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Research Report

Supersaturated Design Using Hadamard Matrix

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ABSTRACT

The Hadamard matrix is found to be useful in constructing supersaturated designs. In this paper, we provide a universal form of supersaturated design using a Hadamard matrix. In addition to new designs that are obtained, it is shown that most of the recent work in this area, including Lin (1993a), Wu (1993) and Tang and Wu (1993), can be viewed as special cases of such a universal form. Properties of such a supersaturated design are discussed. In particular, designs given here will always reach the minimum $E(s^2)$ value within a class of the same size. To further distinguish these designs, a new criterion – resolution rank, based upon the estimability of projective design – is proposed, justified, and used. Hadamard matrices of the order $n = 12$ and $n = 16$ are used as examples.

KEY WORDS: Column balance; Projective designs; Resolution rank.

1. Introduction

When the number of factors is large, and when a small number of runs is desired, a supersaturated design can save considerable cost. A supersaturated design is a fraction of a factorial design with n observations in which the number of factor k is larger than $n-1$. The usefulness of such a supersaturated design relies upon the realism of effect sparsity; namely, the number of the dominant active factors is small. The goal here is to identify these active factors, with so-called screening experimentation. (For a brief review of early work in supersaturated design, see Lin (1991).)

Apart from some ad hoc procedures and computer-generated designs, the construction problem has not been addressed until very recently. See, Lin (1993a), Wu (1993), and Tang and Wu (1993). All of these supersaturated designs, though constructed based upon different viewpoints, show that a Hadamard matrix (see, for example, Hedayat and Wallis, 1978) is indeed a useful tool for constructing a supersaturated design. In this paper, we provide a universal form of a supersaturated design using a Hadamard matrix, which will cover all of the above construction methods as special cases. Moreover, we show that a supersaturated design constructed with such a universal form is superior to others, in terms of various criteria.

Booth and Cox (1962) first proposed the $E(s^2)$ criterion to evaluate the goodness of a supersaturated design, and it has been intensively used by others. The designs given here will always reach the minimum value of $E(s^2)$ within a class of the same size (i.e., the class of fixed n and k). This implies that all of the designs given here are optimal in terms of the criterion $E(s^2)$. Note that once these active factors are identified, the initial design is then projected into a much smaller dimension (see Cheng, 1993 and Lin, 1993b). A criterion based upon such an important projection property, called resolution rank, is proposed to further differentiate among designs.

The paper is organized as follows. In Section 2, we introduce a universal form of supersaturated design using a Hadamard matrix. Some interesting properties of these designs are revealed in Section 3. The new criterion, r-rank is also given there. Two specific examples based on the Hadamard matrices of order n=12 and n=16 are presented in Section 4. Finally, in Section 5, aided by the new supersaturated design form, we revisit a half-fraction Hadamard matrix (HFHM) and extend our results to a much more general class of supersaturated design. For the simplicity of the presentation, all proofs are given in the Appendix.

2. The Construction Method

Let $\mathbf{H}$ be a normalized Hadamard matrix of the dimension $n \times n$. Namely, the first column of $\mathbf{H}$ consists of $\pm\mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)'$. One way to construct a Hadamard matrix of the dimension $2n \times 2n$ is as follows:

$$\mathbf{H}_* = \begin{pmatrix} \mathbf{H} & \mathbf{RH} \\ -\mathbf{R}'\mathbf{H} & \mathbf{H} \end{pmatrix} \tag{2.1}$$

where $\mathbf{R}$ is a $n \times n$ orthogonal matrix satisfying the condition that $\mathbf{RH}$ has all its entries ± 1. Two popular choices of the $\mathbf{R}$ matrix are (i) $\mathbf{R} = \mathbf{I}$, the identity matrix (this is also known as a "fold-over" method), and (ii) $\mathbf{R} = \mathbf{D}(\mathbf{h}_i)$, i.e., a diagonal matrix whose diagonal elements correspond to the i-th column vector $\mathbf{h}_i$ of the original $\mathbf{H}$ matrix. We shall refer to this method as a product method.

If we take the first half-fraction of $\mathbf{H}_*$, then we have the matrix

$$\mathbf{X} = [\mathbf{H}, \mathbf{RH}]. \tag{2.2}$$

This is similar to the half-fraction Hadamard matrix (HFHM) method proposed by Lin (1993a) for constructing a supersaturated design. By selecting a column vector in $\mathbf{H}$ as a "branching" column, Lin (1993a) proved that the resulting supersaturated design is column balanced. A design matrix of dimension $N \times m$ is said to be *column balanced*, if the sum of the entries of every column vector is either 0 or $\pm N$.

There are two requirements for the matrix $\mathbf{X}$ to be considered as a supersaturated design:

(1) $\mathbf{RH}$ in $\mathbf{X}$ must be column balanced.

(2) Columns in $\mathbf{RH}$ that are fully aliased with $\mathbf{H}$ must be removed from $\mathbf{X}$. Two column vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be *fully aliased*, if $\mathbf{u} = \pm\mathbf{v}$.

It is clear that a normalized Hadamard matrix is always column balanced. The following theorem gives a necessary and sufficient condition for $\mathbf{RH}$ to be column balanced.

THEOREM 1. *Let $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$, be an $n \times n$ Hadamard matrix, let $\mathbf{R}$ be an orthogonal matrix such that all entries in $\mathbf{RH}$ are ± 1. A necessary and sufficient condition for $\mathbf{R}$ such that $\mathbf{RH}$ is column balanced is*

$$\mathbf{R}'\mathbf{1} = \pm\mathbf{h}_i \quad \text{for some } 1 \le i \le n. \tag{2.3}$$

∎

Now, consider a supersaturated design of the form

$$\mathbf{X}_c = [\mathbf{H}, \mathbf{RHC}], \tag{2.4}$$

where

(1) $\mathbf{H}$ is a column-balanced Hadamard matrix,

(2) $\mathbf{R}$ is an orthogonal matrix satisfying Theorem 1, and

(3) $\mathbf{C}$ is an $n \times (n - c)$ matrix representing the operation of column selection.

Some examples of the matrix $\mathbf{C}$ follow:

(1) Suppose the first column of $\mathbf{H}$ is to be removed. Then we can simply select

$$\mathbf{C} = \begin{pmatrix} \mathbf{0}' \\ \mathbf{I}_{n-1} \end{pmatrix}.$$

In this case, we can easily verify that

$$\mathbf{HC} = (\mathbf{h}_2, \ldots, \mathbf{h}_n)\,.$$

By moving the $\mathbf{0}'$ in $\mathbf{C}$ to the i-th row, the new $\mathbf{C}$ is the matrix deleting the i-th column of $\mathbf{H}$.

(2) Similarly, suppose columns 1 and n of $\mathbf{H}$ are to be removed, then we can simply select

$$\mathbf{C} = \begin{pmatrix} \mathbf{0}' \\ \mathbf{I}_{n-2} \\ \mathbf{0}' \end{pmatrix}.$$

In this case, we can easily verify that

$$\mathbf{HC} = (\mathbf{h}_2, \ldots, \mathbf{h}_{n-1})\,.$$

(3) By moving the $\mathbf{0}'$'s in $\mathbf{C}$ to the i-th and the j-th rows, the new $\mathbf{C}$ is the matrix deleting the i-th and j-th columns of $\mathbf{H}$.

(4) In general, if c columns of $\mathbf{H}$ are to be removed, the corresponding $\mathbf{C}$ can be constructed with c zero row vectors of dimension $1 \times (n-c)$ and $\mathbf{I}_{n-c}$.

The supersaturated design $\mathbf{X}_c$ proposed here consists of several interesting special cases:

(1) $\mathbf{R} = \mathbf{D}(\mathbf{h}_i), 1 \leq i \leq n, \mathbf{h}_i \neq \pm\mathbf{1}$, where $\mathbf{D}(\mathbf{h}_i)$ is the diagonal matrix with the diagonal elements of $\mathbf{h}_i$, and $\mathbf{C}$ as the matrix corresponds to the deletion of the first and the i-th columns. The supersaturated design matrix, $\mathbf{X}_c$ in this case, is the same as the product method proposed by Wu (1993). It is easy to verify that $\mathbf{R} = \mathbf{D}(\mathbf{h}_i)$ satisfies the condition in Theorem 1.

On the other hand, if we choose

$$\mathbf{R} = \mathbf{D}(\mathbf{v})$$

for any vector $\mathbf{v}$ with entries ± 1, and suppose that $\mathbf{RH}$ is column balanced, then, according to Theorem 1,

$$\mathbf{v} = \mathbf{D}(\mathbf{v})'\mathbf{1} = \pm \mathbf{h}_i,$$

for some $1 \leq i \leq n$. This implies that if $\mathbf{R}$ is a diagonal matrix, then its diagonal elements must form a column of the $\mathbf{H}$ matrix. Namely, the product method will only work when using the product of the original $\mathbf{H}$ columns, an important fact that was not addressed in Wu(1993).

(2) $\mathbf{R} = \mathbf{P}$, where $\mathbf{P}$ is a matrix corresponding to the permutation of the row vectors in $\mathbf{H}$. Clearly, $\mathbf{P}'\mathbf{1} = \mathbf{1}$ and it satisfies the condition in Theorem 1. If we further choose $\mathbf{C}$ to be a matrix that will delete the first column and all fully aliased columns that may appear, then this supersaturated design matrix, $\mathbf{X}_c$, is similar to the Tang and Wu (1993) method. As is well-known, the product method and the HFHM method fail under certain situations (e.g., $n = 16$ runs). In this case, permuting the rows of the Hadamard matrix can be used.

(3) $\mathbf{R} = \mathbf{PD}(\pm \mathbf{h}_i)$. This corresponds to the operation of permuting rows using the product method. It is easy to verify that $\mathbf{R} = \mathbf{PD}(\pm \mathbf{h}_i)$ satisfies condition (2.3). On the other hand, using an orthogonal matrix $\mathbf{R} = (r_{ij})$ with all entries $r_{ij} = \pm 1, 0$ will make $\mathbf{RH}$ column balanced. We will call the transformation $\mathbf{R}$ the *equivalence transformation* because it will make $\mathbf{RH}$ and $\mathbf{H}$ equivalent. Two Hadamard matrices are equivalent if one can be obtained from the other by permuting rows and columns and by complementing rows and columns. We can prove that any equivalence transformation $\mathbf{R}$ with $\mathbf{RH}$ column balanced must be of the form, $\mathbf{PD}(\pm \mathbf{h}_i)$. The proof is as follows. Note that we can write

$$\mathbf{R} = \mathbf{PD}(\mathbf{v}),$$

where $\mathbf{P} = (|r_{ij}|)$ is a permutation matrix and $\mathbf{v} = \mathbf{R1}$ is a column vector with entries ± 1. From (2.3), we know that for some $i = 1, 2, ..., n$,

$$\pm \mathbf{h}_i' = \mathbf{1}'\mathbf{R} = \mathbf{1}'\mathbf{PD}(\mathbf{v}) = \mathbf{1}'\mathbf{D}(\mathbf{v}) = \mathbf{v}'.$$

Hence, we show that any equivalence transformation $\mathbf{R}$ that makes $\mathbf{RH}$ column balanced can be only of the form $\mathbf{R} = \mathbf{PD}(\pm \mathbf{h}_i)$. Furthermore, this includes both the product method (with $\mathbf{P} = \mathbf{I}$) and the permutation method (with $\mathbf{D}(\mathbf{h}_1) = \mathbf{D}(\mathbf{1}) = \mathbf{I}$) as special cases.

(4) There are some examples of $\mathbf{R}$ which are not equivalence transformations. For example, let $\mathbf{H}_a$ be any column balanced Hadamard matrix of the dimension $n \times n$, then

$$\mathbf{R} = \frac{1}{n}\mathbf{H}_a\mathbf{H}'$$

is an orthogonal transformation. Furthermore, if $\mathbf{H}_a$ is not equivalent to $\mathbf{H}$, then $\mathbf{R}$ can neither be a permutation nor a product and will satisfy $\mathbf{RH} = \mathbf{H}_a$. Designs of this type are apparently new.

3. Some properties

Consider a supersaturated design $\mathbf{X}_c$ of the form in (2.4) with $\mathbf{R}$ defined the same as in Section 2, and $\mathbf{C}$ as an $n \times (n - c)$ matrix after doing column selection, such that the full alias columns are deleted from the $\mathbf{RH}$.

It then can be shown that

$$\mathbf{X}_c'\mathbf{X}_c = \begin{pmatrix} n\mathbf{I}_n & \mathbf{H}'\mathbf{RHC} \\ \mathbf{C}'\mathbf{H}'\mathbf{R}'\mathbf{H} & n\mathbf{I}_{n-c} \end{pmatrix} = \begin{pmatrix} n\mathbf{I}_n & \mathbf{WC} \\ \mathbf{C}'\mathbf{W}' & n\mathbf{I}_{n-c} \end{pmatrix}, \tag{3.1}$$

where

$$\mathbf{W} = \mathbf{H}'\mathbf{RH} = (w_{ij}) = (\mathbf{h}_i'\mathbf{Rh}_j). \tag{3.2}$$

Let s_{ij} denote the (i,j) entry of $\mathbf{X}_c'\mathbf{X}_c$, $E(s^2)$ of $\mathbf{X}_c$, proposed by Booth and Cox (1962), and be defined as:

$$E(s^2) = \sum_{i \neq j} s_{ij}^2 / \binom{k}{2},$$

where k is the number of columns in $\mathbf{X}_c$. Clearly, $E(s^2) = K \times \sum_{j \in \mathbf{C}} \sum_{i=1}^{n} w_{ij}^2$, which stays constant here as shown in Theorem 2.

THEOREM 2. *Let* $\mathbf{H}$ *be a Hadamard matrix of dimension* $n \times n$, *and* $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r)$ *be a* $n \times r$ *matrix with entries* ± 1 *and*

$$\mathbf{V} = \mathbf{H}'\mathbf{A} = (v_{ij}) = \mathbf{h}_i'\mathbf{a}_j.$$

(1) *For any fixed* $1 \leq j \leq r$, $n^2 = \sum_{i=1}^{n} v_{ij}^2$.

(2) *In particular, let* $\mathbf{A} = \mathbf{RH}$, $\mathbf{W} = \mathbf{H}'\mathbf{RH} = (w_{ij})$,

(a) $\frac{1}{n}\mathbf{W}$ *is a* $n \times n$ *orthogonal matrix.*

(b) $n^2 = \sum_{i=1}^{n} w_{ij}^2 = \sum_{j=1}^{n} w_{ij}^2$.

(c) w_{ij} *is always a multiple of* 4.

(d) *If* $\mathbf{H}'$ *is column balanced, then* $\pm n = \sum_{i=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ij}$.

∎

COROLLARY 1. *For any* $\mathbf{R}$ *and* $\mathbf{C}$ *such that*

(1) $\mathbf{R}'\mathbf{R} = \mathbf{I}$

(2) $rank(\mathbf{C}) = n - c$,

then $\mathbf{X}_c$ *in (2.4) has the same* $E(s^2)$. ∎

This implies that the popular criterion $E(s^2)$ used in supersaturated design is invariant for any choice of $\mathbf{R}$ and $\mathbf{C}$. Therefore, it is not effective in selecting a supersaturated design. In fact, following the argument in Tang and Wu (1993), the design given here will always have the minimum $E(s^2)$ value within a class of the same size. One important feature of the goodness of a supersaturated design is its projective property (see Lin, 1993b and Cheng, 1993). Wu (1993) and Deng, Lin and Wang (1993) extend the classical design optimalities and propose to compute the average D_f (D-optimal) and A_f (A-optimal) criteria over the projected submatrices of f columns to select a better supersaturated design. Specifically, let

$$V_f(\mathbf{X}) = \frac{1}{\binom{k}{f}} \cdot \sum_{|s|=f} v(\mathbf{X}_s),$$

where the summation $|s| = f$ over all submatrices $\mathbf{X}_s$ of size $n \times f$ and $v(\mathbf{X}_s)$ is a function to measure the "orthogonality" of $\mathbf{X}_s$. Some natural choices of $v(\mathbf{X}_s)$ are:

(1) $v(\mathbf{X}_s) = \det(\mathbf{X}_s'\mathbf{X}_s)^{-1}$ (D_f criterion).

(2) $v(\mathbf{X}_s) = \text{trace } (\mathbf{X}_s'\mathbf{X}_s)^{-1}$ (A_f criterion).

A large average D_f does not ensure that every projective design is nonsingular, namely identifiable in terms of the projective design. Since the estimability of the projective design is the primary concern, we thus consider the r-rank property as defined below.

DEFINITION. *Let* $\mathbf{X}$ *be a column-balanced design matrix. We define the resolution-rank (r-rank, for short) of* $\mathbf{X}$ *as* $f = d - 1$, *where* d *is the minimum number subset columns (excluding* $\pm\mathbf{1}$*) that will be linearly dependent.*

Clearly, if a supersaturated design, $\mathbf{X}$, has an r-rank of f, then when $\mathbf{X}$ is projected to any submatrix of f (or less) factors, the main effects of the projected design are all estimable. Moreover, in many situations where the r-ranks are very different for two supersaturated designs, their D_f and A_f values are nearly identical (The maximum difference is around 1%.).

THEOREM 3. *If no columns in any supersaturated design,* $\mathbf{X}$, *are fully aliased, then its r-rank is at least 3.* ▮

One way to check the r-rank of a supersaturated design is to check $\binom{k}{f}$ sub-matrices for all $f = 2, 3...$ etc. This is very time-consuming, even for a moderate k and f. For evaluating D_f and A_f, the computing time is even longer. Obviously, columns within $\mathbf{H}$ or $\mathbf{RH}$ are orthogonal and hence are independent of each other. We need only to consider the relationship between columns in $\mathbf{H}$ and columns in $\mathbf{RH}$. Note that the j-column of $\mathbf{W}$ can inform us about how $\mathbf{Rh}_j$, the j-th column of $\mathbf{RH}$, is related with $\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_n$ as stated in the following lemma.

LEMMA 1. *Let* $\mathbf{W} = \mathbf{H}'\mathbf{RH} = (w_{ij})$, *where* $\mathbf{H}$ *is a* $n \times n$ *Hadamard matrix and* $\mathbf{R}$ *is an orthogonal matrix.*

$$n\mathbf{Rh}_j = \sum_{i=1}^{n} w_{ij}\mathbf{h}_i. \tag{3.3}$$

∎

As we can see from Lemma 1, if many w_{ij} are zero, then $\mathbf{Rh}_j$ will be dependent with only a few $\mathbf{h}_i$.

We will next propose a good estimate of the r-rank that is easy and fast to compute and at the same time reflects the r-rank property.

For a fixed $j \in \mathbf{C}$, let

$$r_1 = \min\{|U_j|, j \in \mathbf{C}\},$$

where

$$U_j = \{i | w_{ij} \neq 0, i = 1, 2, ..., n\},$$

$|S|$ is the number of elements in a set S, and $j \in \mathbf{C}$ means j-th column of $\mathbf{W}$ selected by $\mathbf{C}$.

Similarly, for fixed $j_1 \neq j_2 \in \mathbf{C}$, let

$$r_2 = \min\{|U^+_{j_1,j_2}|, |U^-_{j_1,j_2}|, j_1, j_2 \in \mathbf{C}\} + 1,$$

where

$$U^+_{j_1,j_2} = \{i | w_{ij_1} + w_{ij_2} \neq 0, i = 1, 2, ..., n\}$$

and

$$U^{-}_{j_1,j_2} = \{i | w_{ij_1} - w_{ij_2} \neq 0, i = 1, 2, ..., n\}.$$

We have

THEOREM 4.

$$r\text{-}rank \leq \min\{r_1, r_2\} = r_e.$$

∎

Theorem 4 provides an upper bound of the r-rank.

There are some obvious advantages about the value r_e, an upper bound of r-rank:

(1) It is very easy to compute. According to our study, it is at least 10,000 times faster than calculating the actual r-rank and computing the D_f, A_f criteria.

(2) r_e can easily screen out many undesirable (e.g., low r-rank) supersaturated designs $\mathbf{X}_c$ of the form in (2.4) with the identical r-rankA_f and D_f values.

(3) A potential good supersaturated design is one with a large r_e. Although a large r_e cannot be guaranteed to have a higher r-rank, our empirical study shows that r_e is a good estimate of the r-rank. The supersaturated design with a large r_e, in general, has a higher r-rank.

Note that the property of r-rank, $D_f, A_f, ...$ is determined mainly by $\mathbf{W}$ (and to a minor extent by $\mathbf{C}$). We will formally define two $\mathbf{W}$'s that are equivalent to each other.

DEFINITION. *Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two orthogonal matrices satisfying (2.3) and let*

$$\mathbf{W}_1 = \mathbf{H}'\mathbf{R}_1\mathbf{H}, \quad \mathbf{W}_2 = \mathbf{H}'\mathbf{R}_2\mathbf{H}.$$

If there is a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}'\mathbf{W}_1\mathbf{P} = \mathbf{W}_2,$$

then $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *are said to be equivalent to each other.*

Clearly, if $\mathbf{W}_1$ and $\mathbf{W}_2$ are equivalent, then their corresponding supersaturated designs will have exactly the identical r-rank, A_f and D_f values.

One way to construct a Hadamard matrix is by cyclic generation (see, for example, Lin and Draper, 1993). The cyclic generation method in terms of a matrix representation can be described as follows. For a Hadamard matrix $\mathbf{H}$ generated using this method, we can see that

$$\mathbf{H} = \left(\mathbf{1}, \mathbf{h}, \mathbf{S}\mathbf{h}, \mathbf{S}^2\mathbf{h}, ..., \mathbf{S}^{n-2}\mathbf{h}\right), \tag{3.4}$$

where

(1) The first column of $\mathbf{H}$ is $\mathbf{1}$.

(2) The second column in $\mathbf{H}$ is a pre-specified column $\mathbf{h} = (h_1, h_2, \cdots, h_{n-1}, -1)'$.

(3) The operation that rotating the first (n-1) entries and maintaining the last entry of $\mathbf{h}$ can be written as

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{n-1} & \mathbf{0} \\ \mathbf{0}' & 1 \end{pmatrix}$$

with the $(n-1) \times (n-1)$ matrix

$$\mathbf{S}_{n-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

THEOREM 5. *Let* $\mathbf{H}$ *be a Hadamard matrix and* $\mathbf{D}(\mathbf{h}_i)$ *be the diagonal matrix associated with* $\mathbf{h}_i$, *the* i*-th column vector of* $\mathbf{H}$. *If* $\mathbf{H}$ *is obtained by cyclic generation as in (3.4), for any choice of* $\mathbf{h}_i \neq \pm\mathbf{1}$, *then* $\mathbf{W} = \mathbf{H}'\mathbf{D}(\mathbf{h}_i)\mathbf{H}$ *and* $\mathbf{H}'\mathbf{D}(\mathbf{h})\mathbf{H}$ *are equivalent to each other. That is,*

$$\mathbf{W} = \mathbf{H}'\mathbf{D}(\mathbf{h}_i)\mathbf{H} = \mathbf{P}'\mathbf{H}'\mathbf{D}(\mathbf{h})\mathbf{H}\mathbf{P},$$

for some permutation matrix $\mathbf{P}$. ∎

COROLLARY 2. *Let* $\mathbf{H}^* = \mathbf{R}^*\mathbf{H}$, *where* $\mathbf{H}$ *is a Hadamard matrix and let* $\mathbf{R}^*$ *be a matrix representing the row permutation. Let* $\mathbf{D}(\mathbf{h}_i^*)$ *be the diagonal matrix associated with* $\mathbf{h}_i^*$, *the* i*-th column vector of* $\mathbf{H}^*$. *If* $\mathbf{H}$ *is obtained by cyclic generation as in (3.4), then for any choice of* $\mathbf{h}_i^* \neq \pm\mathbf{1}$,

$$\mathbf{W} = \mathbf{H}^{*\prime}\mathbf{D}(\mathbf{h}_i^*)\mathbf{H}^* = \mathbf{P}'\mathbf{H}'\mathbf{D}(\mathbf{h})\mathbf{H}\mathbf{P},$$

for some permutation matrix $\mathbf{P}$.

∎

THEOREM 6. *Let* $\mathbf{W} = \mathbf{H}'\mathbf{D}(\mathbf{h}_l)\mathbf{H}$. $\mathbf{H}$ *is a Hadamard matrix of size* $n = 4t$ *and* $\mathbf{D}(\mathbf{h}_l)$ *is the diagonal matrix associated with* $\mathbf{h}_l$, *the* l*-th column vector of* $\mathbf{H}$, $\mathbf{h}_l \neq \pm\mathbf{1}$.

(1) *If* t *is odd, then there can be exactly three 0 in each row or each column of* $\mathbf{W}$. *The rest of* w_{ij} *in* $\mathbf{W}$ *can only be of the form* $\pm 8k + 4$, *for some non-negative integer* k.

(2) *If* t *is even, then every entry* w_{ij} *in* $\mathbf{W}$ *can be of the form* $\pm 8k$, *for some non-negative integer* k.

∎

There are some implications for the product method from the above theorem:

(1) When t is even, let

$$w_{ij} = 8u_{ij},$$

where u_{ij} is an integer. From Theorem 2, u_{ij} will satisfy

$$\sum_{i=1}^{n} u_{ij}^2 = \sum_{j=1}^{n} u_{ij}^2 = t^2/4.$$

Hence, there are at most $t^2/4$ of w_{ij} that will be non-zero. For example, for any Hadamard matrix with $n = 16 (t = 4)$, there are at most $4^2/4 = 4$ non-zero elements in any column of $\mathbf{W}$. Furthermore, every non-zero element must be ± 8 or ± 16. The case of $w_{ij} = \pm 16$ corresponds to the j-column of $\mathbf{RH}$ when it is fully aliased with the i-th column of $\mathbf{H}$. Therefore, there are exactly 4 elements in each column vector in $\mathbf{WC}$ with values ± 8, and its r-rank is at most 4. In general, the product method will have a smaller r_1 than that from the permutation method. Hence, the product method is not recommended when t is even.

(2) When t is odd, there are $(n-3)$ $w_{ij} \neq 0$ which may in turn make r_1 large. For example, for a Hadamard matrix with $n = 12 (t = 3)$, there are exactly $n - 3 = 9$ non-zero elements in any column in $\mathbf{WC}$. Furthermore, every non-zero entry of w_{ij} must be ± 4. In this case, we can easily see that the value of r_1 by the product method can be the maximum value.

4. Examples

4.1 Example 1: n=12 runs ($t = 3$, odd).

It is well-known that, for $n = 12$, there is only one equivalent class of Hadamard matrices. The Hadamard matrix $\mathbf{H}$ is given below.

$$\mathbf{H} = \begin{pmatrix} + & - & - & + & - & - & - & + & + & + & - & + \\ + & - & + & - & - & - & + & + & + & - & + & - \\ + & + & - & - & - & + & + & + & - & + & - & - \\ + & - & - & - & + & + & + & - & + & - & - & + \\ + & - & - & + & + & + & - & + & - & - & + & - \\ + & - & + & + & + & - & + & - & - & + & - & - \\ + & + & + & + & - & + & - & - & + & - & - & - \\ + & + & + & - & + & - & - & + & - & - & - & + \\ + & + & - & + & - & - & + & - & - & - & + & + \\ + & - & + & - & - & + & - & - & - & + & + & + \\ + & + & - & - & + & - & - & - & + & + & + & - \\ + & + & + & + & + & + & + & + & + & + & + & + \end{pmatrix}$$

Since $\mathbf{H}$ satisfies the condition in Theorem 4, we know that any choice of $\mathbf{h}_i \neq \pm \mathbf{1}$ will yield a $\mathbf{W}$ matrix of equivalents. In this study, we chose $\mathbf{h}_9$ from $\mathbf{H}$. And the corresponding $\mathbf{W}$ is

$$
\mathbf{W} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 4 & 4 & 4 & 4 & -4 & -4 & 0 & 4 & 4 & -4 \\
0 & 4 & 0 & 4 & -4 & 4 & 4 & 4 & 0 & -4 & 4 & -4 \\
0 & 4 & 4 & 0 & -4 & 4 & -4 & 4 & 0 & 4 & -4 & 4 \\
0 & 4 & -4 & -4 & 0 & 4 & 4 & -4 & 0 & 4 & 4 & 4 \\
0 & 4 & 4 & 4 & 4 & 0 & 4 & -4 & 0 & -4 & -4 & 4 \\
0 & -4 & 4 & -4 & 4 & 4 & 0 & 4 & 0 & -4 & 4 & 4 \\
0 & -4 & 4 & 4 & -4 & -4 & 4 & 0 & 0 & 4 & 4 & 4 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -4 & 4 & 4 & -4 & -4 & 4 & 0 & 0 & 4 & 4 \\
0 & 4 & 4 & -4 & 4 & -4 & 4 & 4 & 0 & 4 & 0 & -4 \\
0 & -4 & -4 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & -4 & 0
\end{pmatrix}
$$

Using $\mathbf{R} = \mathbf{D}(\mathbf{R}_1)$, both the product method and our method given here yield the identical design. The values of $r_1 = 9, r_2 = 7$. Hence the estimated r-rank $r_e = 7$. This is, in fact, the r-rank of the corresponding supersaturated design. The corresponding D_f and A_f values are given below.

f	A_f	D_f
2	0.175	7.961
3	0.277	6.526
4	0.395	5.378
5	0.535	4.460
6	0.713	3.729
7	0.964	3.149

4.2 Example 2: n=16 runs ($t = 4$, even).

Hall (1961) gave a complete listing of all 5 non-equivalent Hadamard matrices of order 16. The one given by Plackett and Burman will fail under the product method because all columns produced by $\mathbf{D}(\mathbf{h}_i)\mathbf{H}$ are fully aliased with $\mathbf{H}$. The product method will still work for the last group ("V. 3/8 Group") of the Hadamard matrix proposed by Hall (1961) with the Hadamard matrix $\mathbf{H}$ given as

$$
\mathbf{H} = \begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & + & - & - & - & - & + & + & + & - & + & - & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & + & - & + & - & + & - & + \\
+ & - & + & - & + & - & + & - & + & + & - & - & - & - & + & + \\
+ & - & + & - & + & - & + & - & - & - & + & + & + & + & - & - \\
+ & - & + & - & - & + & - & + & + & - & - & + & - & + & + & - \\
+ & - & + & - & - & + & - & + & - & + & + & - & + & - & - & + \\
+ & - & - & + & + & - & - & + & + & - & + & - & - & + & - & + \\
+ & - & - & + & + & - & - & + & - & + & - & + & + & - & + & - \\
+ & - & - & + & - & + & + & - & + & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - & - & + & + & - & - & + & + & -
\end{pmatrix}
$$

Note that this Hadamard matrix does not satisfy the condition (3.4) mentioned in Theorem 5 and not all choices of $\mathbf{h}_i$ will yield a matrix $\mathbf{W}$ of equivalent values. From the computer search, we found that with h_i selected from columns 9–16 will yield the same equivalent $\mathbf{W}$. In this example, we chose $\mathbf{h}_{11}$ from $\mathbf{H}$ and the $\mathbf{W}$ matrix is

$$
\mathbf{W} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 8 & 0 & 0 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 8 & -8 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & -8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & -8 & 0 & 8 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 8 & 8 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -8 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 8 & 0 & 0 & 8 & 8 & 0 \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 8 & 0 & 8 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 8 & 0 & -8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 8 & -8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & -8 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 8 & -8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & -8 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 8 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Clearly, $r_1 = 4$ and $r_2 = 6$. Therefore, $r_e = 4$ which is also the r-rank of the corresponding supersaturated design.

The permutation matrix $\mathbf{R} = ((10, 15, 14, 12, 3, 9, 16, 7, 4, 6, 13, 5, 1, 2, 8, 11))$ is used here, where

$$\mathbf{R} = ((j_1, j_2, ...,))$$

denotes the permutation matrix with the first row $\mathbf{e}'_{j_1}$ (the j_1-th row of $\mathbf{I}_n$), the second row $\mathbf{e}'_{j_2}$ (the j_2-th row of $\mathbf{I}_n$), ... The design constructed by the proposed method yields the values of $r_1 = 7, r_2 = 8$ and hence $r_e = 7$. This is indeed the r-rank of the supersaturated design. The corresponding A_f and D_f values are given in Table 1. Our method is clearly superior when compared the values for the product method.

Table 1. Comparisons on Product Method and Proposed Method ($n = 16$ case)

	Product Method		Proposed Method	
f	A_f	D_f	A_f	D_f
2	0.130	11.340	0.130	11.339
3	0.205	9.583	0.204	9.581
4	0.290	8.123	0.287	8.118
5			0.380	6.903
6			0.489	5.886
7			0.619	5.024

5. Extensions

The construction method given here provides another look at the HFHM. Recall that the HFHM method uses one column of $\mathbf{H}$, say $\mathbf{h}_i$, as its "branching" column and selects only rows of $\mathbf{H}$ corresponding to the entries of $\mathbf{h}_i = +1$. After $n/2$ rows are selected, the "branching" column $\mathbf{h}_i$ is deleted. The resulting supersaturated design can be written as

$$\mathbf{S} = \mathbf{RHC}, \tag{5.1}$$

where

(1) $\mathbf{R}$ is $n/2 \times n$ matrix representing a row selecting operation. For example, the first entry of $\mathbf{h}_i$ is $+1$, then the row vector $\mathbf{e}'_1 = (1, 0, 0, \ldots, 0)$ is included in $\mathbf{R}$.

(2) $\mathbf{C}$ is a $n \times (n-c)$ matrix representing a column selecting operation.

The following theorem gives the necessary and sufficient conditions for $\mathbf{RH}$ to be column balanced.

THEOREM 7. *Let* $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$ *be a Hadamard matrix and let* $\mathbf{R}$ *be a* $n/g \times n$ *matrix with row vectors that are orthogonal to each other,*

$$\mathbf{R}\mathbf{R}' = \mathbf{I}_{n/g \times n/g},$$

where g *is an integer that can divide* n. *A necessary and sufficient condition for* $\mathbf{RH}$ *to be column balanced is*

$$\mathbf{R}'\mathbf{1} = \frac{1}{g}\sum_{j=1}^{g} \pm \mathbf{h}_{k_j} \quad \text{for some } 1 \le k_1 < k_2 < \ldots < k_g \le n. \tag{5.2}$$

∎

Furthermore, for a more general class of supersaturated design, consider the following:

$$\mathbf{S}_{g,K} = (\mathbf{R}_1\mathbf{H}\mathbf{C}_1, \quad \mathbf{R}_2\mathbf{H}\mathbf{C}_2, \quad \ldots, \quad \mathbf{R}_K\mathbf{H}\mathbf{C}_K) \tag{5.3}$$

where

(1) g is an integer that can divide n,

(2) $\mathbf{R}_k$ is a $n/g \times n$ matrix with row vectors that are orthogonal to each other, as in

$$\mathbf{R}_k\mathbf{R}_k' = \mathbf{I}_{n/g \times n/g}, \quad \text{for } k = 1, 2, .., K \tag{5.4}$$

and, of course, all the entries of $\mathbf{RH}$ are ± 1.

(3) $\mathbf{C}_k$ is a $n \times r_k$ matrix representing the operation of column selection. The main purpose of $\mathbf{C}_k$ is to delete fully aliased columns from the design. The $\mathbf{C}_k$ can be defined similarly in Section 2. Another way of defining $\mathbf{C}_k$ is

$$\mathbf{C}_k = \left(\mathbf{e}_{k_1}, \mathbf{e}_{k_2}, \ldots, \mathbf{e}_{k_{r_k}}\right), \tag{5.5}$$

where $(k_1, k_2, \ldots)$ are the column indices to be selected and $\mathbf{e}_i$ is the i-th column of the identity matrix

$$\mathbf{I}_{n\times n} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n).$$

5.1. Case 1: $g = 1$.

When $g = 1$, the dimension of $\mathbf{R}_k$ is $n \times n$ and it is an orthogonal matrix. In this case,

$$\mathbf{S}_{1,K} = (\mathbf{R}_1\mathbf{H}\mathbf{C}_1, \quad \mathbf{R}_2\mathbf{H}\mathbf{C}_2, \quad \ldots, \quad \mathbf{R}_K\mathbf{H}\mathbf{C}_K). \tag{5.6}$$

According to Theorem 7, a necessary and sufficient condition for $\mathbf{S}_{1,K}$ in (5.6) to be column balanced is for all $k = 1, 2, \ldots, K$

$$\mathbf{R}_k'\mathbf{1} = \pm\mathbf{h}_i \quad \text{for some } 1 \le i \le n. \tag{5.7}$$

Note that $\mathbf{S}_{1,K}$ consists of several interesting special cases as mentioned in Section 2.

5.2. Case 2: $g = 2$.

We can show that HFHM is a special case of $\mathbf{S}_{2,K}$ with $K = 1$. When $g = 2, K = 1$, according to Theorem 7, the necessary and sufficient condition for $\mathbf{S}$ in (1) to be column balanced is

$$\mathbf{R}'\mathbf{1} = \frac{\pm\mathbf{h}_i \pm \mathbf{h}_j}{2}. \tag{5.8}$$

Note that the entries in the right-hand side of (5.8) are always $\pm 1, 0$. Lin (1993a) starts with a column balanced Hadamard matrix (hence $\pm\mathbf{1}$ is in $\mathbf{H}$), and takes $\mathbf{h}_i = \mathbf{1}$ in (5.8). In this case, all entries in $\mathbf{R}'\mathbf{1}$ can be 0 or 1. Selecting those rows corresponding to each entry equals 1 will yield the supersaturated design proposed by Lin (1993a).

When $g = 2, K > 1$, this corresponds to the method of permuting a Hadamard matrix $\mathbf{H}$ as described in Tang and Wu (1993). The only difference is that Tang and Wu's method can only be used for $n = 4t$ runs, and the current method can be used also for any n even runs.

5.3. Case 3: $g > 2$.

Note that we have not made additional assumptions on the entries of $\mathbf{R}$. To consider another cases of g, let us assume that the entries of $\mathbf{R}$ can be only $\pm 1, 0$. It is easy to show that g cannot be an odd integer larger than 1. For the case of $g = 4$, it will correspond to the case of a quarter-fraction of the Hadamard matrix (QFHM). According to Theorem 7 again, the necessary and sufficient condition for QFHM to be column balanced is

$$\mathbf{R}'\mathbf{1} = \frac{\pm \mathbf{h}_{k_1} \pm \mathbf{h}_{k_2} \pm \mathbf{h}_{k_3} \pm \mathbf{h}_{k_4}}{4}, \tag{5.9}$$

for some columns $k_1 < k_2 < k_3 < k_4$. If such columns exist (we have found they do not always), then the columns corresponding to (k_1, k_2, k_3, k_4) will be the branching columns. All rows with $\mathbf{R}'\mathbf{1} = \pm 1$ should be selected and the branching columns removed.

APPENDIX: PROOFS

Proof of Theorem 1: If $\mathbf{RH}$ is column balanced, then $\mathbf{RH}$ contains a $\pm\mathbf{1}$. Therefore,

$$\mathbf{1}'\mathbf{RH} = (0, 0, \ldots, \pm n, \ldots, 0).$$

Multiplying both sides by $\mathbf{H}'$, we can see that

$$\mathbf{1}'\mathbf{RHH}' = (0, 0, \ldots, \pm n, \ldots, 0)(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)' = \pm n\mathbf{h}_i', \quad \text{for some } 1 \le i \le n.$$

Since $\mathbf{HH}' = n\mathbf{I}$,

$$\mathbf{1}'\mathbf{R} = \pm \mathbf{h}_i', \quad \text{for some } 1 \le i \le n.$$

On the other hand, if $\mathbf{1}'\mathbf{R} = \pm \mathbf{h}_i'$, for some $1 \le i \le n$, then

$$\mathbf{1}'\mathbf{RH} = \mathbf{h}_i'(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n) = (0, 0, \ldots, \pm n, \ldots, 0).$$

This completes the proof of Theorem 1. ∎

PROOF OF THEOREM 2: Since $\mathbf{h}_i'\mathbf{a}_j = \mathbf{a}_j'\mathbf{h}_i$, we have

$$\begin{aligned}\sum_{i=1}^{n} v_{ij}^2 &= \sum_{i=1}^{n} \mathbf{a}_j'\mathbf{h}_i\mathbf{h}_i'\mathbf{a}_j \\ &= \mathbf{a}_j'\left(\sum_{i=1}^{n} \mathbf{h}_i\mathbf{h}_i'\right)\mathbf{a}_j \\ &= \mathbf{a}_j'\mathbf{H}\mathbf{H}'\mathbf{a}_j \\ &= n\mathbf{a}_j'\mathbf{I}\mathbf{a}_j \\ &= n^2.\end{aligned}$$

This proves Part (1). Parts (2a) and (2b) are true because

$$\begin{aligned}\mathbf{W}'\mathbf{W} &= \mathbf{H}'\mathbf{R}\left(\mathbf{H}\mathbf{H}'\right)\mathbf{R}'\mathbf{H} \\ &= n\mathbf{H}'\left(\mathbf{R}\mathbf{R}'\right)\mathbf{H} \\ &= n\mathbf{H}'\mathbf{H} \\ &= n^2\mathbf{I}.\end{aligned}$$

Let $\mathbf{h}_i \neq \pm\mathbf{1}$ be the i-th column of $\mathbf{H}$ and $\mathbf{h}_j^* \neq \pm\mathbf{1}$ be the j-th column of $\mathbf{RH}$.

$$\begin{aligned}w_{ij} = \mathbf{h}_i'\mathbf{h}_j^* &= \sum_{m=1}^{n} h_{im}h_{jm}^* \\ &= \sum_{m\in S_{++}}(+1) + \sum_{m\in S_{+-}}(-1) + \sum_{m\in S_{-+}}(-1) + \sum_{m\in S_{--}}(+1),\end{aligned}$$

where

$$\begin{aligned}S_{++} &= \{m | h_{im} = +1, h_{jm}^* = +1\}, \\ S_{+-} &= \{m | h_{im} = +1, h_{jm}^* = -1\}, \\ S_{-+} &= \{m | h_{im} = -1, h_{jm}^* = +1\}, \\ S_{--} &= \{m | h_{im} = -1, h_{jm}^* = -1\}.\end{aligned}$$

Since $\mathbf{1}'\mathbf{h}_i = 0$ and $\mathbf{1}'\mathbf{h}_j^* = 0$, we have

$$\begin{aligned}|S_{++}| + |S_{+-}| = |S_{-+}| + |S_{--}| &= \frac{n}{2} \\ |S_{++}| + |S_{-+}| = |S_{+-}| + |S_{--}| &= \frac{n}{2},\end{aligned}$$

where $|S|$ is the number of elements in a set S. From this, we can easily see that

$$|S_{+-}| = |S_{-+}| = P.$$

Hence,

$$\begin{aligned} w_{ij} &= |S_{++}| - |S_{+-}| - |S_{-+}| + |S_{--}| \\ &= (|S_{++}| + |S_{+-}| + |S_{-+}| + |S_{--}|) - 2(|S_{+-}| + |S_{-+}|) \\ &= n - 2(|S_{+-}| + |S_{-+}|) \\ &= n - 4P = 4(t - P). \end{aligned}$$

This proves Part (2c).

If $\mathbf{H}'$ is column balanced, then

$$\begin{aligned} \mathbf{1}'\mathbf{W} = (\mathbf{1}'\mathbf{H}')(\mathbf{RH}) &= (0, 0, .., \pm n, 0, ..., 0)\mathbf{RH} \\ &= \pm n\mathbf{g}'_*, \end{aligned}$$

where $\mathbf{g}'_*$ is a row vector of $\mathbf{RH}$. Similarly, we can see that

$$\mathbf{W1} = \pm n\mathbf{g},$$

for some column vector $\mathbf{g}$ with entries ± 1. This proves Part (2d). ∎

Proof of Lemma 1: Pre-multiplying $\mathbf{H}$ in both sides of equation

$$\mathbf{W} = \mathbf{H}'\mathbf{RH},$$

and using the fact $\mathbf{HH}' = n\mathbf{I}$, we have

$$n\mathbf{RH} = \mathbf{HW} = \mathbf{H}(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n) = (\mathbf{Hw}_1, \mathbf{Hw}_2, ..., \mathbf{Hw}_n),$$

where $\mathbf{w}_j$ is the j-th column of $\mathbf{W}$. Therefore the j-th column of $n\mathbf{RH}$ is

$$\begin{aligned} n\mathbf{R}\mathbf{h}_j &= \mathbf{H}\mathbf{w}_j \\ &= (\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_n)\,(w_{1j}, w_{2j}, ..., w_{nj})' \\ &= \sum_{i=1}^{n} w_{ij}\mathbf{h}_i. \end{aligned}$$

∎

Proof of Theorem 4: From Lemma 1, we have

$$n\mathbf{R}\mathbf{h}_j = \sum_{i \in U_j} w_{ij}\mathbf{h}_i. \tag{3.3}$$

Therefore, r-rank $\leq |U_j|$ for each $j \in \mathbf{C}$ and

$$\text{r-rank} \leq r_1.$$

Applying Lemma 1, for fixed $j_1 \neq j_2$, we have

$$n\mathbf{R}(c_1\mathbf{h}_{j_1} + c_2\mathbf{h}_{j_2}) = \sum_{i=1}^{n}(c_1 w_{ij_1} + c_2 w_{ij_2})\mathbf{h}_i.$$

If we choose $c_1 = 1, c_2 = 1$, then

$$n\mathbf{R}(\mathbf{h}_{j_1} + \mathbf{h}_{j_2}) = \sum_{i \in U^+_{j_1 j_2}} (w_{ij_1} + w_{ij_2})\mathbf{h}_i$$

and for $c_1 = 1, c_2 = -1$,

$$n\mathbf{R}(\mathbf{h}_{j_1} - \mathbf{h}_{j_2}) = \sum_{i \in U^-_{j_1, j_2}} (w_{ij_1} - w_{ij_2})\mathbf{h}_i.$$

Consequently, for each $j_1 \neq j_2 \in \mathbf{C}$,

$$\text{r-rank} \leq |U^+_{j_1, j_2}| + 1, \quad \text{r-rank} \leq |U^-_{j_1, j_2}| + 1.$$

Theorem 4 now follows easily from the fact that r-rank of the supersaturated design is also less than r_2. ▮

PROOF OF THEOREM 5: Let $\mathbf{h}_i$ be the i-th column of $\mathbf{H}$ ($i \geq 2$), then

$$\mathbf{h}_i = \mathbf{S}^{i-2}\mathbf{h}.$$

Note that $\mathbf{S}$ is a $n \times n$ orthogonal matrix of order $n-1$, that is,

$$\mathbf{S}\mathbf{S}' = \mathbf{I}, \quad \mathbf{S}^{-i} = \mathbf{S}^{n-1-i}.$$

Therefore,

$$\begin{aligned}\mathbf{H}'\mathbf{D}(\mathbf{h}_i)\mathbf{H} &= \mathbf{H}'\mathbf{D}(\mathbf{S}^{(i-2)}\mathbf{h})\mathbf{H} \\ &= \mathbf{H}'\left[\mathbf{S}^{(i-2)}\mathbf{D}(\mathbf{h})\mathbf{S}'^{(i-2)}\right]\mathbf{H} \\ &= \mathbf{H}'\mathbf{S}^{(i-2)}\mathbf{D}(\mathbf{h})\mathbf{S}^{-(i-2)}\mathbf{H}.\end{aligned}$$

Here we use a simple fact that $\mathbf{D}(\mathbf{R}\mathbf{h}) = \mathbf{R}\mathbf{D}(\mathbf{h})\mathbf{R}'$ for any permutation matrix $\mathbf{R}$. Since

$$\begin{aligned}\mathbf{S}^{-(i-2)}\mathbf{H} &= \mathbf{S}^{-(i-2)}\left(-\mathbf{1}, \mathbf{h}, \mathbf{S}\mathbf{h}, \mathbf{S}^2\mathbf{h}, ..., \mathbf{S}^{n-2}\mathbf{h}\right) \\ &= \left(-\mathbf{1}, \mathbf{S}^{-(i-2)}\mathbf{h}, \mathbf{S}^{-(i-1)}\mathbf{h}, \mathbf{S}^{-i}\mathbf{h}, ..., \mathbf{S}^{n-i}\mathbf{h}\right) \\ &= \mathbf{H}\mathbf{P},\end{aligned}$$

where $\mathbf{P}$ is a matrix representing the column permutation. From this, we can see that

$$\mathbf{H}'\mathbf{D}(\mathbf{h}_i)\mathbf{H} = \mathbf{P}'\mathbf{H}'\mathbf{D}(\mathbf{h})\mathbf{H}\mathbf{P}.$$

This proves Theorem 4. ▮

PROOF OF COROLLARY 1: Since $\mathbf{h}_i^* = \mathbf{R}^*\mathbf{h}_i$ and $\mathbf{R}^*$ is a permutation matrix, we have

$$\mathbf{D}(\mathbf{h}_i^*) = \mathbf{D}(\mathbf{R}^*\mathbf{h}_i) = \mathbf{R}^*\mathbf{D}(\mathbf{h}_i)\mathbf{R}^{*\prime}.$$

Hence,

$$\mathbf{H}^{*\prime}\mathbf{D}(\mathbf{h}_i^*)\mathbf{H}^* = \mathbf{H}'\mathbf{R}^{*\prime}(\mathbf{R}^*\mathbf{D}(\mathbf{h}_i)\mathbf{R}^{*\prime})\mathbf{R}^*\mathbf{H} = \mathbf{H}'\mathbf{D}(\mathbf{h}_i)\mathbf{H}.$$

The proof Corollary 1 is now completed using Theorem 4. ∎

Proof of Theorem 6: Without loss of generality, we assume $\mathbf{h}_1 = \pm\mathbf{1}$ is the first column of $\mathbf{H}$. For a fixed j column ($j \geq 2$) of $\mathbf{W}$, it is easy to see that when $i = 1, i = j$ and $i = l$ will satisfy

$$w_{ij} = \mathbf{h}_i'\mathbf{D}(\mathbf{h}_l)\mathbf{h}_j = 0.$$

Suppose now (i, j, l) are all different and they all are not equal to 1,

$$\begin{aligned} w_{ij} = \mathbf{h}_i'\mathbf{D}(\mathbf{h}_l)\mathbf{h}_j &= \sum_{m=1}^{n} h_{im}h_{lm}h_{jm} \\ &= \sum_{m\in S_{++}} h_{jm} - \sum_{m\in S_{+-}} h_{jm} - \sum_{m\in S_{-+}} h_{jm} + \sum_{m\in S_{--}} h_{jm}, \end{aligned}$$

where

$$S_{++} = \{m | h_{im} = +1, h_{lm} = +1\},$$

$$S_{+-} = \{m | h_{im} = +1, h_{lm} = -1\},$$

$$S_{-+} = \{m | h_{im} = -1, h_{lm} = +1\},$$

$$S_{--} = \{m | h_{im} = -1, h_{lm} = -1\}.$$

Using the conditions

$$\mathbf{h}_i'\mathbf{h}_l = 0, \quad \mathbf{h}_i'\mathbf{1} = 0, \quad \mathbf{h}_l'\mathbf{1} = 0,$$

we can see that

$$|S_{++}| = |S_{+-}| = |S_{-+}| = |S_{--}| = n/4 = t.$$

Let

$$p_{++} = \{m \in S_{++} | h_{jm} = +1\},$$

$$p_{+-} = \{m \in S_{+-} | h_{jm} = +1\},$$

$$p_{-+} = \{m \in S_{-+} | h_{jm} = +1\},$$

$$p_{--} = \{m \in S_{--} | h_{jm} = +1\}.$$

Using the conditions

$$\mathbf{h}_j'\mathbf{h}_i = 0, \quad \mathbf{h}_j'\mathbf{h}_l = 0, \quad \mathbf{h}_j'\mathbf{1} = 0,$$

we can see that

$$|p_{++}| = |p_{--}| = P, \quad |p_{+-}| = |p_{-+}| = t - P.$$

$$\begin{aligned} w_{ij} &= \mathbf{h}_i'\mathbf{D}(\mathbf{h}_l)\mathbf{h}_j = \sum_{m=1}^{n} h_{im}h_{lm}h_{jm} \\ &= [P-(t-P)] + [P-(t-P)] + [P-(t-P)] + [P-(t-P)] \\ &= 8P - 4t = 4(2P-t). \end{aligned}$$

If t is odd, say $t = 2L - 1$, then $w_{ij} = \pm 8k + 4$ with $k = |P - L|$. Clearly,

$$w_{ij} \neq 0.$$

This proves Part (1).

If t is even, say $t = 2L$, then $w_{ij} = \pm 8k$ with $k = |P - L|$. Therefore w_{ij} can only be a multiple of 8. This proves Part (2). ∎

Proof of Theorem 7: If $\mathbf{RH}$ is column balanced, then

$$\mathbf{1}'\mathbf{RH} = (0, 0, \pm\frac{n}{g}, 0, ..., 0, \pm\frac{n}{g}, ..., \pm\frac{n}{g}, 0, 0),$$

where $\mathbf{1}$ is a n/g-dimension column vector. Let us assume that $\pm\frac{n}{g}$ appears at c positions and their indices are $k_1 < k_2 < ... < k_c$. Multiplying both sides by the inverse of $\mathbf{H}$

$$\mathbf{H}^{-1} = \frac{1}{n}\mathbf{H}' = \frac{1}{n}\begin{pmatrix} \mathbf{h}_1' \\ \mathbf{h}_2' \\ \vdots \\ \mathbf{h}_n' \end{pmatrix},$$

we can now easily see that

$$\mathbf{R}'\mathbf{1} = \frac{1}{g}\sum_{j=1}^{c} \pm\mathbf{h}_{k_j}.$$

Since $\mathbf{RR}' = \mathbf{I}_{n/g \times n/g}$,

$$\begin{aligned}\frac{n}{g} = \mathbf{1}'\mathbf{1} &= \mathbf{1}'\mathbf{RR}'\mathbf{1} \\ &= \left(\frac{1}{g}\sum_{j=1}^{c} \pm \mathbf{h}_{k_j}\right)' \left(\frac{1}{g}\sum_{j=1}^{c} \pm \mathbf{h}_{k_j}\right) \\ &= \frac{1}{g^2}\sum_{j=1}^{c} n \\ &= \frac{nc}{g^2}.\end{aligned}$$

Thus, $c = g$ and the proof of Theorem 7 is completed. ∎

REFERENCES

Booth, K. H. V. and Cox, D. R. (1962), "Some Systematic Supersaturated Designs," *Technometrics* **4**, 489-495.

Cheng, C. S. (1993), "Some Projection Properties of Orthogonal Arrays," preprint.

Deng, L. Y, Lin, D. K. J. and Wang, J. (1993), "A Measurement of Multi-factor Orthogonality," preprint.

Hall, M. J. (1961), "Hadamard matrix of order 16," *Jet Propulsion Lab Summary* **1**, 21-26.

Hedayat, A. and Wallis, W. D. (1978), "Hadamard matrices and Their Applications," *Annals of Stat* **6**, 1184-1238.

Lin, D. K. J. (1991), "Systematic Supersaturated Designs," Working Paper, No. 264, College of Business Administration, University of Tennessee.

Lin, D. K. J. (1993a), "A New Class of Supersaturated Designs," *Technometrics* **35**, 28-31.

Lin, D. K. J. (1993b), "Another Look at First-Order Saturated Designs: The p-efficient Designs," *Technometrics* **35**, 284-292.

Lin, D. K. J. (1994), "Generating Systematic Supersaturated Designs," *Technometrics* (to appear).

Lin, D. K. J. and Draper, N. R. (1993), "Generating alias relationships for two-level Plackett and Burman designs," *Computational Statistics and Data Analysis* **15**, 147-157.

Plackett, R. L. and Burman, J. P. (1946), "The Design of Optimum Multifactorial Experiments," *Biometrika* **33**, 303-325.

Tang, B. and Wu, C. F. J. (1993), "Construction of Supersaturated Designs," preprint.

Wu, C. F. J. (1993), "Construction of Supersaturated Designs Through Partially Aliased Interactions," *Biometrika* **80**, 661-669.