(\frac{1}{d_i}\left(\frac{f_i}{d_i}\right)^{q-1}\right)^{p'}\right)^{1/p'}} = \hat{\sigma}.$$

As $\lambda \uparrow \infty$, the associated g_i in (5.5) decrease to zero, and

$$\lambda\left(\frac{g_i^{q-1}}{d_i}\right)^{\frac{1}{p-1}} \approx f_i, \quad i = 1, \dots, n.$$

Thus

$$\frac{g_i^{q-1}}{d_i} \approx \left(\frac{f_i}{\lambda}\right)^{p-1}, \quad i = 1, \dots, n$$

and

$$g_i^q \approx d_i^{q'}\left(\frac{f_i}{\lambda}\right)^{(p-1)q'}, \quad i = 1, \dots, n$$

while

$$\sigma = \frac{\left(\sum_{i=1}^{n} g_i^q\right)^{1/q'}}{\left(\sum_{i=1}^{n}\left(\frac{g_i^{q-1}}{d_i}\right)^{p'}\right)^{1/p'}} \approx \frac{\left(\sum_{i=1}^{n} d_i^{q'} f_i^{(p-1)q'}\right)^{1/q'}}{\left(\sum_{i=1}^{n} f_i^p\right)^{1/p'}} = \tilde{\sigma}.$$

That is, as λ increases from zero to infinity, using Proposition 3.2, we see that σ increases from $\hat{\sigma}$ to $\tilde{\sigma}$.

Case 2. $p = 1,\ 1 < q < \infty$

Let

$$\min_{i=1,\ldots,n} \frac{f_i}{d_i^{q'}} \le c \le \max_{i=1,\ldots,n} \frac{f_i}{d_i^{q'}}.$$

Set

$$g_i = c\, d_i^{1/(q-1)},\ h_i = 1 \ \text{ if } \frac{f_i}{d_i^{q'}} \ge c$$

$$g_i = \frac{f_i}{d_i},\ h_i = \frac{f_i^{q-1}}{c^{q-1} d_i^q} \quad \text{if } \frac{f_i}{d_i^{q'}} < c.$$

Note that if $\frac{f_i}{d_i^{q'}} < c$, then $h_i \le 1$. Thus $\|\mathbf{h}\|_\infty = 1$. Furthermore,

$$d_i h_i = \frac{g_i^{q-1}}{c^{q-1}} \quad \text{for } i = 1, \ldots, n.$$

Thus

$$(\mathbf{g},\ D\mathbf{h}) = ||\mathbf{g}||_q ||D\mathbf{h}||_{q'}.$$

Now if $\dfrac{f_i}{d_i^{q'}} \geq c$

$$f_i - d_i g_i = f_i - c d_i^{1/(q-1)} d_i = f_i - c d_i^{q'} \geq 0$$

while if $\dfrac{f_i}{d_i^{q'}} < c$, then

$$f_i - d_i g_i = 0.$$

Thus

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_1 ||\mathbf{h}||_\infty.$$

The above g solves (5.1) with

$$\sigma = \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_\infty} = ||D\mathbf{h}||_{q'} = \left(\sum_{f_i/d_i^{q'} \geq c} d_i^{q'} + \sum_{f_i/d_i^{q'} < c} (f_i/cd_i)^q \right)^{1/q'}.$$

For $c = \min\limits_{i=1,\dots,n} \dfrac{f_i}{d_i^{q'}}$,

$$\sigma = \left(\sum_{i=1}^{n} d_i^{q'} \right) = \tilde{\sigma}.$$

For $c = \max_{i=1,\ldots,n} \dfrac{f_i}{d_i^{q'}}$,

$$\sigma = \left(\sum_{i=1}^{n} \left(\frac{f_i}{d_i} \right)^q \right)^{1/q'} = \frac{\left(\sum_{i=1}^{n} \left(\frac{f_i}{d_i} \right)^q \right)^{1/q'}}{c^{q/q'}}$$

$$= \frac{\left(\sum_{i=1}^{n} \left(\frac{f_i}{d_i} \right)^q \right)^{1/q}}{\max_{i=1,\ldots,n} \left(\frac{f_i^{q-1}}{d_1^q} \right)} = \hat{\sigma}.$$

Case 3. $p = \infty$, $1 < q < \infty$.

Let $0 \leq \lambda < \max_{i=1,\ldots,n} f_i = \|f\|_\infty$. Set

$$g_i = 0,\ h_i = 0, \text{ if } \lambda \geq f_i$$

$$g_i = \frac{f_i - \lambda}{d_i},\ h_i = \alpha \frac{(f_i - \lambda)^{q-1}}{d_i^q}, \text{ if } \lambda < f_i.$$

Note that if $h_i \neq 0$, then

$$f_i - d_i g_i = \lambda$$

while if $h_i = 0$, then

$$f_i - d_i g_i = f_i \leq \lambda.$$

Thus

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_\infty ||\mathbf{h}||_1.$$

Furthermore,

$$d_i h_i = \alpha g_i^{q-1}, \quad \text{for all} \quad i = 1, \dots, n.$$

Thus

$$(\mathbf{g}, D\mathbf{h}) = ||\mathbf{g}||_q ||D\mathbf{h}||_{q'}.$$

The above g solves (5.1) for

$$\sigma = \frac{||D\mathbf{h}||_{q'}}{||\mathbf{h}||_1} = \frac{\left(\sum_{\{i : f_i > \lambda\}} \left(\frac{f_i - \lambda}{d_i} \right)^q \right)^{1/q'}}{\sum_{\{i : f_i > \lambda\}} \frac{(f_i - \lambda)^{q-1}}{d_i^q}}.$$

For $\lambda = 0$,

$$\sigma = \frac{\left(\sum_{i=1}^n \left(\frac{f_i}{d_i} \right)^q \right)^{1/q'}}{\left(\sum_{i=1}^n \frac{f_i^{q-1}}{d_i^q} \right)} = \hat{\sigma}.$$

For $\lambda_0 < \lambda \leq \max_{i=1,\dots,n} f_i$, where $\lambda_0 = \max_{f_i \neq ||\mathbf{f}||_\infty} f_i$, set $J = \{j : f_j = ||\mathbf{f}||_\infty\}$. Then

$$\sigma = \frac{\left(\sum_{j \in J} \frac{(\|\mathbf{f}\|_q - \lambda)^q}{d_j^q}\right)^{1/q'}}{\sum_{j \in J} \frac{(\|\mathbf{f}\|_q - \lambda)^{q-1}}{d_j^q}} = \left(\sum_{j \in J} \frac{1}{d_j^q}\right)^{-1/q} = \tilde{\sigma}.$$

Case 4. $1 < p < \infty,\ q = \infty.$

Let $0 \le \lambda < \max_{j=1,\dots,n} \frac{f_j}{d_j}$. Set

$$g_i = \lambda,\ h_i = (f_i - \lambda d_i)^{p-1},\ \text{ if } \lambda < \frac{f_i}{d_i}$$

$$g_i = \frac{f_i}{d_i},\ h_i = 0,\ \text{ if } \lambda \ge \frac{f_i}{d_i}.$$

Since $h_i = (f_i - d_i g_i)^{p-1}$, for all i, we have

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = \|\mathbf{f} - D\mathbf{g}\|_p \|\mathbf{h}\|_{p'}.$$

Now $\|\mathbf{g}\|_\infty = \lambda$ and $d_i h_i = 0$ if $g_i < \lambda$. Thus

$$(\mathbf{g}, D\mathbf{h}) = \|\mathbf{g}\|_\infty \|D\mathbf{h}\|_1.$$

Thus $\mathbf{g}$ is a solution to (5.1) for

$$\sigma = \frac{\|D\mathbf{h}\|_1}{\|\mathbf{h}\|_{p'}} = \frac{\sum_{\{i : \frac{f_i}{d_i} > \lambda\}} d_i(f_i - d_i\lambda)^{p-1}}{\left(\sum_{\{i : \frac{f_i}{d_i} > \lambda\}} (f_i - d_i\lambda)^p \right)^{1/p'}} .$$

For $\lambda = 0$,

$$\sigma = \frac{\sum_{i=1}^{n} d_i f_i^{p-1}}{\left(\sum_{i=1}^{n} f_i^p \right)^{1/p'}} = \tilde{\sigma}.$$

Set

$$\tilde{J} = \{j : \frac{f_j}{d_j} = \max_{i=1,\ldots,n} \frac{f_i}{d_i} \}.$$

Let

$$\max_{i \notin \tilde{J}} \frac{f_i}{d_i} < \lambda < \max_{i-1,\ldots,n} \frac{f_i}{d_i} .$$

Then

$$\sigma = \frac{\sum_{j \in \tilde{J}} d_j^p}{\left(\sum_{j \in \tilde{J}} d_j^p \right)^{1/p'}} = \left(\sum_{j \in \tilde{J}} d_j^p \right)^{1/p} = \hat{\sigma}.$$

Case 5. $1 < p < \infty,\ q = 1.$

Let

$$0 \le \lambda \le \max_{i=1,\ldots,n} d_i f_i^{p-1}.$$

Set

$$g_i = 0,\ h_i = f_i^{p-1},\ \text{ if } d_i f_i^{p-1} \le \lambda.$$

If $d_i f_i^{p-1} > \lambda$, define g_i by

$$(f_i - d_i g_i)^{p-1} = \frac{\lambda}{d_i},$$

and set $h_i = \dfrac{\lambda}{d_i}$.

Since $h_i = (f_i - d_i g_i)^{p-1}$ for all i, we have

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_p ||\mathbf{h}||_{p'}.$$

Now $d_i h_i = \lambda$, if $g_i \ne 0$. Thus

$$(\mathbf{g}, D\mathbf{h}) = ||\mathbf{g}||_1 ||D\mathbf{h}||_\infty.$$

The vector $\mathbf{g}$ is a solution to (5.1) for

$$\sigma = \frac{||D\mathbf{h}||_\infty}{||\mathbf{h}||_{p'}} = \frac{\lambda}{||\mathbf{h}||_{p'}}$$

$$= \frac{\lambda}{\left(\sum_{d_i f_i^{p-1} \le \lambda} f_i^{(p-1)p'} + \sum_{d_i f_i^{p-1} > \lambda} \left(\frac{\lambda}{d_i} \right)^{p'} \right)^{1/p'}}$$

$$= \frac{1}{\left(\sum_{d_i f_i^{p-1} \le \lambda} \lambda^{-p'} f_i^{p} + \sum_{d_i f_i^{p-1} > \lambda} \left(\frac{1}{d_i} \right)^{p'} \right)^{1/p'}} .$$

For $\lambda \le \min_{i=1,\ldots,n} d_i f_i^{p-1}$

$$\sigma = \frac{1}{\left(\sum_{i=1}^{n} \left(\frac{1}{d_i} \right)^{p'} \right)^{1/p'}} = \hat{\sigma}.$$

For $\lambda \le \max_{i=1,\ldots,n} d_i f_i^{p-1}$

$$\sigma = \frac{1}{\lambda^{-1} \left(\sum_{i=1}^{n} f_i^{p} \right)^{1/p'}} = \frac{\lambda}{\left(\sum_{i=1}^{n} f_i^{p} \right)^{1/p'}} = \frac{||D\mathbf{f}^{p-1}||_\infty}{||\mathbf{f}||_p^{p-1}} = \tilde{\sigma}.$$

Case 6. $p = \infty,\ q = 1.$

Let $J = \{j: f_j = ||\mathbf{f}||_\infty\}$ and $\min_{i=1,\ldots,n} f_i \le \lambda \le \max_{i=1,\ldots,n} f_i$. Set

$$g_i = 0, \qquad h_i = 0 \text{ if } f_i < \lambda$$

$$g_i = \frac{f_i - \lambda}{d_i}, \quad h_i = \frac{1}{d_i} \text{ if } f_i > \lambda$$

and

$$g_i = 0, \; h_i = \frac{\mu}{d_i} \text{ if } f_i = \lambda$$

where $\mu \in [0, 1]$ (and could also depend on i).

Now $h_i \neq 0$ only if $f_i - d_i g_i = \lambda$. Thus

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_\infty ||\mathbf{h}||_1.$$

If $g_i \neq 0$, then $d_i h_i = 1$ (while $d_i h_i \leq 1$ for all i). Thus

$$(\mathbf{g}, D\mathbf{h}) = ||\mathbf{g}||_1 ||D\mathbf{h}||_\infty.$$

Thus $\mathbf{g}$ is a solution to (5.1) for

$$\sigma = \frac{||D\mathbf{h}||_\infty}{||\mathbf{h}||_1} = \frac{1}{\left(\sum_{f_i > \lambda} \frac{1}{d_i} + \mu \sum_{f_i = \lambda} \frac{1}{d_i}\right)}.$$

For $\lambda = \min_{i = 1, \ldots, n} f_i$ and $\mu = 1$,

$$\sigma = \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{d_i}\right)} = \hat{\sigma}.$$

For $\lambda = \max_{l=1,\dots,n} f_l$ and $\mu = 1$

$$\sigma = \frac{1}{\left(\sum_{j \in J} \frac{1}{d_j}\right)} = \tilde{\sigma}.$$

The $\mu \in [0, 1]$ allows us to cover the full range $[\hat{\sigma}, \tilde{\sigma}]$ as λ varies (i.e., if $\lambda = f_i$, some i, let μ also vary). Thus the same **g** may be a solution to (5.1) for an interval of values of σ.

Case 7. $p = 1,\ q = \infty$.

Let $\tilde{J} = \{j : \frac{f_j}{d_j} = \max_{l=1,\dots,n} \frac{f_l}{d_l}\}$ and $\min_{j=1,\dots,n} \frac{f_j}{d_j} \le \lambda < \max_{j=1,\dots,n} \frac{f_j}{d_j}$.

Set

$$g_i = \frac{f_i}{d_i},\ h_i = 0 \text{ if } \frac{f_i}{d_i} < \lambda$$

$$g_i = \lambda,\ h_i = 1 \text{ if } \frac{f_i}{d_i} > \lambda$$

$$g_i = \frac{f_i}{d_i},\ h_i = \mu,\ \text{ if } \frac{f_i}{d_i} = \lambda$$

for $\mu \in [0, 1]$.

If $f_l - d_l g_l \neq 0$, then $h_l = 1 = ||\mathbf{h}||_\infty$. Thus

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_1\ ||\mathbf{h}||_\infty.$$

If $d_l h_l \neq 0$, then $g_l = \lambda = ||\mathbf{g}||_\infty$. Thus

$$(\mathbf{g}, D\mathbf{h}) = \|\mathbf{g}\|_\infty \|D\mathbf{h}\|_1.$$

Thus $\mathbf{g}$ is a solution to (5.1) for

$$\sigma = \frac{\|D\mathbf{h}\|_1}{\|\mathbf{h}\|_\infty} = \|D\mathbf{h}\|_1 = \Big(\sum_{\frac{f_i}{d_i} > \lambda} d_i + \sum_{\frac{f_i}{d_i} = \lambda} \mu\, d_i\Big).$$

For $\lambda = \max_{j=1,\dots,n} \dfrac{f_j}{d_j}$, $\mu = 1$ (or λ slightly less).

$$\sigma = \sum_{j \in \tilde{J}} d_j = \hat{\sigma}.$$

For $\lambda = \min_{j=1,\dots,n} \dfrac{f_i}{d_j}$, $\mu = 1$,

$$\sigma = \left(\sum_{i=1}^{n} d_i\right) = \tilde{\sigma}.$$

The μ allows us to cover the interval $[\hat{\sigma}, \tilde{\sigma}]$.

Case 8. $p = 1$, $q = 1$.

Let $\min_{i=1,\dots,n} d_i \le \lambda \le \max_{i=1,\dots,n} d_i$. Set

$$g_i = 0,\ h_i = \frac{1}{\lambda} \quad \text{if } d_i < \lambda$$
$$g_i = \frac{f_i}{d_i},\ h_i = \frac{1}{d_i} \quad \text{if } d_i > \lambda$$
$$g_i = \frac{\mu f_i}{d_i},\ h_i = \frac{1}{d_i} \quad \text{if } d_i = \lambda$$

for $\mu \in [0, 1]$. As is easily checked

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = ||\mathbf{f} - D\mathbf{g}||_1 ||\mathbf{h}||_\infty$$

and

$$(\mathbf{g}, D\mathbf{h}) = ||\mathbf{g}||_1 ||D\mathbf{h}||_\infty.$$

Thus $\mathbf{g}$ is a solution to (5.1) for

$$\sigma = \frac{||D\mathbf{h}||_\infty}{||\mathbf{h}||_\infty}.$$

Now, $||\mathbf{h}||_\infty = \frac{1}{\lambda}$ and $||D\mathbf{h}||_\infty = 1$. Thus

$$\sigma = \frac{||D\mathbf{h}||_\infty}{||\mathbf{h}||_\infty} = \lambda.$$

Note that for some λ (where $d_i = \lambda$) the solution $\mathbf{g}$ of (5.1) is not unique (by choosing different $\mu \in [0, 1]$).

Case 9. $p = \infty$, $q = \infty$.

Let

$$J = \{j: f_j = ||\mathbf{f}||_\infty\}$$

$$\tilde{J} = \{j: \frac{f_j}{d_j} = \max_{i=1,\dots,n} \frac{f_i}{d_i}\},$$

and for $0 \le \lambda \le \|\mathbf{f}\|_\infty$,

$$J_\lambda = \{j\colon \frac{f_j - \lambda}{d_j} = \max_{i=1,\dots,n} \frac{f_i - \lambda}{d_i}\}.$$

For $0 \le \lambda < \max_{i=1,\dots,n} f_i = \|\mathbf{f}\|_\infty$, set

$$g_i = 0,\ h_i = 0,\quad \text{if } f_i \le \lambda$$
$$g_i = \frac{f_i - \lambda}{d_i},\ h_i = \alpha_i,\quad \text{if } f_i > \lambda \text{ and } i \in J_\lambda$$
$$g_i = \frac{f_i - \lambda}{d_i},\ h_i = 0,\quad \text{if } f_i > \lambda \text{ and } i \notin J_\lambda$$

where $\alpha_i \ge 0$ in the above, and not all zero, i.e., $\mathbf{h} \ne \mathbf{0}$. It is easily checked that

$$(\mathbf{f} - D\mathbf{g}, \mathbf{h}) = \|\mathbf{f} - D\mathbf{g}\|_\infty \|\mathbf{h}\|_1$$

since $f_i - d_i g_i = \lambda$ if $h_i = \alpha_i$. Furthermore, if $h_i \ne 0$, then $g_i = \dfrac{f_i - \lambda}{d_i}$ and $i \in J_\lambda$. Thus $g_i = \|\mathbf{g}\|_\infty$.
Therefore

$$(\mathbf{g}, D\mathbf{h}) = \|\mathbf{g}\|_\infty \|D\mathbf{h}\|_1.$$

Thus $\mathbf{g}$ is a solution to (5.1) with

$$\sigma = \frac{\|D\mathbf{h}\|_1}{\|\mathbf{h}\|_1} = \left(\frac{\sum_{j \in J_\lambda} d_j \alpha_j}{\sum_{j \in J_\lambda} \alpha_j} \right)$$

If J_λ is not a singleton, this quantity spans the interval $[\min_{j \in J_\lambda} d_j, \ \max_{j \in J_\lambda} d_j]$..
For $\lambda = 0$, $J_\lambda = \tilde{J}$. Thus we can get

$$\sigma = \frac{\left(\sum_{j \in \tilde{J}} d_j \alpha_j\right)}{\left(\sum_{j \in \tilde{J}} \alpha_j\right)} = \max_{j \in \tilde{J}} d_j = \hat{\sigma}$$

by letting $\alpha_{j_0} = 1$ for $d_{j_0} = \max_{j \in J} d_j$, and $\alpha_j = 0$ for $j \neq j_0$.

For $\max_{j \notin J} f_j < \lambda < \|\mathbf{f}\|_\infty$

$$J_\lambda = \{j : \frac{f_j - \lambda}{d_j} = \max_{i = 1, \ldots, n} \frac{f_i - \lambda}{d_i}\} = \{j : j \in J, d_j = \min_{i \in J} d_i\},$$

and we get $\sigma = \tilde{\sigma} = \min_{j \in J} d_j$.

Remark 5.2. The cases (p, q) and (q, p) are related. As is easily checked,

$$\min_{\mathbf{g}} \|\mathbf{f} - D\mathbf{g}\|_p + \sigma\|\mathbf{g}\|_q = \sigma \min_{\mathbf{h}} [\|D^{-1}\mathbf{f} - D^{-1}\mathbf{h}\|_q + \frac{1}{\sigma} \|\mathbf{h}\|_p].$$

6. Semi-Discrete Kernel

In this section we investigate problems in two general settings.

A. Let $v_1, \ldots, v_n \in C(\Omega)$, where Ω is a compact subset of $\mathbb{R}^m$. For $1 \leq p \leq \infty$, $1 < q \leq \infty$, $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{R}^n$, and $\sigma > 0$, define

$$(6.1) \qquad E_{pq}(\sigma) = \min_{g \in L^q} \left(\sum_{i=1}^{n} |f_i - \int_\Omega v_i(y)g(y)dy|^p\right)^{1/p} + \sigma\|g\|_q$$

where $L^q = L^q(\Omega)$ is the usual L^q-space with associated Lebesgue measure on Ω: That the minimum is attained and Assumption I holds is easily checked. (The only case demanding some attention is where $q = \infty$). The dual problem is

$$(6.2) \qquad \max_{\substack{\|h\|_{p'} \le 1 \\ \|\Sigma_{i=1}^{n} h_i v_i\|_{q'} \le \sigma}} \sum_{i=1}^{n} f_i h_i$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

The analogue of (6.1) for $q = 1$ is not correct. (The analogue of (6.2) in this case is valid.) Instead of (6.1) we consider (abusing somewhat our notation)

$$(6.3) \qquad E_{p1}(\sigma) = \min_{\mu \in C^*(\Omega)} \left(\sum_{i=1}^{n} |f_i - \int_{\Omega} v_i(y) d\mu(y)|^p \right)^{1/p} + \sigma \|\mu\|_{T.V.}$$

where $C^*(\Omega)$ denotes the dual space of $C(\Omega)$. We identify this as the space of all Borel measures μ on Ω, with norm denoted $\|\mu\|_{T.V.}$, the total variation of μ over Ω.

For ease of exposition we assume that the $v_1, \dots, v_n$ are in $C(\Omega)$ (and thus in $L^{q'}(\Omega)$ for all $q' \ge 1$). To further simplify matters, we assume that the $v_1, \dots, v_n$ are linearly independent over $C(\Omega)$ (and every $L^{q'}(\Omega)$, $1 \le q' < \infty$). We first investigate the possible solution set. This plays an important role in this problem. We are in fact identifying the possible choices of g in (6.1) (or μ in (6.3)) satisfying Proposition 3.2 *(v)*, or (3.13).

To deal with the various cases we state a more general known result which is one of the simpler consequences of the Hahn-Banach Theorem. For completeness, we provide the proof thereof.

Proposition 6.1. *Let $W = \text{span}\{w_1, \dots, w_n\}$ be a finite-dimensional subspace of a normed linear space Z. Let Z^* denote the continuous dual of Z. Given real numbers $e_1, \dots, e_n$, assume there exists a $\ell_0 \in Z^*$ such that $\ell_0(w_i) = e_i$, $i = 1, \dots, n$. There then exists a solution $\ell^* \in Z^*$ to the problem*

(6.4)
$$\inf_{\substack{\ell(w_i) = e_i \\ i=1,\dots,n}} \|\ell\|_{Z^*}.$$

Furthermore $\ell^* \in Z^*$ *is a solution to* (6.4) *if and only if* $\ell^*(w_i) = e_i$, $i = 1, \dots, n$, *and there exists a* $w^* \in W$, $w^* \neq 0$, *such that*

(6.5)
$$\ell^*(w^*) = \|\ell^*\|_{Z^*}\|w^*\|_Z.$$

Proof ($\Leftarrow$). Assume $\ell^*(w_i) = e_i$, $i = 1, \dots, n$, and (6.5) holds. Let $\hat{\ell}$ be any other element of Z^* satisfying $\hat{\ell}(w_i) = e_i$, $i = 1, \dots, n$. Then

$$\|\ell^*\|_{Z^*} = \frac{\ell^*(w^*)}{\|w^*\|_Z} = \frac{\hat{\ell}(w^*)}{\|w^*\|_Z} \leq \|\hat{\ell}\|_Z.$$

Thus ℓ^* attains the minimum (6.4).

($\Rightarrow$). By assumption, there exists a linear functional ρ defined on W such that $\rho(w_i) = e_i$, $i = 1, \dots, n$. Set

$$\|\rho\|_{W^*} = \max_{w \neq 0} \frac{\rho(w)}{\|w\|_Z}.$$

We write maximum because it is in fact attained by some $w^* \in W$, $w^* \neq 0$. By the Hahn-Banach Theorem there exists a $\ell^* \in Z^*$ such that $\ell^*|_W = \rho$ (thus $\ell^*(w_i) = e_i$, $i = 1, \dots, n$) and $\|\ell^*\|_{Z^*} = \|\rho\|_{W^*}$. Thus

$$\ell^*(w^*) = \rho(w^*) = \|\rho\|_{W^*}\|w^*\|_Z = \|\ell^*\|_{Z^*}\|w^*\|_Z$$

and (6.5) holds. □

A different general approach is given by the following.

Let X and Y be Banach spaces and T a bounded linear operator from X to Y. T is said to be a semi-Fredholm operator provided that

$$\dim \operatorname{Ker} T < \infty \tag{6.6}$$

where Ker $T :=$ null space of T and

$$\text{Range of } T = R(T) \text{ is closed.} \tag{6.7}$$

In view of the latter property we may assume without loss of generality that T is onto. Every semi-Fredholm operator admits a "regularizer" which is a bounded linear operator L_0 from Y to X.

$$L_0 T f = f - L_1 f, \quad f \in X, \tag{6.8}$$

where L_1 is a projection of X onto Ker T. L_0 may be taken to be the inverse of T restricted to a complementary subspace of Ker T in X. For instance, in the special choice $Tf = f^{(n)}$, $X = W^n_\infty$ with norm $\|f\| = \|f\|_\infty + \|f^{(n)}\|_\infty$, and $Y = L^\infty[0, 1]$, the regularizer of T is given by the integral operator

$$(L_0 g)(x) = \frac{1}{(n-1)!} \int_0^1 (x-t)_+^{n-1} g(t)dt$$

and the projection onto Ker T is simply

$$(L_1 f)(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j.$$

Hence (6.8) becomes in this case Taylor's formula with remainder.

Let $F_1, \dots, F_N$ be linearly independent continuous linear functionals on X and $y = (y_1, \dots, y_N)$ be a prescribed nontrivial vector in R^N. We introduce the notation $If = (F_1 f, F_2 f, \dots, F_N f)$ and consider the variational problem

$$\rho = \inf_{If = y} \|Tf\|_Y \tag{6.9}$$

Note that I is a bounded linear operator from X onto R^N. It may occur that $I^{-1}(y) \cap \operatorname{Ker} T \neq \phi$ which implies $\rho = 0$. Let us exclude this trivial case and assume always that

$$I^{-1}(y) \cap \operatorname{Ker} T \neq \phi. \tag{6.10}$$

In the following theorem the combination IL_0 and IL_1 appear frequently. We denote these operators by I_0 and I_1, respectively. We also use the notation $x \cdot y$ to denote the inner product of two vectors in R^N.

Theorem 6.2 *Let T be a semi-Fredholm operator such that $I^{-1}(y) \cap \operatorname{Ker} T = \phi$ then*

$$\rho^{-1} = \inf_{\substack{\lambda \in R(I_1)^{\perp} \\ \lambda \cdot y = 1}} \|\lambda \cdot I_0\|_{Y^*}. \tag{6.11}$$

Proof. The proof of this result relies on the basic duality principle expressing the distance of a point to a subspace, viz.

$$\inf_{y \in \mathcal{M}} \|x - y\|_X = \sup_{\substack{F \in X^* \\ \|F\| \leq 1 \\ F \in \mathcal{M}^{\perp}}} |F(x)|. \tag{6.12}$$

Here, as before, X is a normed linear space and $\mathcal{M}$ a closed subspace of X.

To apply (6.12) we choose any $f_0 \in X$ for which $If_0 = y$ and define

$$\mathcal{M} = \{g : g = Tf \text{ for some } f \in I^{-1}(0)\} \tag{6.13}$$

then $\rho = \inf_{g \in \mathcal{M}} \|Tf_0 - g\|$. We claim that $\mathcal{M} = I_0^{-1}(R(I_1))$. To see this, first suppose that $g = Tf$ for some $f \in I^{-1}(0)$. Then in view of (6.8) we have $I_0 g = IL_0 g = IL_1(-f) = I_1(-f)$ and so $g \in I_0^{-1}(R(I_1))$. Conversely, suppose $g \in I_0^{-1}(R(I_1))$. Then there exist $f_1, f_2 \in X$ such that $g = Tf_1$ and $IL_0 g + IL_1 f_2 = 0$ (recall that $R(T) = Y$). Setting $h = f_1 + L_1(f_2 - f_1)$, we may compute $Th = Tf_1 = g$ and $Ih = I(f_1 - L_1 f_1 + L_1 f_2) = I(L_0 Tf_1 + L_1 f_2) = I(L_0 g + L_1 f_2) = 0$. Hence $g \in \mathcal{M}$ and our assertation is verified.

We now apply (6.12) which gives the equation

$$\rho = \sup_{\substack{F \in Y^* \\ \|F\| \le 1 \\ F \in \mathcal{M}^\perp}} F\,Tf_0. \tag{6.14}$$

To identify $\mathcal{M}^\perp$ we let Q be the orthogonal projection of R^N onto $R(I_1)^\perp$. Thus $Qx = 0$ if and only if $x \in R(I_1)$. Hence we have $\mathcal{M}^\perp = \text{Ker } (Q\,I_0)^\perp = R((QI_0)^*)$. Therefore $F \in \mathcal{M}^\perp$ if and only if there exists a $\mu \in R^N$ for which $F = \mu \cdot QI_0 = Q\mu \cdot I_0$. Equivalently $F \in \mathcal{M}^\perp$ means there exists $\lambda \in R^N$ so that $F = \lambda \cdot I_0$ and $\lambda \in R(I_1)^\perp$. Substituting this equation into (6.14) gives

$$\rho = \sup_{\substack{\lambda \in R(I_1)^\perp \\ \|\lambda \cdot I_0\|_{Y^*} \le 1}} \lambda \cdot I_0\, Tf_0 \tag{6.15}$$

which in view of (6.8) simplifies to

$$= \sup_{\substack{\lambda \in R(I_1)^{\perp} \\ \|\lambda \cdot I_0\|_{Y*} \leq 1}} \lambda \cdot y$$

(6.16)
$$= \frac{1}{\inf_{\substack{\lambda \in R(I_1)^{\perp} \\ \lambda \cdot y = 1}} \|\lambda \cdot I_0\|_{Y*}}.$$

The value of the denominator is attained by some $\lambda_0 \in R(I_1)^{\perp}$ with $\lambda_0 \cdot y = 1$ and necessarily $\|\lambda_0 \cdot I_0\| > 0$. For otherwise $\lambda_0 \cdot I\, L_0 g = 0$ for all $g \in Y$. Hence for any $f \in X$ we have

$$\lambda_0 \cdot If = \lambda_0 \cdot I(L_1 f + L_0\, Tf) = \lambda_0 \cdot I\, L_1 f = \lambda_0 \cdot I_1 f = 0$$

and so $\lambda_0 \in R(I)^{\perp} = 0$ contradicting the relation $\lambda_0 \cdot y = 1$. This completes the proof of Theorem 6.2.. □

The value of Theorem 6.2 lies in the fact that it replaces the (possibly) infinite dimensional variational problem (6.9) by a finite dimensional one given by (6.11). Existence, uniqueness and characterization of solutions to (6.9) may be discussed through the dual problem (6.11). We present three corollaries to Theorem 6.2. The first may be interpreted as a characterization of the solutions to (6.9).

Corollary 6.3. *Suppose T is a semi-Fredholm operator. Assume (6.10) holds and let f_0 be a solution to the variational problem* (6.9). *Then there exists a solution λ_0 to the dual variational problem (6.11) such that if $F_0 = \lambda_0 \cdot I_0$, then*

(6.17)
$$F_0(Tf_0) = \|F_0\|_{Y*}\ \|Tf_0\|_{Y}.$$

Proof: Since f_0 is a solution to the variational problem (6.9) we have $\|Tf_0\| = \inf_{g \in \mathcal{M}} \|Tf_0 - g\|_Y > 0$ where $\mathcal{M} = I_0^{-1}(R(I_1))$. By the Hahn-Banach theorem there exists a nontrivial $F \in \mathcal{M}^{\perp}$ such that

$F(Tf_0) = \|F\|_{Y^*} \|Tf_0\|_Y$. In view of our discussion in Theorem 6.2, $F = \mu_0 \bullet I_0$ for some $\mu_0 \in R(I_1)^{\perp}$. Furthermore, we have for any $\lambda \in R(I_1)^{\perp}, f \in I^{-1}(y)$.

(6.18)
$$\begin{aligned} \lambda \bullet y = \lambda \bullet If &= \lambda \bullet I(L_1 f + L_0\, Tf) \\ &= \lambda \bullet I_0(Tf) \end{aligned}$$

and so

(6.19)
$$1 \leq \|\lambda \bullet I_0\|_{Y^*}\ \|Tf\|_Y$$

for any $\lambda \in R(I_1)^{\perp}$, $\lambda \bullet y = 1$ and $f \in I^{-1}(y)$. Hence by choosing $\lambda = \mu_0$ and $f = f_0$ in (6.18) we see that $\mu_0 \bullet y = \|F\|_{Y^*} \|Tf_0\|_Y > 0$. If we define $F_0 = \lambda_0 \bullet I_0$ where $\lambda_0 = \mu_0 / \mu_0 \bullet y$ then $1 = F_0(Tf_0) = \|F_0\|_{Y^*} \|Tf_0\|_Y$ and from (6.19) $\|\lambda_0 \bullet I_0\| \leq \|\lambda \bullet I_0\|$ for all $\lambda \in R(I_1)^{\perp}, \lambda \bullet y = 1$. □

The following result is a partial converse to Corollary 6.3. It gives sufficient conditions for the existence of a solution to (6.9).

Corollary 6.4. *Suppose T is a semi-Fredholm operator, and assume (6.10) holds. Suppose there exists a solution λ_0 to the dual problem* (6.11), *and $F_0 = \lambda_0 \bullet I_0$ has the property that there exists a unique $g \in Y$, of norm one, which satisfies the equation $F_0(g) = \|F_0\|_{Y^*}$. Then $g = \rho^{-1} T f_0$ where f_0 is a solution to* (6.9).

Proof: It is well-known that under our hypothesis the norm on Y^* is differentiable at F_0 and the derivative is given by

(6.20)
$$\frac{d}{dt}\ \|F_0 + t\, G\|_{Y^*}\, |_{t=0} = G(g).$$

Now, recall that Q, defined in Theorem 6.2 is the orthogonal projection of R^N onto $R(I_1)^{\perp}$. Hence for any $t \in R, \mu \in R^N$ with $\mu \bullet Qy = 0$ we have $\lambda_0 + tQ\mu \in R(I_1)^{\perp}$ and $(\lambda_0 + tQv) \bullet y = 1$. Therefore

$$\|\lambda_0 \cdot I_0\|_{Y*} \leq \|\lambda_0 \cdot I_0 + t\mu \cdot Q\, I_0\|_{Y*}$$

for all $t \in R, \mu \in R^N, \mu \cdot Qy = 0.$

Thus from (6.20) we conclude that $\mu \cdot Q\, I_0 g = 0$. This implies that there exists a real number δ such that $Q\, I_0 g = \delta Q y$. Since Ker $Q = R(I_1)$ we conclude that there exists an $f_1 \in X$ such that

$$I_0 g = \delta y - I_1 f_1. \tag{6.21}$$

Also, since T is onto we may find an $f_2 \in X$ such that

$$g = T f_2. \tag{6.22}$$

Substituting this expression into (6.21) we obtain

$$I\,(f_2 - L_1 f_2 + L_1 f_1) = \delta y. \tag{6.23}$$

Using the fact that $\lambda_0 \in R(I_1)^{\perp}$, $\lambda_0 \cdot y = 1$ and $(\lambda_0 \cdot I_0)g = \|\lambda_0 \cdot I_0\| = \rho^{-1}$ we find from (6.22) that $\delta = \rho^{-1}$. Now define $f_0 = \rho(f_2 - L_1 f_2 + L_1 f_1)$ then $T f_0 = \rho\, T f_2 = \rho g$. Since $(\lambda_0 \cdot I_0) T f_0 = \|\lambda_0 \cdot I_0\|_{Y*} \|T f_0\|_Y$ it also follows as in Corollary 6.3 that f_0 is a solution to (6.9). □

As an example of the above, we specialize the range of T to be $L^q[0, 1]$, for some $q, 1 < q < \infty$. In this case by the Riesz representation theorem there exists $h_1, h_2, \ldots, h_N \in L^{q'}[0, 1], \frac{1}{q} + \frac{1}{q'} = 1$ such that $I_0 g = (\int_0^1 h_1(t) g(t) dt, \int_0^1 h_2(t) g(t) dt, \ldots, \int_0^1 h_N(t) g(t) dt)$.

Hence $\|\lambda \cdot I_0\| = \|\sum_{j=1}^{N} \lambda_j \cdot h_j\|_{L^{q'}[0, 1]}$ and there exists a unique $g \in L^q[0, 1]$ of norm one which satisfies $(\lambda \cdot I_0) g = \|\sum_{j=1}^{N} \lambda_j h_j\|_{L^{q'}[0, 1]}$. Thus Corollary 6.3 implies that (6.9) has a solution. The dual variational problem (6.11) in this case becomes

$$\min_{\substack{\lambda \in R(I_1)^{\perp} \\ \lambda \bullet y = 1}} \left(\int_0^1 \left| \sum_{j=1}^{N} \lambda_j h_j(t) \right|^{q'} dt \right)^{\frac{1}{q'}} .$$

This problem has a unique solution which we denote by λ_0. Now, by Corollary 6.3 if (6.9) had two solutions, say f_1 and f_2, they both must satisfy the equations

$$\|Tf_1\| = \|Tf_2\|,$$
$$(\lambda_0 \bullet I_0)Tf_1 = \|\lambda_0 \bullet I_0\| \; \|Tf_1\|,$$

and

$$(\lambda_0 \bullet I_0)Tf_2 = \|\lambda_0 \bullet I_0\| \; \|Tf_2\|.$$

Hence it follows that $Tf_1 = Tf_2$ and so $f_1 - f_2 \in \text{Ker}\, T$. We summarize these facts in the following corollary.

Corollary 6.5. *Suppose T is a semi-Fredholm operator mapping a Banach space X onto $L^q[0, 1]$, $1 < q < \infty$. Then the variational problem* (6.9) *has a solution. Furthermore any two solutions differ by an element in* Ker T.

We now return to the minimization problem (6.1). As consequences of either Proposition 6.1 or Corollary 6.5, we have,

Corollary 6.6. *Assume $1 < q < \infty$. In the solution of $E_{pq}(\sigma)$, $\sigma > 0$, it is both necessary and sufficient to consider only those g of the form*

(6.24) $$g(y) = |\nu(y)|^{q'-1} \operatorname{sgn}(\nu(y))$$

for some $v \in V = \text{span}\{v_1, \dots, v_n\}$.

Remark 6.1. Note that for each choice of $e_1, \dots, e_n$ as in Proposition 6.1, the solution exists (by the assumption of linear independence of the v_i) and is unique (because of when equality holds in Hölder's inequality for $1 < q < \infty$). To apply Corollary 6.5, let T be the identity operator.

Corollary 6.7. *When considering* $E_{p\infty}(\sigma)$ *for* $\sigma > 0$, *it is necessary to consider only those* $g \in L^\infty$ *for which*

(6.25)
$$\int_\Omega g(y)v(y)dy = \|g\|_\infty \|v\|_1$$

for some $v \in V$, $v \neq 0$. *Furthermore it suffices to consider only those* g *satisfying* (6.25) *and such that* $|g(y)| = \|g\|_\infty$ *for all* $y \in \Omega$.

Proof. Equation (6.25) is simply (6.5). The latter statement of the corollary is a consequence of Liapounoff's Theorem which implies the following:

Given any $g_0 \in L^\infty(\Omega)$, there exists a $\widetilde{g} \in L^\infty(\Omega)$ satisfying

1. $\int_\Omega \widetilde{g}(y)v(y)dy = \int_\Omega g_0(y)v(y)dy$ for all $v \in V$.
2. $|\widetilde{g}(y)| = \|\widetilde{g}\|_\infty$ for all $y \in \Omega$.
3. $\|\widetilde{g}\|_\infty \leq \|g_0\|_\infty$.

It is also worth noting that if each $v \in V$, $v \neq 0$, has a zero set of measure zero, then (6.25) uniquely defines (a.e.) the g therein. □

Finally in the case $q = 1$, we have

Corollary 6.8. *When considering* $E_{p1}(\sigma)$ *for* $\sigma > 0$, *it is necessary to consider only those* $\mu \in C^*(\Omega)$ *for which*

(6.26)
$$\int_{\Omega} v(y)d\mu(y) = \|v\|_{\infty}\|\mu\|_{T.V.}$$

for some $v \in V$, $v \neq 0$. *Furthermore, it suffices to consider* μ *satisfying* (6.26) *with at most* n *points of support. That is,* μ *of the form*

(6.27)
$$d\mu = \sum_{i=1}^{k} \lambda_i (\mathrm{sgn}\, v(y_i))\delta_{y_i}$$

where $k \leq n$, $\lambda_i > 0$, $i = 1, \ldots, k$, *the* v *is as in* (6.26), *the* y_i*'s are in*

$$A = \{y : |v(y)| = \|v\|_{\infty}\}$$

for this same v, *and* δ_{yi}, *is the dirac-delta measure at* y_i.

Proof. For given $e = (e_1, \ldots, e_n)$, let μ^* satisfy $\int_{\Omega} v_i(y)d\mu^*(y) = e_i$, $i = 1, \ldots, n$, and

$$\|\mu^*\|_{T.V.} = \min_{\substack{\int v_i(y)d\mu(y) = e_i \\ i=1,\ldots,n}} \|\mu\|_{T.V.}$$

Assume $v^* \in V$, $v^* \neq 0$, is such that

$$\int_{\Omega} v^*(y)d\mu^*(y) = \|v^*\|_{\infty}\|\mu\|^*_{T.V.}$$

Set

$$A = \{y : |v^*(y)| = \|v^*\|_\infty\}.$$

Thus $d\tilde{\mu}(y) = \operatorname{sgn}(v^*(y))d\mu^*(y)$ is a nonnegative measure with support in A. Let B be the convex cone generated by the set

$$\{(v_1(y)\operatorname{sgn}(v^*(y)), \dots, v_n(y)\operatorname{sgn}(v^*(y))) : y \in A\}.$$

Since

$$\int_\Omega v_i(y)\operatorname{sgn}(v^*(y))d\tilde{\mu}(y) = e_i, \quad i = 1, \dots, n,$$

it follows that $\mathbf{e} \in B$. Thus from Caratheodory's Theorem, there exists a measure $\hat{\mu}$ satisfying $\int_\Omega v_i(y)d\hat{\mu}(y) = e_i$, $i = 1, \dots, n$, of the form (6.27), but with $k \le n+1$. A standard linear algebra argument shows that we can assume that $k \le n$. It remains to prove that $\|\hat{\mu}\|_{T.V.} = \|\mu^*\|_{T.V.}$. Using (6.27) it easily follows that $\|\hat{\mu}\|_{T.V.} = \Sigma_{i=1}^k \lambda_i$. Thus

$$\int_\Omega v^*(y)d\hat{\mu}(y) = \|v^*\|_\infty \|\hat{\mu}\|_{T.V.},$$

and (6.5) holds. Our corollary is proved. □

Remark 6.2. Not every atomic measure with at most n atoms satisfies (6.26) for some $v \in V$, $v \ne 0$. However from (6.27) it does follow that

$$(6.28) \qquad E_{p1}(\sigma) = \min_{a_j, y_j} \left(\sum_{i=1}^n |f_i - \sum_{j=1}^n a_j v_i(y_j)|^p \right)^{1/p} + \sigma \sum_{j=1}^n |a_j|.$$

In the previous section we delineated very explicitly, in the simple case considered therein, the values of $\tilde{\sigma}, \hat{\sigma}$ and how to obtain g^{σ}. We could do much the same here, albeit not quite as successfully. For example, for $1 < p < \infty$, $1 < q < \infty$,

$$\tilde{\sigma} = \frac{\|\Sigma_{i=1}^{n} h_i v_i\|_{q'}}{\|\mathbf{h}\|_{p'}}$$

where the h_i are uniquely determined as solutions to

$$f_i = \int_{\Omega} v_i(y) \mid \sum_{j=1}^{n} h_j v_j(y) \mid^{q'-1} \operatorname{sgn}\left(\sum_{j=1}^{n} h_j v_j(y)\right) dy, \quad i = 1, \dots, n.$$

(See Corollary 6.6)

Certain particular cases do stand out and we will here consider some of these cases and some of their properties in more detail.

Case 1. $n = 1$.

Since $n = 1$, the problem is totally independent of p. Moreover, it is easily checked that for all $1 \le q \le \infty$,

$$\tilde{\sigma} = \hat{\sigma} = \|v_1\|_{q'}.$$

Case 2. $q = 2$.

From Corollary 6.6, in the determination of solutions for $E_{p2}(\sigma)$, we should be considering only g in $V = \operatorname{span}\{v_1, \dots, v_n\}$. Thus,

$$E_{p2}(\sigma) = \min_{g \in L^2} \left(\sum_{i=1}^{n} |f_i - \int_{\Omega} v_i(y) g(y) dy|^p \right)^{1/p} + \sigma \|g\|_2$$

$$= \min_{b_1, \dots, b_n} \left(\sum_{i=1}^{n} |f_i + \sum_{j=1}^{n} b_i \int_\Omega v_i(y) v_j(y) dy|^p \right)^{1/p} + \sigma \left(\sum_{j,k=1}^{n} b_j b_k \int_\Omega v_j(y) v_k(y) dy | \right)^{1/2}.$$

Let C denote the $n \times n$ Grammian matrix given by $C = (c_{ij})$,

$$c_{ij} = \int_\Omega v_i(y) v_j(y) dy, \quad i, j = 1, \dots, n.$$

Thus

$$E_{p2}(\sigma) = \min_{\mathbf{b}} \|\mathbf{f} - C\mathbf{b}\|_p + \sigma(\mathbf{b}, C\mathbf{b})^{1/2}.$$

By our assumption of this section, C is nonsingular. Furthermore C is a Grammian matrix and hence positive definite. Let D denote the square root of C (which exists). Then

$$E_{p2}(\sigma) = \min_{\mathbf{a}} \|\mathbf{b} - D\mathbf{a}\|_p + \sigma\|\mathbf{a}\|_2.$$

For $p = 2$, this problem was considered in Section 4. For $p \neq 2$, we can get quite a bit of information from the theory developed in the last few sections, but nothing of particular interest to us here. If however the $\{v_i\}_{i=1}^n$ are orthogonal, i.e. $\int_\Omega v_i(y) v_j(y) dy = c_i \delta_{ij}$, $i, j = 1, \dots, n$, then both C and D are diagonal matrices. In other words, we have reduced our problem to a particular case of the results of Section 5.

Case 3. $q = 1$, $\hat{\sigma}$ and Descartes systems.

From Remark 6.2, we have that for each $1 \le p \le \infty$,

$$E_{pi}(\sigma) = \min_{a_j, y_j} \left(\sum_{i=1}^{n} |f_i - \sum_{j=1}^{n} a_j v_i(y_j)|^p \right)^{1/p} + \sigma \sum_{j=1}^{n} |a_j| .$$

We show how to calculate $\hat{\sigma}$ for a class of $v_1, \ldots, v_n$, $\mathbf{f}$, and all $1 \le p \le \infty$. We recall that

$$(6.29)\quad \hat{\sigma} = \max \left\{ \frac{\|\Sigma_{i=1}^{n} h_i v_i\|_\infty}{\|\mathbf{h}\|_{p'}} : \int_\Omega \sum_{i=1}^{n} h_i v_i(y) d\mu^*(y) = \|\sum_{i=1}^{n} h_i v_i\|_\infty \|\mu^*\|_{T.V.} \right\}$$

where μ^* satisfies

$$(6.30)\quad \begin{aligned} &\text{a) } f_i = \int_\Omega v_i(y) d\mu^*(y), \quad i = 1, \ldots, n \\ &\text{b) } \|\mu^*\|_{T.V.} \le \|\mu\|_{T.V.} \text{ for all } \mu \in C^*(\Omega) \text{ satisfying } (a). \end{aligned}$$

This is not in general a simple problem.

We make two assumptions. Let us assume that $(-1)^i f_i > 0$, $i = 1, \ldots, n$, and $\{v_1, \ldots, v_n\}$ is a Descartes system on $\Omega = [a, b] \subseteq \mathbb{R}$. By a Descartes system we mean that for every choice of $1 \le j_1 < \cdots < j_k \le n$ and $k \in \{1, \ldots, n\}$, the functions $v_{j_1}, \ldots, v_{j_k}$ form a positively oriented Chebyshev (T-) system on $[a, b]$. That is, for every $a \le x_1 < \cdots < x_k \le b$,

$$\det\,(v_{j_\ell}(x_m))_{\ell, m=1}^{k} > 0.$$

(These assumptions may be considerably weakened in what follows.)

Under these assumptions it is readily verified that there exists a unique function of the form $v^* = \Sigma_{i=1}^{n} b_i^* v_i$, and points $a \le y_1 < \cdots < y_n \le b$ (not necessarily unique) such that

$$(-1)^j v^*(y_j) = \|v^*\|_\infty = 1, \quad j = 1, \dots, n.$$

Furthermore, since the $\{v_1, \dots, v_n\}$ form a Descartes system, $(-1)^i b_i^* > 0$, $i = 1, \dots, n$. Let $\{a_j^*\}_{j=1}^n$ satisfy

$$f_i = \sum_{j=1}^n a_j^* v_i(y_j), \quad i = 1, \dots, n,$$

where the $\{y_j\}_{j=1}^n$ are as above. Since $(-1)^i f_i > 0$, $i = 1, \dots, n$, we may again deduce that $(-1)^j a_j^* > 0, j = 1, \dots, n$. Thus the measure

$$d\mu^* = \sum_{j=1}^n a_j^* \delta_{y_j}$$

satisfies

$$\text{a) } f_i = \int_a^b v_i(y) d\mu^*(y), \quad i = 1, \dots, n$$

$$\text{b) } \int_a^b v^* \, d\mu^* = \left(\sum_{j=1}^n | a_j^* | \right) \|v^*\|_\infty = \|\mu^*\|_{T.V.} \|v^*\|_\infty$$

and therefore from Corollary 6.8,

$$\sum_{j=1}^n | a_j^* | = \|\mu^*\|_{T.V.} = \min_{\substack{f_i = \int_a^b v_i(y) d\mu(y) \\ i = 1, \dots, n}} \|\mu\|_{T.V.}$$

We now prove :

Proposition 6.9. *Under the above assumptions,*

$$\hat{\sigma} = \frac{1}{\|\mathbf{b}^*\|_{p'}} .$$

Remark 6.3. Note that the $\mathbf{b}^*$ is independent of p and the particular $\mathbf{f}$ satisfying our assumptions.

Proof. It easily follows that the maximum (6.29) (over $\mathbf{h}$) does not depend on the particular choice of the μ satisfying *(a)* and *(b)* of (6.30). Thus let us choose μ^* as above. For this μ^*, the unique (up to multiplication by a positive constant) $v \in \text{span}\{v_1, \dots, v_n\}$, $v \neq 0$, satisfying

$$\int_a^b v(y)d\mu^*(y) = \|v\|_\infty \|\mu^*\|_{T.V.}$$

is the above $v^* = \Sigma_{i=1}^n b_i^* v_i$ since $v^*(y_j) = \text{sgn}(a_j)$, $j = 1, \dots, n$, and $\|v^*\|_\infty = 1$. (The interpolation conditions uniquely determine v^*). Substitute in (6.29) and our result follows. □

The above vector $\mathbf{b}_1^*$ has an additional property well worth noting. We state it in this next proposition.

Proposition 6.10. *Let* $\mathbf{b}^* = (b_i^*, \dots, b_n^*)$ *be as above. Then for any* $\mathbf{b} = (b_1, \dots, b_n) \neq 0$,

$$\frac{\|\Sigma_{i=1}^n b_i v_i\|_\infty}{\|\mathbf{b}\|_{p'}} \leq \frac{\|\Sigma_{i=1}^n b_i^* v_i\|_\infty}{\|\mathbf{b}^*\|_{p'}}$$

for every $1 \leq p' \leq \infty$.

Proof. Assume $\|\Sigma_{i=1}^n b_i^* v_i\|_\infty > \|\Sigma_{i=1}^n b_i v_i\|_\infty$. Thus for any choice of $\pm$, we have

$$(-1)^j \sum_{i=1}^{n} (b_i^* \pm b_i)\, v_i(y_j) > 0, \quad j = 1, \dots, n.$$

Since $\{v_1, \dots, v_n\}$ is a Descartes system, this implies that

$$(-1)^i (b_i^* \pm b_i) > 0, \quad i = 1, \dots, n.$$

Therefore $|b_i^*| > |b_i|$, $i = 1, \dots, n$. Hence if $\|\Sigma_{i=1}^n b_i^* v_i\|_\infty > \|\Sigma_{i=1}^n b_i v_i\|_\infty$ then $\|\mathbf{b}^*\|_{p'} > \|\mathbf{b}\|_{p'}$. This implies our result. □

B. Let $u_1, \dots, u_n \in C(\Omega)$, where Ω is a compact subset of $\mathbb{R}^n$. For $1 \le p \le \infty$, $1 \le q \le \infty$, $f \in C(\Omega)$ and $\sigma > 0$, define

$$E_{pq}(\sigma) = \min_{\mathbf{a} = (a_1, \dots, a_n)} \|f - \sum_{i=1}^{n} a_i u_i\|_p + \sigma \|\mathbf{a}\|_q, \tag{6.31}$$

where $L^p = L^p(\Omega)$ is the usual L^p-space with associated Lebesgue measure on Ω, and $\|\cdot\|_q$ denotes the ℓ_q^n norm on $\mathbb{R}^n$. Assumption I holds, and it may be shown that the dual problem for $1 \le p < \infty$, $1 \le q \le \infty$ is

$$\max_{\substack{\|h\|_{p'} \le 1 \\ \|((u_1, h), \dots, (u_n, h))\|_{q'} \le \sigma}} (f, h), \tag{6.32}$$

while for $p = \infty$, $1 \le q \le \infty$, the dual problem is given by

$$\max_{\substack{\|v\|_{T.V.} \le 1 \\ \|((u_1, v), \dots, (u_n, v))\|_{q'} \le \sigma}} (f, v), \tag{6.33}$$

where both $((u_1, h), \ldots, (u_n, h))$ and $((u_1, v), \ldots, (u_n, v))$ are vectors in $\mathbb{R}^n$. For a precise proof of the equivalence of (6.31) and (6.32) and (6.33), see Section 7. Problem (6.31) is a very natural problem. Find a best approximation from the finite dimensional subspace span$\{u_1, \ldots, u_n\}$ where there is a penalty involved in terms of the size of the coefficients. For $1 < p < \infty$, $1 < q < \infty$, we know that $\mathbf{a} = (a^*_1, \ldots, a^*_n)$ is a solution to $E_{pq}(\sigma)$ if and only if

$$\int_\Omega |f - \sum_{i=1}^n a_i^* u_i|^{p-1} \operatorname{sgn}(f - \sum_{i=1}^n a_i^* u_i) u_k = \lambda(\sigma) \, | a_k^* |^{q-1} \operatorname{sgn} a_k^*, \quad k = 1, \ldots, n,$$

for some $\lambda(\sigma) \ge 0$. Since this problem is generally intractable (for the case $p = q = 2$, see Section 4), we will not deal with it at all.

In the next section we consider a framework which is in a certain sense more general (although in order to keep things simple, we do not there provide this general framework). In particular we there demand that the kernel K be strictly totally positive. The analogue in this setting is the demand that $\{u_1, \ldots, u_n\}$ form a Descartes system. The major results of Section 7 have analogues in this setting. As such we will not present them here. It should, however, be mentioned that the descrete analogue to Theorem 7.1, central to the development of the results in Case 1 of Section 7, may be found in Pinkus, Strauss [21, Theorem 4.2].

7. Totally Positive Kernels

We consider in this section the particular case of our general problem where the operator T^* is given by an integral operator whose kernel is strictly totally positive (STP). Before dealing with the specifics of this problem, we first present the general framework.

We let $[a, b]$ or $[c, d]$ denote closed nontrivial intervals of $\mathbb{R}$. For $1 \le p \le \infty$, L^p will be the usual space with associated Lebesgue measure on either $[a, b]$ or $[c, d]$. As previously, for any interval $[a, b]$, $C^*[a, b]$ will denote the dual space of $C[a, b]$ which we identify as all Borel measures μ on $[a, b]$ normed with total variation $\|\mu\|_{T.V.}$.

Let $K(x, y)$ be a jointly continuous kernel for $(x, y) \in [a, b] \times [c, d]$, i.e., $K \in C([a, b] \times [c, d])$. For functions h and g defined on $[a, b]$ and $[c, d]$, respectively, set

$$(Th)(y) = (K^T h)(y) = \int_a^b K(x, y)h(x)dx$$

and

$$(T^*g)(x) = (Kg)(x) = \int_c^d K(x, y)g(y)dy.$$

(K^T here denotes the transpose of K.) For $\nu \in C^*[a, b]$ and $\mu \in C^*[c, d]$, we set

$$(T\nu)(y) = (K^T \nu)(y) = \int_a^b K(x, y)d\nu(x)$$

and

$$(T^*\mu)(x) = (K\mu)(x) = \int_c^d K(x, y)d\mu(y).$$

In what follows, we also assume that $f \in C[a, b]$.

From Theorem 3.1, and since Assumption I holds (as may be shown), we get the following.

a) Set $X = L^{p'}$, $Y = L^{q'}$, for $1 < p < \infty$, $1 < q \leq \infty$. (Note that X is a dual space.) Then for $\sigma > 0$

$$\max_{\substack{\|h\|_{p'} \leq 1 \\ \|K^T h\|_{q'} \leq \sigma}} (f, h) = \min_{g \in L^q} \|f - Kg\|_p + \sigma\|g\|_q.$$

b) Set $X = L^{p'}$, $Y = C[c, d]$, for $1 < p < \infty$. Then for $\sigma > 0$

$$\max_{\substack{\|h\|_{p'} \leq 1 \\ \|K^T h\|_\infty \leq \sigma}} (f, h) = \min_{\mu \in C^*[a, b]} \|f - K\mu\|_p + \sigma\|\mu\|_{T.V.}$$

c) Let $X = L^\infty$, $Y = L^{q'}$, $1 < q \leq \infty$. Then for $\sigma > 0$,

$$\max_{\substack{\|h\|_\infty \leq 1 \\ \|K^T h\|_{q'} \leq \sigma}} (f, h) = \min_{g \in L^q} \|f - Kg\|_{(L^\infty)^*} + \sigma\|g\|_q.$$

Now $f - Kg \in C[a, b]$, and therefore

$$\|f - Kg\|_{(L^\infty)^*} = \|f - Kg\|_1.$$

Thus

$$\max_{\substack{\|h\|_\infty \leq 1 \\ \|K^T h\|_{q'} \leq \sigma}} (f, h) = \min_{g \in L^q} \|f - Kg\|_1 + \sigma\|g\|_q.$$

d) Let $X = L^\infty$, $Y = C[c, d]$. Combining (b) and (c), we get for $\sigma > 0$

$$\max_{\substack{\|h\|_\infty \leq 1 \\ \|K^T h\|_\infty \leq \sigma}} (f, h) = \min_{\mu \in C^*[c,d]} \|f - K\mu\|_1 + \sigma\|\mu\|_{T.V.}$$

e) Let $X = C^*[a, b]$, $Y = L^{q'}$, $1 < q \leq \infty$. Then for $\sigma > 0$

$$\max_{\substack{\|v\|_{T.V.} \le 1 \\ \|K^T v\|_{q'} \le \sigma}} (f, v) = \min_{g \in L^q} \|f - Kg\|_{C^{**}[a,b]} + \sigma \|g\|_q .$$

As in (c), $f - Kg \in C[a, b]$ and therefore

$$\|f - Kg\|_{C^{**}[a,b]} = \|f - Kg\|_\infty .$$

Thus

$$\max_{\substack{\|v\|_{T.V.} \le 1 \\ \|K^T v\|_{q'} \le \sigma}} (f, v) = \min_{g \in L^q} \|f - Kg\|_\infty + \sigma \|g\|_q .$$

f) Let $X = C^*[a, b]$, $Y = C[c, d]$. Combining (b) and (e), we get for $\sigma > 0$,

$$\max_{\substack{\|v\|_{T.V.} \le 1 \\ \|K^T v\|_\infty \le \sigma}} (f, v) = \min_{\mu \in C^*[c,d]} \|f - K\mu\|_\infty + \sigma \|\mu\|_{T.V.}$$

As usual, we set

$$E_{pq}(\sigma) = \min_{g \in L^q} \|f - Kg\|_p + \sigma \|g\|_q$$

for all $p, q \in [1, \infty]$, where if $q = 1$ we really mean

$$E_{p1}(\sigma) = \min_{\mu \in C^*[c,d]} \|f - K\mu\|_p + \sigma \|\mu\|_{T.V.}$$

Certain facts are immediate consequences of previous results. For example, from Proposition 3.3, if $1 < p < \infty$, then

$$\tilde{\sigma} = \frac{\|K^T \tilde{h}\|_{q'}}{\|\tilde{h}\|_{p'}}$$

where $\tilde{h}(x) = |f(x)|^{p-1} \operatorname{sgn}(f(x))$. Similarly if $1 < p < \infty$, $1 < q < \infty$ and $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, then the solution $g_\sigma \in L^q$ to $E_{pq}(\sigma)$ is unique (Proposition 3.10), and is uniquely given by a solution to

$$K^T(|f - Kg_\sigma|^{p-1} \operatorname{sgn}(f - Kg_\sigma))(y) = \lambda |g_\sigma(y)|^{q-1} \operatorname{sgn}(g_\sigma(y))$$

a.e. on $[c, d]$ for a specific choice of $\lambda > 0$ (see Proposition 3.6) depending on σ.

In what follows we assume that the kernel K is STP. That is, for every $a \leq x_1 < \cdots < x_n \leq b$, $c \leq y_1 < \cdots < y_n \leq d$ and $n = 1, 2, \ldots$

$$\det(K(x_i, y_j))_{i,j=1}^n > 0.$$

Much of the theory of STP kernels may be found in Karlin [13]. The two main facts connected with STP kernels which we shall use are the following.

a) For any $h \in L^1[c, d]$,

$$\tilde{Z}(Kh) \leq S(h)$$

where $\tilde{Z}(f)$ is the number of zeros of $f(f \in C[a, b])$ on $[a, b]$, with the convention that a zero of f in (a, b) is counted twice if f does not change sign at that zero. $S(h)$ is the number of essential sign changes of the function h. This is called the *variation diminishing property*.

b) If

$$\widetilde{Z}(Kh) = S(h),$$

and h is positive near d, then $(Kh)(b) > 0$, or $(Kh)(b) = 0$ and $(Kh)(b - \varepsilon) < 0$ for all $\varepsilon > 0$ sufficiently small. We then say that Kh and h have the same sign *orientation*.

Thus, for example, for the g_σ as defined above, we get from (a) that $\widetilde{Z}(Kg_\sigma) \le S(g_\sigma) \le \widetilde{Z}(g_\sigma) \le S(f - Kg_\sigma)$ for each $\sigma \in (\hat{\sigma}, \widetilde{\sigma})$.

There are various motivations for the study of this problem for K STP. We mention one. For $p \in [1, \infty]$, $q \in (1, \infty]$, consider the problem

$$\min_{s \in W_q^n[a,b]} \|f - s\|_p + \sigma \|s^{(n)}\|_q$$

where $W_q^n[a, b] = \{s: s \in C^{(n-1)}[a, b], s^{(n)} \in L^q\}$. Each $s \in W_q^n[a, b]$ may be written in the form

$$s(x) = \sum_{i=0}^{n-1} \frac{s^{(i)}(a)(x - a)^i}{i!} + \frac{1}{(n-1)!} \int_a^b (x - y)_+^{n-1} s^{(n)}(y)dy.$$

We can rewrite this as

$$s(x) = \sum_{i=0}^{n-1} a_i u_i(x) + \int_a^b K(x, y)g(y)dy,$$

with the obvious identifications. The difference between this and Kh for K STP is the presence of the $u_0, \dots, u_{n-1}$, and the fact that the kernel $K(x,y) = (x - y)_+^{n-1}/(n-1)!$ is not STP, but only TP (totally positive). However to a large extent both the differences are technical. By considering Kh with K STP, we are simplifying our situation in order to study and highlight the essential features. Almost all the results go over to the Sobolev space setting with at most minor modifications.

Case 1. For $f \in C[a, b]$ and $\sigma > 0$, set

$$E(\sigma) = (E_{\infty\infty\infty}(\sigma) =) \min_{g \in L^\infty} \|f - Kg\|_\infty + \sigma\|g\|_\infty. \tag{7.1}$$

From (e), we have that

$$E(\sigma) = \min_{\substack{\|\mu\|_{T.V.} \leq 1 \\ \|K^T\mu\|_1 \leq \sigma}} (f, \mu). \tag{7.2}$$

The following result was proved in Pinkus [19].

Theorem 7.1. *Let K be STP, $f \in C[a, b]$, and $\tau > 0$. Assume $f \notin A_\tau = \{Kg\colon \|g\|_\infty \leq \tau\}$. Then there exists a unique solution g^τ to*

$$\min\{\|f - Kg\|_\infty\colon \|g\|_\infty \leq \tau\}. \tag{7.3}$$

It is uniquely characterized as follows:

There exists a nonnegative integer $m(\tau)$, points (knots)

$$c = \xi_0 < \xi_1 < \cdots < \xi_{m(\tau)} < \xi_{m(\tau)+1} = d$$

and points

$$a \leq \eta_1 < \cdots \leq \eta_{m(\tau)+1} \leq b$$

(all dependent on τ), and some $\varepsilon \in \{-1, 1\}$ such that

$$(7.4) \quad \begin{aligned} &a)\ \varepsilon(-1)^i g^\tau(y) = \|g^\tau\|_\infty = \tau \text{ a.e. for } y \in (\xi_{i-1}, \xi_i), \quad i = 1, \ldots, m(\tau)+1 \\ &b)\ \varepsilon(-1)^i (f - Kg^\tau)(\eta_i) = \|f - Kg^\tau\|_\infty, \quad i = 1, \ldots, m(\tau)+1. \end{aligned}$$

As noted in Section 2, the two problems (7.1) and (7.3) are much related. The solution g_σ to $E(\sigma)$ is equal to g^τ of Theorem 7.1 for $\tau = \|g_\sigma\|_\infty$. Furthermore g^τ also solves $E(\sigma)$ for some σ. It is worth doing the short analysis involved as there is a point we wish to highlight. Let g^τ be as in Theorem 7.1 with associated $\{\xi_i\}_{i=1}^{m(\tau)}$ and $\{\eta_i\}_{i=1}^{m(\tau)+1}$. Since K is STP, there exists a unique set of coefficients $\{a^*_i\}_{i=1}^{m(\tau)+1}$ satisfying

$$(7.5) \quad \begin{aligned} &a)\ \left[\sum_{i=1}^{m(\tau)+1} a_i^* K(\eta_i, y)\right] g^\tau(y) \geq 0 \text{ a.e. } y \in [c, d] \\ &b)\ \sum_{i=1}^{m(\tau)+1} |a_i^*| = 1 \\ &c)\ \varepsilon(-1)^i a_i^* > 0, \quad i = 1, \ldots, m(\tau)+1. \end{aligned}$$

Define $\mu^* \in C^*[a, b]$ by

$$(7.6) \qquad d\mu^* = \sum_{i=1}^{m(\tau)+1} a_i^* \delta_{\eta_i}.$$

Then $\|\mu^*\|_{T.V.} = \sum_{i=1}^{m(\tau)+1} |a_i^*| = 1$, and from (7.4) and (7.5),

$$(f - Kg^\tau, \mu^*) = \|f - Kg^\tau\|_\infty \|\mu^*\|_{T.V.}$$

and

$$(g^{\tau}, K^{T}\mu^*) = \|g^{\tau}\|_{\infty} \|K^{T}\mu^*\|_1.$$

Thus from Proposition 3.6, we get that g^{τ} solves (7.1) for $\sigma = \|K^T\mu^*\|_1$. In addition, it follows that $\hat{\sigma} \le \sigma \le \tilde{\sigma}$.

Now, it need not be that the $\{\eta_i\}_{i=1}^{m(\tau)+1}$ are uniquely defined for a given τ. That is, $f - Kg^{\tau}$ may attain its norm at more than $m(\tau) + 1$ points. In this case the μ^* of (7.6) is not uniquely defined. As such there may exist a range of values for $\|K^T\mu^*\|_1$ where μ^* is associated with a fixed g^{τ}, as above. This range of values is necessarily an interval. On this interval the same g^{τ} is a solution to $E(\sigma)$, and thus $E(\sigma)$ is linear. This corresponds to a lack of differentiability at a point of the associated Gagliardo diagram. On the other hand, it may be that distinct τ give rise to the same μ^*. In this case we will have nonuniqueness of the solution to (7.1) for the associated σ. We discuss this point later in some detail.

Let us now consider the relationship between σ, τ and $m(\tau)$ for $\sigma \in (\hat{\sigma}, \tilde{\sigma})$. From Proposition 3.2 (iv), if $0 < \sigma_1 < \sigma_2 < \infty$, then

$$\text{(7.7)} \qquad \begin{array}{l} a)\ \|g_{\sigma_1}\|_{\infty} \ge \|g_{\sigma_2}\|_{\infty} \\ b)\ \|f - Kg_{\sigma_1}\|_{\infty} \le \|f - Kg_{\sigma_2}\|_{\infty}. \end{array}$$

Let $\hat{\sigma} < \sigma_1 < \sigma_2 < \tilde{\sigma}$. Denote by g_{σ_i} a solution to (7.1) for $\sigma = \sigma_i$, $i = 1, 2$. By our previous remarks, $g_{\sigma_i} = g^{\tau_i}$, where $\tau_i = \|g_{\sigma_i}\|_{\infty}$, $i = 1, 2$. Thus from (7.7) (a) $\tau_1 \ge \tau_2$. Associated with each $g_{\sigma_i} = g^{\tau_i}$ is the number $m(\tau_i)$ as given in (7.4). That is, the number of "knots" of g^{τ_i}.

Proposition 7.2. *For* $\hat{\sigma} < \sigma_1 < \sigma_2 < \tilde{\sigma}$, $m(\tau_1) \ge m(\tau_2)$.

Proof. Assume $g_{\sigma_1} \ne g_{\sigma_2}$. Since $\|g_{\sigma_1}\|_{\infty} \ge \|g_{\sigma_2}\|_{\infty}$ and $|g_{\sigma_1}(y)| = \|g_{\sigma_1}\|_{\infty}$ for all y, with $S(g_{\sigma_1}) = m(\tau_1)$, we have that

$$S(g_{\sigma_1} - g_{\sigma_2}) \le m(\tau_1).$$

Now $\|f - Kg_{\sigma_2}\|_\infty \geq \|f - Kg_{\sigma_1}\|_\infty$, and from (7.4)(b), $f - Kg_{\sigma_2}$ attains its norm, alternately, on at least $m(\tau_2) + 1$ points. Thus

$$\widetilde{Z}((f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})) \geq m(\tau_2).$$

Thus

$$(7.8) \quad m(\tau_2) \leq \widetilde{Z}((f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})) = \widetilde{Z}(K(g_{\sigma_1} - g_{\sigma_2})) \leq S(g_{\sigma_1} - g_{\sigma_2})) \leq m(\tau_1).$$

This proves the proposition. □

A more detailed analysis provides us with this next result.

Proposition 7.3. *Assume $m(\tau_1) = m(\tau_2)$, $\hat{\sigma} < \sigma_1 < \sigma_2 < \widetilde{\sigma}$, and $g_{\sigma_1} \neq g_{\sigma_2}$. Then*

1) $\tau_1 = \|g_{\sigma_1}\|_\infty > \|g_{\sigma_2}\|_\infty = \tau_2$,

2) $\|f - Kg_{\sigma_1}\|_\infty < \|f - Kg_{\sigma_2}\|_\infty$,

3) *g_{σ_1} and g_{σ_2} have the same orientation,*

4) *$f - Kg_{\sigma_2}$ attains its norm, alternately, at exactly $m(\tau_2) + 1$ points.*

Proof. Both (1) and (2) are immediate consequences of the uniqueness of the solution in (7.3). We know that $\tau_1 \geq \tau_2$. If $\tau_1 = \tau_2$, then g_{σ_1} and g_{σ_2} are distinct solutions to (7.3) for the same τ. Thus (1) holds. From (7.7) (b), $\|f - Kg_{\sigma_1}\|_\infty \leq \|f - Kg_{\sigma_2}\|_\infty$. If equality holds, then g_{σ_1} and g_{σ_2} would again both be solutions to (7.3) for $\tau = \tau_1$. This proves (2).

To prove (4), note that equality holds in (7.8), and thus $f - Kg_{\sigma_2}$ cannot attain its norm, alternately, at more than $m(\tau_2) + 1$ points.

Equality in (7.8) also implies that $K(g_{\sigma_1} - g_{\sigma_2})$ and $g_{\sigma_1} - g_{\sigma_2}$ have the same sign orientation. The sign orientation of $K(g_{\sigma_1} - g_{\sigma_2}) = (f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})$ is determined by the sign orientation of $f - Kg_{\sigma_2}$, and this, by (7.4), is in turn determined by the sign pattern of g_{σ_2}. The sign orientation of $g_{\sigma_1} - g_{\sigma_2}$ is determined by the sign pattern of g_{σ_1}. As such a little bookkeeping shows that g_{σ_1} and g_{σ_2} have the same orientation. This proves (3). □

For fixed $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, the above analysis does *not* imply that there is a unique solution to (7.1). Nonuniqueness may occur. But only when the following is satisfied.

Proposition 7.4. *Let $\sigma \in (\hat{\sigma}, \tilde{\sigma})$. Assume g_σ^1 and g_σ^2 are two distinct solutions to $E(\sigma)$ ((7.1)). Then*

1) $g_\sigma^1(y) = \alpha g_\sigma^2(y)$ *a.e. y for some $\alpha > 0$.*

2) *If $S(g_\sigma^1) = S(g_\sigma^2) = m$, then there exist $a \leq \eta_1 < \cdots < \eta_{m+1} \leq b$ and $\varepsilon \in \{-1, 1\}$ such that*

$$\varepsilon(-1)^i (f - Kg_\sigma^j)(\eta_i) = \|f - Kg_\sigma^j\|_\infty$$

for $i = 1, \ldots, m+1$ and $j = 1, 2$.

3)

$$\sigma = \frac{\varepsilon(-1)^i (Kg_\sigma^j)(\eta_i)}{\|g_\sigma^j\|_\infty}, \quad i = 1, \ldots, m+1;\ j = 1, 2.$$

Proof. Let $\|g_\sigma^j\|_\infty = \tau_j$, $j = 1, 2$. Since (7.3) has a unique solution and $g_\sigma^1 \neq g_\sigma^2$, it follows that $\tau_1 \neq \tau_2$.

Set $g_\lambda = \lambda g_\sigma^1 + (1 - \lambda) g_\sigma^2$ for any fixed $\lambda \in (0, 1)$. Then

$$
\begin{aligned}
E(\sigma) &\le \|f - Kg_\lambda\|_\infty + \sigma\|g_\lambda\|_\infty \\
&\le \lambda(\|f - Kg_\sigma^1\|_\infty + \sigma\|g_\sigma^1\|_\infty) + (1-\lambda)(\|f - Kg_\sigma^2\|_\infty + \sigma\|g_\sigma^2\|_\infty) \\
&= E(\sigma).
\end{aligned}
$$

Equality implies that g_λ is also a solution to $E(\sigma)$. Thus

$$\|\lambda g_\sigma^1 + (1-\lambda)g_\sigma^2\|_\infty = \lambda\|g_\sigma^1\|_\infty + (1-\lambda)\|g_\sigma^2\|_\infty \tag{7.9}$$

and

$$\|\lambda(f - Kg_\sigma^1) + (1-\lambda)(f - Kg_\sigma^2)\|_\infty = \lambda\|f - K_\sigma^1\|_\infty + (1-\lambda)\|f - Kg_\sigma^2\|_\infty. \tag{7.10}$$

Since g_λ is a solution to $E(\sigma)$ for $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, it is also a solution to (7.3), and thus $|g_\lambda(y)| = \|g_\lambda\|_\infty$ a.e. y. This fact together with (7.9), and the form of g_σ^1 and g_σ^2, implies that

$$g_\sigma^1(y) = \alpha g_\sigma^2(y), \quad \text{a.e. } y$$

for $\alpha = \|g_\sigma^1\|_\infty / \|g_\sigma^2\|_\infty > 0$.

Let $S(g_\lambda) = m(= S(g_\sigma^1) = S(g_\sigma^2))$. There then exist $a \le \eta_1 < \cdots < \eta_{m+1} \le b$ and $\varepsilon \in \{-1, 1\}$ (determined as in Theorem 7.1) such that

$$\varepsilon(-1)^i(f - Kg_\lambda)(\eta_i) = \|f - Kg_\lambda\|_\infty$$

for all $i = 1, \dots, m+1$. From this fact and (7.10) we get

$$\varepsilon(-1)^i(f - Kg_\sigma^j)(\eta_i) = \|f - Kg_\sigma^j\|_\infty$$

for $i = 1, \ldots, m+1$ and $j = 1, 2$.

Finally

$$\begin{aligned}\sigma[\|g_\sigma^2\|_\infty - \|g_\sigma^1\|_\infty] &= \|f - Kg_{1\sigma}\|_\infty - \|f - Kg_\sigma^2\|_\infty \\ &= \varepsilon(-1)^i[(f - Kg_\sigma^1)(\eta_i) - (f - Kg_\sigma^2)(\eta_i)] \\ &= \varepsilon(-1)^i K(g_\sigma^2 - \alpha g_\sigma^2)(\eta_i) \\ &= \left[\frac{\|g_\sigma^2\|_\infty - \|g_\sigma^1\|_\infty}{\|g_\sigma^2\|_\infty}\right]\varepsilon(-1)^i Kg_\sigma^2(\eta_i)\end{aligned}$$

and thus

$$\sigma = \frac{\varepsilon(-1)^i Kg_\sigma^2(\eta_i)}{\|g_\sigma^2\|_\infty}.$$

Substituting from (1), we get the same equality but with g_σ^1 replacing g_σ^2. □

Case 2. For $f \in C[a, b]$ and $\sigma > 0$, set

$$\widetilde{E}(\sigma) = (E_{11}(\sigma) =)\min_{\mu \in C^*[c,d]} \|f - K\mu\|_1 + \sigma\|\mu\|_{T.V.}.$$

From (d), we have that

$$\widetilde{E}(\sigma) = \max_{\substack{\|h\|_\infty \le 1 \\ \|K^T h\|_\infty \le \sigma}} (f, h).$$

We here review some results from Micchelli and Pinkus [15]. The results apply to the set of $f \in C[a, b]$ which are in the *convexity cone of K*. By that we mean that for all $m \geq 0$, and all points $a \leq x_1 < \cdots < x_{m+1} \leq b$ and $a \leq y_1 < \cdots < y_m \leq d$,

$$\begin{vmatrix} K(x_1, y_1) & \cdots & K(x_1, y_m) & f(x_1) \\ \vdots & & \vdots & \vdots \\ K(x_{m+1}, y_1) & \cdots & K(x_{m+1}, y_m) & f(x_{m+1}) \end{vmatrix} > 0.$$

For $m = 0$, we take this to simply mean that $f(x) > 0$ for all $x \in [a,b]$. What we will explain is how for each $\sigma \in (0, \widetilde{\sigma})$ (here $\widetilde{\sigma} = \|\int_c^d K(\cdot, y)dy\|_\infty$) an optimal μ^* is chosen in a simple way which depends linearly on the particular f in the class.

Given any nonnegative integer m, there exist $a = \xi_0 < \xi_1 < \cdots < \xi_m < \xi_{m+1} = b$ and $c \leq \eta_1 < \cdots < \eta_{m+1} \leq d$ such that for h_ξ defined by

$$h_\xi(x) = (-1)^{i+m+1}, \ x \in (\xi_{i-1}, \xi_i), \quad i = 1, \dots, m+1,$$

we have

$$(K^T h_\xi)(\eta_j) = (-1)^{j+m+1} \|K^T h_\xi\|_\infty, \quad j = 1, \dots, m+1.$$

In fact, these conditions uniquely define ξ. Let

$$\sigma_m = \|K^T h_\xi\|_\infty.$$

Then it is easily shown that $\{\sigma_m\}_{m=0}^\infty$ is a strictly decreasing sequence ($\sigma_0 = \widetilde{\sigma}$) and $\lim_{m \to \infty} \sigma_m = 0$. Now for $\sigma \in (\sigma_m, \sigma_{m-1})$, there exist, for $k = 1, 2$, points $a = \xi_0^k < \xi_1^k < \cdots \xi_m^k < \xi_{m+1}^k = b$ and $c \leq \eta_1^k < \cdots < \eta_m^k \leq d$ such that for h_{ξ^k} defined by

$$h_{\xi^k}(x) = (-1)^{i+m+1}, \quad x \in (\xi^k_{i-1}, \xi^k_i), \quad i = 1, \ldots, m+1; \; k = 1, 2, \tag{7.11}$$

we have

$$(K^T h_{\xi^k})(\eta^k_j) = (-1)^{j+m+k} \|K^T h_{\xi^k}\|_\infty, \quad j = 1, \ldots, m, \quad k = 1, 2. \tag{7.12}$$

In fact these conditions uniquely determine the ξ^k, $k = 1, 2$.

For f in the convexity cone of K, an optimal μ^* for $\widetilde{E}(\sigma)$ is given as follows. For $\sigma \in (\sigma_m, \sigma_{m-1})$, let

$$d\mu^* = \sum_{j=1}^{m} a^*_j \delta_{\eta^2_j}, \tag{7.13}$$

where the $\{a^*_j\}_{j=1}^m$ are uniquely determined by the conditions

$$f(\xi^2_i) = \sum_{j=1}^{m} a^*_j K(\xi^2_i, \eta^2_j), \quad i = 1, \ldots, m. \tag{7.14}$$

Conditions (7.13) and (7.14), together with the fact that f is in the convexity cone of K imply that

$$a^*_j (-1)^{j+m} > 0, \quad j = 1, \ldots, m, \tag{7.15}$$

and

$$h_{\xi^2}(x)[f(x) - (K\mu^*)(x)] \geq 0, \tag{7.16}$$

for all $x \in [a, b]$. Thus from (7.12), (7.13) and (7.15),

$$(K^T h_{\xi 2}, \mu^*) = \|K^T h_{\xi 2}\|_\infty \|\mu^*\|_{T.V.},$$

while from (7.11) and (7.16),

$$(h_{\xi 2}, f - K\mu^*) = \|h_{\xi 2}\|_\infty \|f - K\mu^*\|_1.$$

Applying Proposition 3.6, we see that μ^* is indeed optimal for $\widetilde{E}(\sigma)$. For $\sigma = \sigma_m$, we do much the same using the associated $\{\xi_1, \ldots, \xi_m\}$ and $\{\eta_2, \ldots, \eta_{m+1}\}$ (note here that we have an extra η_1).

We use the $\{\xi_i^1\}_{i=1}^m$ and $\{\eta_j^1\}_{j=1}^m$ (or $\{\eta_j\}_{j=1}^m$ for $\sigma = \sigma_m$) in case f satisfies

$$\begin{vmatrix} f(x_1) & K(x_1, y_1) & \cdots & K(x_1, y_m) \\ \vdots & \vdots & & \vdots \\ f(x_{m+1}) & K(x_{m+1}, y_1) & \cdots & K(x_{m+1}, y_m) \end{vmatrix} > 0$$

for all $m \geq 0$, points $a \leq x_1 < \cdots < x_{m+1} \leq b$ and $c \leq y_1 < \cdots < y_m \leq d$. Of course, all these various conditions can be much weakened.

Case 3. $\widetilde{\sigma} = \hat{\sigma}$

If $1 < p < \infty$ and $1 < q < \infty$, and in addition $f = Kg^*$ for some $g^* \in L^q$, then from Proposition 3.9, $\widetilde{\sigma} = \hat{\sigma}$ if and only if

$$K^T(|Kg^*|^{p-1}\operatorname{sgn}(Kg^*))(y) = \lambda |g^*(y)|^{q-1}\operatorname{sgn}(g^*(y))$$

a.e. on $[c, d]$, where

$$\lambda = \frac{\|Kg^*\|_p^p}{\|g^*\|_q^q}.$$

This is certainly valid without the assumption of K being strictly totally positive. However, for $p = q \in (1, \infty)$, and K strictly totally positive, this equation was studied in Pinkus [19]. Its solutions (there are a countable number, up to multiplication by constants) are related to certain n-width problems. In fact for K STP, many of the extremal problems considered in the study of n-widths seem to relate to this exact problem of when $\tilde{\sigma} = \hat{\sigma}$.

Let us consider, under the assumption of K being strictly totally positive, some of the other cases of when $\tilde{\sigma} = \hat{\sigma}$. Given any fixed nonnegative integer m, and $c = t_0 < t_1 < \cdots < t_m < t_{m+1} = d$, set

$$g_t(y) = (-1)^{i+m+1}, \quad t \in (t_{i-1}, t_i), \quad i = 1, \ldots, m+1.$$

For $1 \leq p \leq \infty$, the problem

$$\inf_t \|Kg_t\|_p$$

has a solution $g^* = g_\xi$ (dependent on p) where ξ satisfies $c = \xi_0 < \xi_1 < \cdots < \xi_m < \xi_{m+1} = d$, and

$$\int_a^b |Kg^*(x)|^{p-1} [\operatorname{sgn} Kg^*(x)] \, K(x, \xi_i) dx = 0, \quad i = 1, \ldots, m,$$

if $1 \leq p < \infty$. For $p = \infty$ there exist $a \leq \eta_1 < \cdots < \eta_{m+1} \leq b$ such that

$$(-1)^{i+m+1} Kg^*(\eta_i) = \|Kg^*\|_\infty, \quad i = 1, \ldots, m+1.$$

For $1 \leq p < \infty$ set

$$h^*(x) = |Kg^*(x)|^{p-1} \operatorname{sgn}(Kg^*(x)).$$

Thus

$$(7.17) \qquad (Kg^*, h^*) = \|Kg^*\|_p \|h^*\|_{p'},$$

where $1/p + 1/p' = 1$. From the variation diminishing property of K (and K^T)

$$g^*(y)(K^T h^*)(y) \geq 0$$

for all $y \in [c, d]$. Thus

$$(7.18) \qquad (g^*, K^T h^*) = \|g^*\|_\infty \|K^T h^*\|_1.$$

Applying Proposition 3.9, we get the following result.

Proposition 7.5. *If $f = Kg^*$ where g^* is as above for some m and some $p \in [1, \infty)$, then in the problem $E_{p\infty}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kg^*\|_p}{\|g^*\|_\infty} = \frac{\|K^T h^*\|_1}{\|h^*\|_{p'}}.$$

If $p = \infty$, we define $\mu^* \in C^*[a, b]$ by

$$d\mu^* = \sum_{i=1}^{m+1} a_i^* \delta_{\eta_i},$$

where the $\{a_i^*\}_{i=1}^{m+1}$ are the unique set of coefficients satisfying

$$a) \left[\sum_{i=1}^{m+1} a_i^* K(\eta_i, y) \right] g^*(y) \geq 0 \text{ a.e. } y \in [c, d]$$

$$b) \sum_{i=1}^{m+1} |a_i^*| = 1$$

$$c) (-1)^{i+m+1} a_i^* > 0, \quad i = 1, \dots, m+1.$$

Thus $\|\mu^*\|_{T.V.} = \sum_{i=1}^{m+1} |a_i^*| = 1$. From the above constructions, we get

$$(Kg^*, \mu^*) = \|Kg^*\|_\infty \|\mu^*\|_{T.V.}$$

and

$$(g^*, K^T \mu^*) = \|g^*\|_\infty \|K^T \mu^*\|_1.$$

Applying Proposition 3.9, we get this next result.

Proposition 7.6. *If $f = Kg^*$ where g^* is as above for some m, then in the problem $E_{\infty\infty\infty}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kg^*\|_\infty}{\|g^*\|_\infty} = \frac{\|K^T \mu^*\|_1}{\|\mu^*\|_{T.V.}}.$$

Remark 7.1. Let $p \in [1, \infty]$ and g^* be as above so that either Proposition 7.5 or 7.6 is satisfied. For $0 < \tau < \|g^*\|_\infty$, consider the problem

$$\min\{\|f - Kg\|_p\colon \ \|g\|_\infty \leq \tau\}.$$

For $p = \infty$ it follows from Theorem 7.1 that there is a unique solution g^τ. This g^τ is explicitly given by

$$g^\tau = \frac{\tau g^*}{\|g^*\|_\infty}.$$

This follows since (7.4) of Theorem 7.1 is satisfied. Moreover, this same g^τ is the solution for all other p as well. This is an immediate consequence of the fact that the associated h^* is a nontrivial linear functional on L^p which attains its norm on Kg^*.

Interchanging the roles of K and K^T in the analysis which led to Propositions 7.5 and 7.6, we get corresponding results for $E_{1q}(\sigma)$, $1 \leq q \leq \infty$. More explicitly, for $1 \leq q \leq \infty$, let g^* be the solution to

$$\min_t \|K^T g_t\|_{q'}$$

where $1/q + 1/q' = 1$, and g_t is as previously defined. For $1 < q \leq \infty$, set

$$h^*(y) = |K^T g^*(y)|^{q'-1} \operatorname{sgn}(K^T g^*(y)),$$

---

and for $q = 1$ let ν^* play the corresponding role to the μ^* of the above. Then,

Proposition 7.7. *If $f = Kh^*$ where h^* is as above for some m and some $q \in (1, \infty]$, then in the problem $E_{1q}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kh^*\|_1}{\|h^*\|_q} = \frac{\|K^T g^*\|_{q'}}{\|g^*\|_\infty}.$$

If $f = Kv^*$ where v^* is as above, then in the problem $E_{11}(\sigma)$,

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kv^*\|_1}{\|v^*\|_{T.V.}} = \frac{\|K^T g^*\|_\infty}{\|g^*\|_\infty}.$$

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