A PROOF OF TRANSCENDENCE BASED ON FUNCTIONAL EQUATIONS

by

Alan Cobham

Research Division
Yorktown Heights, New York

ABSTRACT: It is shown that under suitable conditions the values, for rational arguments, of a function which satisfies a functional equation of special type are transcendental.

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I. INTRODUCTION

Fundamental to most proofs of transcendence is the fact that a number which can be too closely approximated by algebraic numbers cannot itself be algebraic. In the method developed by Siegal differential equations are used to obtain successive refinements of an initial algebraic approximation. Ultimately these approach too closely the number approximated, thus demonstrating its transcendence. It is not surprising that functional equations can replace differential equations in this argument with similar consequences. My purpose in presenting this report is to illustrate this point in detail [1].

The functional equations discussed explicitly are of restricted form. Examples arise naturally in the theory of partitions and related areas of analysis. The most interesting source, however, is the developing field of computational analysis, specifically that part concerned with real-time computable numbers, i.e., numbers whose binary expansion (say) can be mechanically generated, digit by digit at an unending, uniform rate. The generation process can be characterized by a system of functional equations which, under certain circumstances, reduces to an equation of the type treated here. Although the method of derivation of such equations lies outside the scope of this report, several of the examples included are drawn from this area [2].

Since both statement and proof for the general case are burdened by detail, I have started off, in Section II, with a simple special case, namely a proof of the transcendence of $\Sigma_{n=0}^{\infty} 2^{-2^n}$, which illustrates the essentials of the method. The proof depends on the fact that this number is the value for $z = \frac{1}{2}$ of the function $G_1(z) = \Sigma_{n=0}^{\infty} z^{2^n}$, which satisfies the functional equation $G_1(z^2) = G_1(z) - z$. The general case deals with functions of several variables, $G(z_1, \ldots, z_p) = G(z)$, which satisfy equations of the form

$$G(z^\theta) = r(z)G(z) + s(z),$$

where z is the p-tuple $(z_1, \ldots, z_p)$, z^θ a p-tuple of monomials in $z_1, \ldots, z_p$, and $r(z)$ and $s(z)$ are rational functions. After some necessary preliminaries the main result concerning the values of such functions is stated at the end of Section III. Section IV is devoted to its proof, Section V to examples.

II. TRANSCENDENCE OF $\Sigma\, 2^{-2^n}$

Let $G(z)$ be the function defined in the interior of the unit circle by the relation

$$(1) \qquad G(z) = \Sigma_{n=0}^{\infty} z^{2^n}.$$

Direct substitution shows that $G(z)$ satisfies the functional equation

$$(2) \qquad G(z^2) = G(z) - z.$$

In this section this fact is used to show the transcendence of $G(\frac{1}{2})$. The proof requires the following basic result [3].

Lemma 0. Let α be an algebraic number of degree g and let m be a positive integer. Then there exists a positive constant $W(\alpha, m)$, which depends only on α and m, such that if $p(y)$ is any polynomial of degree m whose coefficients are integers of absolute value not exceeding A, and if $p(\alpha) \neq 0$, then $A^{g-1} |p(\alpha)| \geq W(\alpha, m)$.

The lemma provides an objective. Given g we wish to find a sequence of integer polynomials $p_k(y)$ of some fixed degree such that $p_k(\alpha) \neq 0$, where $\alpha = G(\frac{1}{2})$, and $A_k^{g-1} p_k(\alpha) \to 0$, where A_k bounds the coefficients of $p_k(y)$. This will establish that α is not algebraic of degree g and hence, if we can find such a sequence for every g, that α is transcendental.

The objective is achieved indirectly. For some m, depending on g, we find integer polynomials $Q_j(z)$, $j = 0, \ldots, m$, such that the Maclaurin series for the function $R(z) = \Sigma_{j=0}^{m} Q_j(z)\, G^j(z)$ begins with a term in z of high exponent. For an appropriate n, $2^n Q_j(\frac{1}{2})$, $j = 0, \ldots, m$, will be an integer and $2^n R(\frac{1}{2}) = \Sigma\, 2^n\, Q_j(\frac{1}{2})\, G^j(\frac{1}{2}) = \Sigma\, 2^n Q_j(\frac{1}{2}) \alpha^j = p_0(\alpha)$ will be small. In other words, some root of $p_0(y) = 0$ will be a good approximation of α. A second, better approximation can be obtained by replacing z by z^2 in the expression defining $R(z)$, using the functional relation (2) to eliminate the $G(z^2)$ which this substitution introduces, and repeating the argument. Since the series for $R(z^2)$ begins with a term of exponent twice that of the first term of $R(z)$, the resulting approximation will be closer to α. Carrying this procedure to the limit we obtain a sequence of polynomials $p_0(y), p_1(y), \ldots$, whose roots approximate α, and the main effort of the proof is directed at showing that these approximations are too close in the sense suggested by Lemma 0.

<u>Lemma 1</u>. Given positive integers m and n there exist polynomials $Q_{0,0}(z), \ldots, Q_{0,m}(z)$, each of which has integer coefficients and is of degree in z not exceeding n, and not all of which are identically zero, such that in the Maclaurin series for

$$(3) \qquad R_0(z) = \Sigma_{j=0}^{m} Q_{0,j}(z)\, G^j(z)$$

the coefficients of all terms of degree in z less than mn vanish.

<u>Proof</u>. We regard the $(m+1)(n+1)$ coefficients of the $Q_{0,j}(z)$ as unknowns. Expanding the right side of (3) and equating all coefficients of degree $< mn$ in z to zero we obtain a system of mn homogeneous linear equations in $(m+1)(n+1)$ unknowns. The coefficients of this system are integral since the coefficients of the series (1) for $G(z)$ are. Such a system always possesses a nontrivial solution in integers; this determines in turn specific polynomials $Q_{0,0}(z), \ldots, Q_{0,m}(z)$ having the required properties.

With $Q_{0,j}(z)$, $j = 0, \ldots, m$, and $R_0(z)$ fixed as in the lemma we define sequences of functions $Q_{k,j}(z)$, $j = 0, \ldots, m$, and $R_k(z)$, $k = 0, 1, \ldots,$ which satisfy the relations

$$(4) \quad R_k(z) = \Sigma_{j=0}^{m} Q_{k,j}(z)\, G^j(z)$$

by setting

$$(5) \quad R_{k+1}(z) = R_k(z^2)$$
$$= \Sigma_{j=0}^{m} Q_{k,j}(z^2)\, G^j(z^2)$$
$$= \Sigma_{j=0}^{m} Q_{k,j}(z^2)(G(z)-z)^j$$
$$= \Sigma_{j=0}^{m}\{\Sigma_{i=j}^{m} \tbinom{i}{j}(-z)^{i-j} Q_{k,i}(z^2)\}\, G^j(z),$$

so that

$$(6) \quad Q_{k+1,j}(z) = \Sigma_{i=j}^{m} \tbinom{i}{j}(-z)^{i-j} Q_{k,i}(z^2).$$

<u>Lemma 2.</u> There exists a constant C_1 such that for all $j = 0, \ldots, m$ and $k = 0, 1, \ldots,$ $2^{2^k(m+n)} Q_{k,j}(\frac{1}{2})$ is an integer and

$$(7) \qquad |Q_{k,j}(\tfrac{1}{2})| \leq C_1 2^{mk}.$$

<u>Proof</u>. We show first that each $Q_{k,j}(z)$ is a polynomial of degree $\leq 2^k(m+n)-m$. For $k = 0$ this is immediate from Lemma 1. Assuming it is true for a given k, we obtain from the recurrence relation (6),

(8) $$\deg Q_{k+1,j}(z) \leq m + \max_{0\leq i\leq m} \deg Q_{k,i}(z^2)$$

$$\leq m + 2(2^k(m+n)-m) = 2^{k+1}(m+n)-m,$$

and the statement follows by induction. Clearly this implies that $2^{2^k(m+n)}Q_{k,j}(\frac{1}{2})$ is an integer.

Let

(9) $$C_1 = \max_{|z|\leq 1,\ 0\leq j\leq m} |Q_{0,j}(z)|.$$

We claim that $|Q_{k,j}(z)| \leq C_1 2^{mk}$ for $|z| \leq 1$. This is true for $k = 0$ and, assuming it for given k, we have from (6),

(10) $$|Q_{k+1,j}(z)| \leq \Sigma_{i=j}^{m} \binom{i}{j} |Q_{k,i}(z^2)| \leq 2^m \cdot C_1 2^{mk} = C_1 2^{m(k+1)},$$

which establishes the claim and hence the special case (7).

Lemma 3. There exists a constant C_2 such that for $k = 0, 1, \ldots,$

(11) $$|R_k(\tfrac{1}{2})| \leq C_2 2^{-2^k mn}.$$

Proof. By construction the function $R_0(z)/z^{mn}$ is analytic in the unit circle, hence bounded by some constant C_2 in the circle about $z = 0$ of radius $\frac{1}{2}$. Thus $|R_0(z)| \leq C_2|z|^{mn}$ for $|z| \leq \frac{1}{2}$. From the recurrence relation (5) it follows easily that $R_k(z) = R_0(z^{2^k})$, hence $|R_k(\frac{1}{2})| = |R_0(2^{-2^k})| \leq C_2\, 2^{-2^k mn}$.

Lemma 4. For all sufficiently large k, $R_k(\frac{1}{2}) \neq 0$.

Proof. If there are arbitrarily large k such that $R_k(\frac{1}{2}) = 0$, then, since $R_0(2^{-2^k}) = R_k(\frac{1}{2})$, the point $z = 0$ is a limit point of zeros of $R_0(z)$. Since $R_0(z)$ is analytic at $z = 0$ it follows that $R_0(z) \equiv 0$. From equation (3) of Lemma 1 we see that this implies that $G(z)$ is an algebraic function. But, recalling that $G(z)$ is a textbook example of a function having the unit circle as natural boundary of definition and that no algebraic function has such a natural boundary, this provides a contradiction [4]. The lemma follows.

Using estimates supplied by the preceding lemmas we are now in a position to prove the main result of this section.

Theorem. The number $\alpha = G(\frac{1}{2}) = \Sigma_{n=0}^{\infty} 2^{-2^n}$ is transcendental.

Proof. Assume on the contrary that α is algebraic of degree g. We select the parameters m and n introduced in Lemma 1 so that

(12) $$\epsilon = mn - g(m+n) > 0.$$

Let

(13) $\omega_{k,j} = 2^{2^k(m+n)} Q_{k,j}(\frac{1}{2})$,

(14) $\xi_k = 2^{2^k(m+n)} R_k(\frac{1}{2})$.

By Lemmas 2 and 3, all $\omega_{k,j}$ are integers and

(15) $|\omega_{k,j}| \leq C_1 \; 2^{mk+2^k(m+n)}$,

(16) $|\xi_k| \leq C_2 \; 2^{-2^k(mn-(m+n))}$.

By (4), (13) and (14),

(17) $\xi_k = \Sigma_{j=0}^{m} \omega_{k,j} \alpha^j$.

Thus, by Lemma 0,

(18) $$(C_1 \; 2^{mk+2^k(m+n)})^{g-1}(C_2 \; 2^{-2^k(mn-m-n)})$$

$$= C_1^{g-1} C_2 \; 2^{(g-1)mk-2^k(mn-g(m+n))}$$

$$= C_1^{g-1} C_2 \; 2^{(g-1)mk-2^k\epsilon} \geq W(\alpha, m)$$

provided $\xi_k \neq 0$, which by (14) and Lemma 4 is the case for all sufficiently large k. On the other hand, since $\epsilon > 0$, $(g-1)mk - 2^k\epsilon \to -\infty$, and thus the inequality (18) fails for all sufficiently large k. From this contradiction we deduce that α cannot be algebraic of degree g, and thus, since g can be chosen arbitrarily, cannot be algebraic; i.e., α is transcendental.

In concluding this section it should be pointed out that the transcendence of $\Sigma\, 2^{-2^n}$ can be obtained as a corollary of the Thue-Siegal-Roth-Mahler-Ridout theorem. The proof given above, however, appears to me to be basically simpler and more direct [5].

III. PRELIMINARIES; MAIN THEOREM

Through the remainder of this report we will be dealing with functions of some fixed, but generally unspecified, number of variables, p. If $z_1, \dots, z_p$ are variables, we write z for the p-tuple $(z_1, \dots, z_p)$. A similar convention is used for p-tuples of constants; in particular, we sometimes use 0 for a p-tuple of zeros. If z is a p-tuple and $\theta = \|\theta_{ij}\|$ a $p \times p$ matrix of integers we write z^θ for the p-tuple of monomials:

$$(z_1^{\theta_{11}} \cdots z_p^{\theta_{1p}}, \dots, z_1^{\theta_{i1}} \cdots z_p^{\theta_{ip}}, \dots, z_1^{\theta_{p1}} \cdots z_p^{\theta_{pp}}).$$

Our concern is mainly with functions having the properties (a) - (e) listed below. A function $G(z) = G(z_1, \dots, z_p)$ having these properties will be called an acceptable function.

(a) $G(z)$ is a function in p complex variables analytic in some region containing the origin, at which it has a power series expansion

$$(1) \qquad G(z) = \Sigma\, a_{i_1 \dots i_p} z_1^{i_1} \cdots z_p^{i_p}$$

in which all coefficients are rational.

(b) There are rational functions $r(z)$ and $s(z)$, and a nonsingular $p \times p$ matrix of nonnegative integers $\theta = \|\theta_{ij}\|$ such that the functional relation

$$(2) \qquad G(z^\theta) = r(z)\, G(z) + s(z)$$

holds at all points z for which both sides are defined.

(c) $r(z) \not\equiv 0$. Both $r(z)$ and $s(z)$ are representable as ratios of polynomials having integer coefficients. (Equivalently: their power series at rational points have rational coefficients.)

(d) The matrix θ is irreducible and has an eigenvalue of modulus greater than one.

(e) The intersection of the domain of definition of $G(z)$ with the set $\mathscr{D}(\theta)$, composed of all points z such that $\lim_{k \to \infty} z^{\theta^k} = 0$, is connected.

Example 1. The function $G_1(z) = \Sigma_{n=0}^{\infty} z^{2^n}$ discussed in Section II is acceptable. Here $p = 1$, $\theta = (2)$, $r(z) = 1$, $s(z) = -z$.

Example 2. Let $G_2(z) = \Sigma_{n=0}^{\infty} a_n z^n$ where $a_n = 0$ or 1 according as the number of 1's in the binary representation of n is even or odd. Then $G_2(z)$ is acceptable since it satisfies the functional relation

$$(3) \qquad G_2(z^2) = \frac{1}{1-z}\, G_2(z) - \frac{z}{(1-z)(1-z^2)} .$$

Example 3. The function $G_3(z_1, z_2) = \Sigma_{n=0}^{\infty} z_1^{f(n+1)} z_2^{f(n)}$, where $f(n)$ = the n-th Fibonacci number ($f(0) = 0$, $f(1) = 1$, $f(n+2) = f(n) + f(n+1)$) is acceptable. It satisfies the relation

$$(4) \qquad G_3(z_1 z_2, z_1) = G_3(z_1, z_2) - z_1.$$

Here

$$(5) \qquad \theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is irreducible and has maximal eigenvalue $\lambda = \frac{1}{2}(1 + \sqrt{5}) > 1$.

Example 4. A rational function in any number of variables which is analytic at the origin and whose power series there has rational coefficients is acceptable.

Neither the functional relation (2) nor the matrix θ is uniquely determined by a given acceptable function. Nevertheless, we will often speak of the functional relation and the associated matrix, always having in mind some fixed ones of these for which conditions (a) - (e) hold.

Concerning condition (d), a nonnegative matrix θ is irreducible if there is no permutation matrix P such that

$$(6) \qquad P\theta P^{-1} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are square matrices and 0 a rectangular matrix of zeros. An irreducible matrix θ always has a positive eigenvalue λ which is a simple root of its characteristic equation and is of maximum modulus among all eigenvalues. Further, if θ has $h \geq 1$ distinct eigenvalues all of modulus λ, then there are h matrices $\Phi = \Phi_0, \ldots, \Phi_{h-1}$ such that for $\ell = 0, \ldots, h-1$ and $k = 0, 1, \ldots,$

$$(7) \qquad \theta^{kh+\ell} \sim \lambda^{kh+\ell} \Phi_\ell,$$

where the asymptotic relation is an elementwise one. Each Φ_ℓ is nonnegative with all row and column sums positive [6]. To avoid exponents on exponents we often write $\theta(k)$ for θ^k and denote the entries of $\theta(k)$ by $\theta_{ij}(k)$: $\theta(k) = ||\theta_{ij}(k)||$.

Associated with each θ is the set $\mathcal{E}(\theta)$ of all points η such that $\eta^{\theta(k)} \to 0$. We refer to this set as the natural domain for θ. The choice of terminology is suggested by the fact that many acceptable functions have natural boundaries of definition which coincide with the boundary of the natural domain of the associated matrix. In all cases an acceptable function can be continued throughout the corresponding natural domain, the only singularities within this domain being removable by multiplication by a polynomial; i.e., if η is a singularity of $G(z)$ in the natural domain, then there is a polynomial $P(z)$ such that the singularity of $P(z)G(z)$ at η is removable.

Example 1, cont. The matrix $\theta = (2)$ has as natural domain the set of points, $|z| < 1$. As noted in the preceding section, the boundary of this domain is the natural boundary of definition of the function $G_1(z)$, as it is also for the function $G_2(z)$ of Example 2 above.

Example 3, cont. The matrix $\theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has the eigenvalues $\lambda = \frac{1}{2}(1 + \sqrt{5})$, $\lambda_1 = \frac{1}{2}(1 - \sqrt{5})$ and $\lambda > |\lambda_1|$ so $h = 1$. For this matrix we have

$$\Phi = \frac{1}{2\lambda - 1}\begin{pmatrix} \lambda & 1 \\ 1 & \lambda-1 \end{pmatrix}. \tag{8}$$

The natural domain for θ is the set of z for which $|z_1|^\lambda |z_2| < 1$. This is exactly the region of convergence of the series for $G_3(z)$ and is the natural domain of definition of this function.

<u>Example 5.</u> The matrix $\theta = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ has the eigenvalues $\lambda = 2$, $\lambda_1 = -2$, hence $h = 2$. We have

$$\Phi = \Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

The natural domain for θ is the set of z for which $|z_1| < 1$, $|z_2| < 1$.

Quite often we refer to the natural domain of a function when we mean the natural domain of the matrix associated with its functional equation. Our results concern only points lying within such domains. Although their shape does not directly concern us, the two preceding examples indicate the general situation: a natural domain is defined in terms of h monomials, possibly with irrational exponents, and these exponents appear as linearly independent rows of the matrix Φ.

The proof given in Section II generalizes readily to establish the transcendence of $G_1(\zeta)$ for any rational point ζ with $0 < |\zeta| < 1$. For acceptable functions of several variables such generality is not always achievable. An acceptable function which is globally transcendental may be rational on certain algebraic manifolds, and for rational points on these manifolds the values of such functions cannot be transcendental.

Example 5, cont. The function

$$G_5(z_1, z_2) = \Sigma_{n=0}^{\infty}(-1)^n(z_1^{2^n} - z_2^{2^n}) \tag{10}$$

is acceptable, satisfying the functional equation

$$G_5(z_2^2, z_1^2) = G_5(z_1, z_2) - z_1 + z_2. \tag{11}$$

The series (10) converges in the natural domain of $G_5(z)$ and represents a transcendental function there. However, it is identically zero on the surface $z_1 - z_2 = 0$, while on the surface $z_1 - z_2^4 = 0$ it coincides with the polynomial function $z_2^2 - z_2$. Thus, e.g., $G_5(\frac{1}{2}, \frac{1}{2})$ and $G_5(\frac{1}{16}, \frac{1}{2})$ are rational numbers.

The point at which the proof in Section II breaks down lies in Lemma 4 in which we used the fact, true for functions of one variable but not for functions of several, that a function which is analytic at a limit point of a sequence of its zeros is necessarily identically zero. By putting a simple restriction on ζ we can, however, obtain an analog of this result for sequences of the type in which we are most interested, namely, those of the form $\{\zeta^{\theta(k)} | k = 0, 1, \dots\}$. For a monomial $m(z) = z_1^{i_1} \dots z_p^{i_p}$ and a point ζ we denote by $m(|\zeta|^{\Phi})$ the product $(\prod_{j=1}^{p} \prod_{\ell=1}^{p} |\zeta|^{\phi_{j\ell}})^{i_j}$, where $\Phi = ||\phi_{ij}||$ is the matrix introduced above. We will call a point ζ in the natural domain of θ a <u>general point</u> of θ if for any two monomials, $m_1(z)$ and $m_2(z)$, each having unit coefficient, the identity $m_1(|\zeta|^{\Phi}) = m_2(|\zeta|^{\Phi})$ implies $m_1(z) \equiv m_2(z)$. It should be noted that if ζ is a general point, then so is ζ^{θ}; for the identity $m_1(|\zeta^{\theta}|^{\Phi}) = m_2(|\zeta^{\theta}|^{\Phi})$ implies $m_1^{\theta}(|\zeta|^{\Phi}) = m_2^{\theta}(|\zeta|^{\Phi})$, where by definition, $m_i^{\theta}(z) = m_i(z^{\theta})$, and because of the nonsingularity of θ, $m_1^{\theta}(z)$ and $m_2^{\theta}(z)$ are distinct if $m_1(z)$ and $m_2(z)$ are. Concerning general points, we will show, as Lemma 4 of the next section, that a function which is analytic at the origin and takes the value zero on a sequence $\{\zeta^{\theta(k)}\}$, where ζ is a general point, is necessarily identically zero.

Example 3, cont. Every point ζ with nonzero coordinates which belongs to the natural domain of $\theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is general. For if $m_1(z) = z_1^{i_1} z_2^{i_2}$, $m_2(z) = z_1^{j_1} z_2^{j_2}$, then $|m_1(\zeta^{\Phi})| = |m_2(\zeta^{\Phi})|$ implies

$$(12) \qquad 1 = |\zeta_1|^{k_1\lambda + k_2} |\zeta_2|^{k_1 + k_2(\lambda - 1)} = (|\zeta_1|^{\lambda} |\zeta_2|)^{k_1 + k_2/\lambda},$$

where $k_1 = j_1 - i_1$, $k_2 = j_2 - i_2$. (Recall, $1/\lambda = \lambda - 1$.) As noted earlier, the expression in parentheses at the right is less than unity; hence for (12) to be satisfied we must have $k_1 + k_2/\lambda = 0$. Since λ is irrational, it follows that $k_1 = k_2 = 0$ and hence that $m_1(z) \equiv m_2(z)$.

Example 5, cont. The general points of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ are just those which belong to its natural domain and have coordinates whose absolute values are multiplicatively independent, i.e., satisfy no nontrivial relation of the form $|\zeta_1|^{k_1}|\zeta_2|^{k_2} = 1$, k_1, k_2 integers. Thus $(\frac{1}{2}, \frac{1}{3})$ is general, $(\frac{1}{2}, \frac{1}{2})$ is not.

We are in position now to state our main result.

Theorem. Let $G(z)$ be an acceptable function satisfying the functional relation

$$G(z^\theta) = r(z)G(z) + s(z),$$

and let ζ be a general point of θ, all of whose coordinates are rational, at which $G(z)$ is defined. If for no $k \geq 0$ is $\zeta^{\theta(k)}$ a singularity of $r(z)$, and if $G(z)$ is transcendental, then $G(\zeta)$ is transcendental.

IV. PROOF OF THE MAIN THEOREM

We assume $G(z)$ is an acceptable function satisfying the relation

$$G(z^\theta) = r(z)G(z) + s(z) \tag{1}$$

and that ζ is a point satisfying all conditions of the theorem stated at the end of the preceding section. Initially we require in addition that for some $\rho < 1$, ζ belong to the polycylinder $\mathcal{C}(\rho)$ consisting of all points z such that $|z_i| < \rho$, $i = 1, \ldots, p$, and that in turn the closure $\bar{\mathcal{C}}(\rho)$ of $\mathcal{C}(\rho)$ lie entirely within the region of convergence of the power series

$$G(z) = \Sigma\, a_{i_1 \ldots i_p} z_1^{i_1} \ldots z_p^{i_p}. \tag{2}$$

Since $\rho < 1$, it follows easily that $z^\theta \in \mathcal{C}(\rho)$ whenever $z \in \mathcal{C}(\rho)$.

By the degree in z_i of a polynomial $P(z)$, which we write $\deg_{z_i} P(z)$, we mean the greatest integer d such that z_i^d occurs in a monomial having nonzero coefficient in $P(z)$. By the total degree of $P(z)$, written $\deg P(z)$, we mean the greatest of the sums $d = d_1 + \ldots + d_p$ such that $z_1^{d_1} \ldots z_p^{d_p}$ occurs with nonzero coefficient in $P(z)$.

<u>Lemma 1.</u> Let m and n be positive integers. Then there exist polynomials $Q_{0,0}(z), \ldots, Q_{0,m}(z)$, and a function $R_0(z)$ such that the following hold.

(i) All $Q_{0,j}(z)$ have integer coefficients and at least one of them is not identically zero.

(ii) For each i and j, $\deg_{z_i} Q_{0,j}(z) \leq m^{(p-1)/p}n$.

(iii) $R_0(z)$ has a power series expansion

$$(3) \qquad R_0(z) = \sum_{i_1+\cdots+i_p \geq mn} r_{i_1\ldots i_p} z_1^{i_1} \cdots z_p^{i_p}$$

which converges at least for $z \in \overline{\mathcal{C}}(\rho)$.

(iv) $R_0(z) = \Sigma_{j=0}^m Q_{0,j}(z) G^j(z), \quad z \in \overline{\mathcal{C}}(\rho)$.

<u>Proof.</u> Consider the polynomials $Q_j^*(z) = \Sigma q^{(j)}_{i_1\ldots i_p} z_1^{i_1} \cdots z_p^{i_p}$ $(j = 0, \ldots, m)$ where the coefficients are indeterminate and the sum is over p-tuples of indices with each coordinate $\leq m^{(p-1)/p}n$. We form the sum $\Sigma_{j=0}^m Q_j^*(z) G^j(z)$, replace $G(z)$ by its series expression (2), and multiply out and collect terms in $z_1, \ldots, z_p$ with like exponents, to obtain a power series in $z_1, \ldots, z_p$ whose coefficients are linear forms in the $q^{(j)}_{i_1\ldots i_p}$. Equating to zero the coefficients of those terms $z_1^{i_1} \cdots z_p^{i_p}$ for which $i_1 + \cdots + i_p < mn$, we obtain a system of $\binom{mn+p-1}{p}$ linear, homogeneous equations. The number of indeterminates is easily calculated as $(m+1)([m^{(p-1)/p}n]+1)^p$. Since

$$(4) \qquad (m+1)([m^{(p-1)/p}n]+1)^p > \frac{m+1}{m}(mn)^p$$
$$> \frac{1}{p!}(mn)(mn+1)\cdots(mn+p-1) = \binom{mn+p-1}{p},$$

the system possesses a nontrivial solution. Furthermore, such a solution can be found in integers since the coefficients of the series (2), and hence of the system, are rational. If we replace the indeterminates in each $Q_j^*(z)$ by the corresponding values in this solution, we obtain polynomials $Q_{0,j}(z)$ for which (i) and (ii) hold. Furthermore, if we use the relation (iv) as a definition of $R_0(z)$ in $\overline{\mathcal{C}}(\rho)$, (iii) is also seen to hold and the lemma follows.

Using the $Q_{0,j}(z)$ and $R_0(z)$ so obtained as initial functions we define inductively functions $Q_{k,j}(z)$, $j = 0, \ldots, m$, and $R_k(z)$ such that for $k = 0, 1, \ldots,$

$$(5) \qquad R_k(z) = \Sigma_{j=0}^m Q_{k,j}(z) G^j(z), \quad z \in \overline{\mathcal{C}}(\rho)$$

by setting

$$(6) \qquad R_{k+1}(z) = R_k(z^\theta),$$

(7) $$Q_{k+1,j}(z) = r^j(z)\, \Sigma_{i=j}^m \binom{i}{j} s^{i-j}(z)\, Q_{k,i}(z^\theta).$$

All these functions satisfy (5). For $k = 0$ this follows from Lemma 1(iv), while if we assume (5) holds for a fixed k then, since $z \in \overline{\mathcal{C}}(\rho)$ implies $z^\theta \in \overline{\mathcal{C}}(\rho)$,

(8) $$R_{k+1}(z) = R_k(z^\theta)$$
$$= \Sigma_{j=0}^m Q_{k,j}(z^\theta) G^j(z^\theta)$$
$$= \Sigma_{j=0}^m Q_{k,j}(z^\theta)[r(z)G(z) + s(z)]^j$$
$$= \Sigma_{j=0}^m \left[r^j(z)\, \Sigma_{i=j}^m \binom{i}{j} s^{i-j}(z) Q_{k,i}(z^\theta)\right] G^j(z)$$
$$= \Sigma_{j=0}^m Q_{k+1,j}(z) G^j(z), \qquad z \in \overline{\mathcal{C}}(\rho).$$

Assuming $r(z)$ and $s(z)$ are expressed as ratios of relatively prime polynomials, we let $q(z)$ be the least common multiple of their denominators. By condition (c) in the definition of an acceptable function, $q(z)$ can be chosen with integer coefficients. Let

(9) $$\mu = \max\,(\deg q(z),\ \deg q(z)r(z),\ \deg q(z)s(z)),$$

(10) $$P_{k,j}(z) = \left(\prod_{\ell=0}^{k-1} q(z^{\theta(\ell)}) \right)^m Q_{k,j}(z),$$

(11) $$\theta^*(k) = \Sigma_{i=1}^p \Sigma_{j=1}^p \theta_{ij}(k).$$

<u>Lemma 2.</u> (i) For $j = 0, \ldots, m$ and $k = 0, 1, \ldots,$ $P_{k,j}(z)$ is a polynomial with

(12) $$\deg P_{k,j}(z) \le m^{(p-1)/p} n\theta^*(k) + m\mu\, \Sigma_{\ell=0}^{k-1} \theta^*(\ell).$$

(ii) There exists a constant M such that

(13) $$|P_{k,j}(\zeta)| \le M^{k+1} \quad \text{for all } j \text{ and } k.$$

<u>Proof.</u> (i) For $j = 0, \ldots, m$, each $P_{0,j}(z) = Q_{0,j}(z)$ is a polynomial. Assuming for given k that each $P_{k,j}(z)$ is a polynomial, we have by (10) and (7),

(14) $$P_{k+1,j}(z) = \left(\prod_{\ell=0}^{k} q(z^{\theta(\ell)}) \right)^m Q_{k+1,j}(z)$$
$$= (q(z)r(z))^j \Sigma_{i=j}^m \binom{i}{j} (q(z)s(z))^{i-j} q^{m-i}(z) \left(\prod_{\ell=1}^{k} q(z^{\theta(\ell)}) \right)^m Q_{k,i}(z^\theta)$$
$$= (q(z)r(z))^j \Sigma_{i=j}^m \binom{i}{j} (q(z)s(z))^{i-j} q^{m-i}(z) P_{k,i}(z^\theta),$$

so, by induction, all $P_{k,j}(z)$ are polynomials. Since $\deg P_{k,j}(z) \le \Sigma_{i=1}^p \deg_{z_i} P_{k,j}(z)$, to establish (12) it is sufficient to show that for $i = 1, \ldots, p$

(15) $\deg_{z_i} P_{k,j}(z) \leq m^{(p-1)/p_n \Sigma_{t=1}^{p} \theta_{ti}(k)} + m\mu \Sigma_{\ell=0}^{k-1} \Sigma_{t=1}^{p} \theta_{ti}(\ell)$.

Clearly (15) holds for $k = 0$, and if it holds for a given k, we have from (14),

(16) $\deg_{z_i} P_{k+1,j}(z) \leq j\mu + \max_{\ell \geq j} \{(\ell - j)\mu + (m-\ell)\mu + \deg_{z_i} P_{k,\ell}(z^\theta)\}$

$$\leq m\mu + \max_{\ell} \{\Sigma_{s=1}^{p} \theta_{si} \deg_{z_s} P_{k,\ell}(z)\}$$

$$\leq m\mu + \Sigma_{s=1}^{p} \theta_{si} \{m^{(p-1)/p_n \Sigma_{t=1}^{p} \theta_{ts}(k)} + m\mu \Sigma_{\ell=0}^{k-1} \Sigma_{t=1}^{p} \theta_{ts}(\ell)\}$$

$$= m^{(p-1)/p_n \Sigma_{t=1}^{p} \theta_{ti}(k+1)} + m\mu \Sigma_{\ell=0}^{k} \Sigma_{t=1}^{p} \theta_{ti}(\ell),$$

and (12) follows.

(ii) Let

(17) $M = (2 \max_{z \in \overline{\mathcal{C}}(\rho)} (|P_{0,0}(z)|, \ldots, |P_{0,m}(z)|, |q(z)|, |q(z) r(z)|, |q(z) s(z)|))^m.$

Clearly $|P_{0,j}(\zeta)| \leq M$. Suppose $|P_{k,j}(\zeta)| \leq M^{k+1}$. From (14) we have

(18) $$|P_{k+1,j}(\zeta)| \leq \frac{M}{2^m} \sum_{i=j}^{m} \binom{i}{j} |P_{k,i}(\zeta^\theta)| \leq M^{k+2}.$$

The lemma follows.

Lemma 3. There exist constants C and D, with $D < 1$, such that for $k = 0, 1, \ldots,$

(19) $$|R_k(\zeta)| \leq CD^{mn\lambda^k}.$$

Proof. Since $\overline{\mathcal{C}}(\rho)$ lies within the region of convergence of the series (3), we have

(20) $$\sum_{i_1+\cdots+i_p \geq mn} |r_{i_1 \ldots i_p}| \rho^{i_1+\cdots+i_p} < \infty$$

so that there is some B such that

(21) $$\sum_{i_1+\cdots+i_p = i} |r_{i_1 \ldots i_p}| \rho^i \leq B, \quad i \geq mn.$$

For $z \in \mathcal{C}(\rho)$ let $\sigma(z) = \max_i |z_i|$. Then

(22) $$|R_0(z)| \leq \sum_{i_1+\cdots+i_p \geq mn} |r_{i_1 \ldots i_p}| |z_1|^{i_1} \cdots |z_p|^{i_p}$$

$$\leq \Sigma_{i=mn}^{\infty} \sum_{i_1+\cdots+i_p = i} |r_{i_1 \ldots i_p}| \sigma^i(z)$$

$$\leq B \Sigma_{i=mn}^{\infty} \left(\frac{\sigma(z)}{\rho}\right)^i = \frac{B(\sigma(z)/\rho)^{mn}}{1 - \sigma(z)/\rho}.$$

The matrices $\Phi_\ell = ||\phi_{ij}^{(\ell)}||$ were introduced in the preceding section. We let

$$\bar{\phi} = \min_{i,\ell} \Sigma_{j=1}^{p} \phi_{ij}^{(\ell)}. \tag{23}$$

Since all row sums of each Φ_ℓ are positive, $\bar{\phi} > 0$, and from the asymptotic relation (III-7) we have

$$\Sigma_{j=1}^{p} \theta_{ij}(kh+\ell) \sim \lambda^{kh+\ell} \Sigma_{j=1}^{p} \phi_{ij}^{(\ell)} \geq \bar{\phi}\lambda^{kh+\ell} \tag{24}$$

for $\ell = 0, \ldots, h-1$. Thus there is a k_0 such that for $k \geq k_0$,

$$\Sigma_{j=1}^{p} \theta_{ij}(k) \geq \frac{1}{2} \bar{\phi}\lambda^{k}, \quad i = 1, \ldots, p. \tag{25}$$

Thus, observing that

$$z^{\theta(k)} = (z_1^{\theta_{11}(k)} \ldots z_p^{\theta_{1p}(k)}, \ldots, z_1^{\theta_{p1}(k)} \ldots z_p^{\theta_{pp}(k)}) \tag{26}$$

we have, for $z \in \mathcal{C}(\rho)$,

$$\begin{aligned} \sigma(z^{\theta(k)}) &\leq \max_i (\sigma(z)^{\Sigma_j \theta_{ij}(k)}) \\ &\leq (\sigma(z))^{\frac{1}{2}\bar{\phi}\lambda^k}, \quad k \geq k_0. \end{aligned} \tag{27}$$

A simple induction using (6) shows that $R_k(z) = R_0(z^{\theta(k)})$ for $z \in \mathcal{C}(\rho)$. Using this and setting

$$D = (\sigma(\zeta))^{\frac{1}{2}\bar{\phi}}, \tag{28}$$

we have from (22), (27), (28),

$$\begin{aligned} |R_k(\zeta)| &\leq B \frac{(\sigma(\zeta^{\theta(k)})/\rho)^{mn}}{1 - \sigma(\zeta^{\theta(k)})/\rho} \\ &\leq \frac{B}{\rho^{mn}(1 - \sigma(\zeta)/\rho)} D^{mn\lambda^k}. \end{aligned} \tag{29}$$

If now we set

$$C = \max\left\{ \frac{B}{\rho^{mn}(1 - \sigma(\zeta)/\rho)}, \ \max_{k<k_0} \frac{|R_k(\zeta)|}{D^{mn\lambda^k}} \right\}, \tag{30}$$

(28) and (30) yield constants C and D satisfying the requirements of the lemma.

Lemma 4. If for all sufficiently large k, $R_0(\zeta^{\theta(k)}) = 0$ then $R_0(z) \equiv 0$ for $z \in \mathcal{C}(\rho)$.

Proof. We order the set of all monomials in $z_1, \ldots, z_p$ which have unit coefficient by putting $m_1(z) \prec m_2(z)$ if $m_1(z)$ and $m_2(z)$ are monomials such that $m_1(|\zeta|^{\Phi}) > m_2(|\zeta|^{\Phi})$. Since ζ is a general point of θ, $\prec$ is a linear order. If $m(z) = z_1^{i_1} \ldots z_p^{i_p}$ then

$$m(|\zeta|^{\Phi}) = (|\zeta_1|^{\phi_{11}} \ldots |\zeta_p|^{\phi_{1p}})^{i_1} \ldots (|\zeta_1|^{\phi_{p1}} \ldots |\zeta_p|^{\phi_{pp}})^{i_p}, \tag{31}$$

where $\Phi = ||\phi_{ij}||$. Since the conditions we have placed on ζ imply $0 < |\zeta_i| < 1$ for $i = 1, \ldots, p$, and since all row sums of the matrix Φ are positive, each of the expressions in parentheses in (31) is positive and less than unity. Thus $m(|\zeta|^{\Phi})$, considered as a function of the exponents $i_1, \ldots, i_p$, is strictly decreasing in each with limit zero. It follows that for any $\epsilon > 0$ there are only finitely many monomials $m(z)$ such that $m(|\zeta|^{\Phi}) > \epsilon$, and consequently $\prec$ is discrete of type ω.

Rearranging the terms of the series (3) we obtain an expression

$$(32) \qquad R_0(z) = \Sigma_{i=0}^{\infty} r_i m_i(z),$$

where the $m_i(z)$ are the monomials in $z_1, \ldots, z_p$ with unit coefficient indexed according to the order $\prec$ defined above, and the r_i are rational numbers. Since (3) is absolutely convergent in $\overline{\mathcal{C}}(\rho)$ the expressions (3) and (32) are equivalent for $z \in \overline{\mathcal{C}}(\rho)$.

Suppose now that the hypotheses of the lemma hold but $R_0(z) \not\equiv 0$. Then there is a least index t such that $r_t \neq 0$ and from (32) we obtain

$$(33) \qquad |r_t| \leq \Sigma_{i>t} |r_i| \left| \frac{m_i(\zeta^{\theta(kh)})}{m_t(\zeta^{\theta(kh)})} \right| ,$$

a relation which holds for all sufficiently large k. Let I_i denote the sum of the exponents of $m_i(z)$, and let $\tau = \max_j (|\zeta_1|^{\phi_{j1}} \ldots |\zeta_p|^{\phi_{jp}})$. From (III-7) we obtain the asymptotic relation $|m_i(\zeta^{\theta(kh)})| = (m_i(|\zeta|^{\Phi}))^{\lambda^{kh}} + o(\lambda^{kh})$, and so by (31),

$$(34) \qquad |m_i(\zeta^{\theta(kh)})| \leq \tau^{\frac{1}{2} I_i \lambda^{kh}},$$

for all $i \geq t$ and all sufficiently large k. As in the proof of Lemma 3, there is a constant B such that

$$(35) \qquad \Sigma_{I_i = I} |r_i| \leq B \rho^{-I}.$$

We note that the ordering of the monomials is such that

$$(36) \qquad \lim_{k \to \infty} \left| \frac{m_i(\zeta^{\theta(kh)})}{m_t(\zeta^{\theta(kh)})} \right| = 0$$

for $i > t$ and so the limit value of the sum on the right of (33) is unaltered by the omission of finitely many terms. Thus, if we choose I^* so that $\tau^{I^*} < (m_t(|\zeta|^{\Phi}))^3$ we have

$$(37) \quad 0 < |r_t| \leq \lim_{k \to \infty} \Sigma_{I_i \geq I^*} |r_i| \left| \frac{m_i(\zeta^{\theta(kh)})}{m_t(\zeta^{\theta(kh)})} \right|$$

(Eq. (37) continued on next page.)

$$\leq B \lim \frac{1}{|m_t(\zeta^{\theta(kh)})|} \; \Sigma_{I \geq I^*} \left(\frac{\tau^{\frac{1}{2}\lambda^{kh}}}{\rho} \right)^I$$

$$= B \lim \frac{1}{(m_t(|\zeta|^{\Phi}))^{\lambda^{kh}} + o(\lambda^{kh})} \; \frac{\left(\tau^{\frac{1}{2}\lambda^{kh}} / \rho\right)^{I^*}}{1 - \left(\tau^{\frac{1}{2}\lambda^{kh}} / \rho\right)}$$

$$\leq \frac{B}{\rho^{I^*}} \lim \frac{(m_t|\zeta|^{\Phi}))^{\frac{3}{2}\lambda^{kh}}}{(m_t|\zeta|^{\Phi}))^{\lambda^{kh}} + o(\lambda^{kh})}$$

$$= 0.$$

This contradiction shows that all $r_i = 0$, which establishes the lemma.

We turn now to the proof of the theorem under the restriction $\zeta \in \mathcal{C}(\rho)$. We shall assume that $\alpha = G(\zeta)$ is algebraic of degree g and obtain a contradiction.

Since the coordinates of the point ζ are rational numbers, they can be expressed in the form $\zeta_i = \nu_i/\nu$, where $\nu_1, \ldots, \nu_p$ and ν are integers and $\nu > 1$. Let

(38) $$\beta = - \log D/\log \nu$$

where D is the constant whose existence was established in Lemma 3. Then $\beta > 0$ since $\nu > 1 > D > 0$. We let

(39) $$\delta(k) = m^{(p-1)/p} n\theta^*(k) + m\mu \Sigma_{\ell=0}^{k-1} \theta^*(\ell),$$

(40) $$\omega_{k,j} = \nu^{\delta(k)} P_{k,j}(\zeta),$$

(41) $$\xi_k = \Sigma_{j=0}^m \omega_{k,j} \alpha^j.$$

By Lemma 2, $\omega_{k,j}$ is an integer with

(42) $$|\omega_{k,j}| \leq M^{k+1} \nu^{\delta(k)}.$$

Using Lemma 3 and (5), (10),

(43) $$|\xi_k| = \nu^{\delta(k)} \, |\Sigma_{j=0}^m P_{k,j}(\zeta) \, G^j(\zeta)|$$

$$= \nu^{\delta(k)} \left| \prod_{i=0}^{k=1} q(\zeta^{\theta(i)}) \right|^m \left| \Sigma_{j=0}^m Q_{k,j}(\zeta) \, G^j(\zeta) \right|$$

$$\leq \nu^{\delta(k)} M_1^k |R_k(\zeta)|$$

$$\leq CM_1^k \nu^{\delta(k)} D^{mn\lambda^k}$$

$$= CM_1^k \nu^{\delta(k) - \beta mn\lambda^k},$$

where

(44) $$M_1 = \max_{z \in \overline{\mathcal{C}}(\rho)} |q(z)|^m.$$

To estimate $\delta(k)$, we have from (III-7),

(45) $$\theta_{ij}(kh+\ell) \sim \lambda^{kh+\ell}\,\phi_{ij}^{(\ell)}.$$

Thus, if we let

(46) $$\tilde{\phi} = \max_{\ell} \Sigma_{i,j}\, \phi_{ij}^{(\ell)},$$

we have

(47) $$\theta^*(k) \le \lambda^k \tilde{\phi} + o(\lambda^k),$$

(48) $$\delta(k) \le \tilde{\phi}(m^{(p-1)/p} n\lambda^k + m\mu\, \Sigma_{\ell=0}^{k-1} \lambda^{\ell}) + o(\lambda^k)$$

$$\le \tilde{\phi}\, \lambda^k (m^{(p-1/p} n + \frac{m\mu}{\lambda - 1}) + o(\lambda^k).$$

We now choose m and n so that

(49) $$\epsilon = \beta mn - g\tilde{\phi}(m^{(p-1)/p} n + \frac{m\mu}{\lambda - 1}) > 0.$$

By Lemma 0 of Section II, there is a positive constant, $W(\alpha, m)$, such that

(50) $$(M^{k+1}\nu^{\delta(k)})^{g-1}(CM_1^k \nu^{\delta(k) - \beta mn\lambda^k}) \ge W(\alpha, m)$$

for all k such that $\xi_k \neq 0$. By (48) and (49)

(51) $$g\delta(k) - \beta mn\lambda^k = -\epsilon\lambda^k + o(\lambda^k);$$

hence

(52) $$CM^{(g-1)(k+1)} M_1^k\, \nu^{-\epsilon\lambda^k + o(\lambda^k)} \ge W(\alpha, m)$$

whenever $\xi_k \neq 0$. However, the left side of (52) approaches zero as k increases so we must have $\xi_k = 0$ for all sufficiently large k.

Referring back to (43), we see that $\xi_k = 0$ implies

(53) $$\left(\prod_{i=0}^{k-1} q(\zeta^{\theta(i)}) \right)^m R_k(\zeta) = 0.$$

The condition that for no i is $\zeta^{\theta(i)}$ a singularity of $r(z)$ implies that no $\zeta^{\theta(i)}$ can be a singularity of $s(z)$ either, as is seen on examining equation (1) and recalling that $G(z)$ is regular at all $\zeta^{\theta(i)}$. But the zeros of $q(z)$ are just the collective singularities of $r(z)$ and $s(z)$ so $q(\zeta^{\theta(i)}) \neq 0$ for all i, and thus from (53), $R_k(\zeta) = 0$ for all sufficiently large k. By Lemma 4, this implies

(54) $$R_0(z) \equiv 0 \qquad z \in \mathcal{C}(\rho),$$

and, by Lemma 1(iv),

(55) $$\Sigma_{j=0}^{m}\, Q_{0,j}(z)\, G^j(z) \equiv 0 \qquad z \in \mathcal{C}(\rho).$$

In other words, $G^j(z)$ is algebraic. But this is contrary to assumption and we conclude that $\alpha = G(\zeta)$ must be transcendental.

We turn now to the case in which ζ is a general point of θ, not necessarily belonging to $\mathcal{C}(\rho)$. Let $\mathcal{D}_0 = \mathcal{C}(\rho)$ and, for $t > 0$, $\mathcal{D}_t = \{z \mid z^{\theta(t)} \in \mathcal{C}(\rho)\}$. Let $G_0(z)$ be the restriction of $G(z)$ to $\mathcal{D}_0$, and for $t > 0$ let $G_t(z)$ be defined for $z \in \mathcal{D}_t$ by

$$(56) \qquad G_t(z) = \frac{1}{\prod_{i=0}^{t-1} r(z^{\theta(i)})} G(z^{\theta(t)}) - \Sigma_{\ell=0}^{t-1} \frac{s(z^{\theta(\ell)})}{\prod_{i=0}^{\ell} r(z^{\theta(i)})} .$$

Using the fact that $G(z)$ is analytic in $\mathcal{D}_0$ and satisfies (1) there, and that the functions $r(z)$ and $s(z)$ are rational, the following facts can be verified. (i) For all t, $G_t(z) = G_0(z)$ for $z \in \mathcal{D}_0$. (ii) For each t, there is a nonzero polynomial $P_t(z)$ such that $G_t(z)$ is analytic at all points of $\mathcal{D}_t$ which are not zeros of $P_t(z)$. (iii) If $u > t$ and $z \in \mathcal{D}_t$, then $z \in \mathcal{D}_u$ and $G_u(z) = G_t(z)$.

Since the natural domain $\mathcal{D} = \mathcal{D}(\theta)$ of θ consists of all points z such that $z^{\theta(t)} \to 0$, we have $\mathcal{D} = \cup_k \mathcal{D}_k$. Let the function $\overline{G}(z)$ be defined for $z \in \mathcal{D}$ by

$$(57) \qquad \overline{G}(z) = G_t(z) \quad \text{if } z \in \mathcal{D}_t.$$

By (iii) above, $\overline{G}(z)$ is single valued at every point in $\mathcal{D}$ at which some $G_t(z)$ is defined.

Now suppose $G(z)$ is defined and analytic at a point $\eta \in \mathcal{D}$. By condition (e) in the definition of acceptable function there is a path in $\mathcal{D}$ along which $G(z)$ is analytic which connects η to the origin. Since $\mathcal{D}$ is an open set, this path must lie entirely within some $\mathcal{D}_t$, and furthermore it can always be chosen so as to avoid all zeros of $P_t(z)$. Thus by (ii) above $\overline{G}(z)$ is also analytic along this path. Since by (i) $G(z) = G_0(z) = \overline{G}(z)$, for $z \in \mathcal{D}_0$, by the principle of analytic continuation $G(z)$ and $\overline{G}(z)$ coincide along this entire path and thus $G(\eta) = \overline{G}(\eta)$. Thus for any point $z \in \mathcal{D}$ at which $G(z)$ is defined we may replace the term $G_t(z)$ in (56) by $G(z)$.

For the particular point ζ, we have $\zeta \in \mathcal{D}_t$, for some t, and thus $\zeta^{\theta(t)} \in \mathcal{C}(\rho)$. Since $\zeta^{\theta(t)}$ is a general point having rational coordinates, $G(\zeta^{\theta(t)})$ is transcendental by the restricted result previously established. From (56) we have

$$(58) \quad G(\zeta^{\theta(t)}) = \left(\prod_{i=0}^{t-1} r(\zeta^{\theta(i)})\right) G(\zeta) + \Sigma_{\ell=0}^{t-1} \left(\prod_{i=\ell+1}^{t-1} r(\zeta^{\theta(i)})\right) s(\zeta^{\theta(\ell)}).$$

Since by hypothesis no $\zeta^{\theta(i)}$ is a singularity of $r(z)$, the first term on the right, hence the second as well must be finite. Thus equation (58) has the form

(59) $$G(\zeta^{\theta(t)}) = c_1 G(\zeta) + c_2,$$

where c_1 and c_2 are rational numbers. Since $G(\zeta^{\theta(t)})$ is transcendental, it follows that $G(\zeta)$ is transcendental. This completes the proof of the theorem.

V. EXAMPLES

Example 2, cont. By inserting a decimal point the two-symbol Thue sequence on $\{0, 1\}$ can be converted into a binary number:

(1) $$0.110100110010110 \cdots = G_2(\tfrac{1}{2}).$$

Since $\zeta = \frac{1}{2}$ is a general point of $\theta = (2)$, this number is transcendental.

Example 3, cont. The number $G_3(\frac{1}{2}, \frac{1}{2}) = \Sigma_{n=1}^{\infty} 2^{-f(n)}$ is transcendental as is any number which is the value of $G_3(z_1, z_2)$ for nonzero rational arguments; e.g., $G_3(\frac{1}{2}, -3) = \frac{1}{2} - \frac{3}{2} - \frac{3}{4} + \frac{9}{8} - \frac{27}{32} - \cdots$.

Example 5, cont. Since $(\frac{1}{2}, \frac{1}{3})$ is a general point of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, the number $G_5(\frac{1}{2}, \frac{1}{3}) = \Sigma_{n=0}^{\infty}(-1)^n(2^{-2^n} - 3^{-2^n})$ is transcendental. Our theorem tells us nothing about the values of $G_5(z_1, z_2)$ at $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{8}, \frac{1}{2})$ and $(\frac{1}{16}, \frac{1}{2})$. As observed earlier, $G_5(\frac{1}{2}, \frac{1}{2}) = 0$ and $G_5(\frac{1}{16}, \frac{1}{2}) = -\frac{1}{4}$ are rational. To determine the nature of $G_5(\frac{1}{4}, \frac{1}{2})$ we set $G_5'(z) = G_5(z^2, z)$. Then $G_5'(z) = -z - 2\Sigma_{n=1}^{\infty}(-1)^n(z^{2^n})$ which is a transcendental function satisfying

(2) $$G_5'(z^2) = -G_5'(z) - z + z^2.$$

By our theorem $G_5(\frac{1}{4}, \frac{1}{2}) = G_5'(\frac{1}{2})$ is transcendental. A similar argument establishes the transcendence of $G_5(\frac{1}{8}, \frac{1}{2})$.

Example 6. The function

$$(3) \qquad G_6(z) = (1-4z)\prod_{n=0}^{\infty}(1+2z^{2^n}) = 1-2z-6z^2-4z^3-\dots$$

is acceptable since it satisfies the functional relation

$$(4) \qquad G_6(z^2) = \frac{1-2z}{1-4z}\, G_6(z).$$

Our theorem tells us that the values of $G_6(z)$ for rational z in the range $0 < |z| < 1$ are transcendental with the possible exceptions of $z = \frac{1}{4}, \pm\frac{1}{2}$. From the product formula for $G_6(z)$ we can determine that $G_6(\frac{1}{4}) = G_6(-\frac{1}{2}) = 0$. From (4) we obtain

$$(5) \qquad G_6(z) = \frac{(1+2z)(1-4z)}{1-2z^2}\, G_6(z^4)$$

so that $G_6(\frac{1}{2}) = -4G_6(\frac{1}{16})$, and since $G_6(\frac{1}{16})$ is transcendental $G_6(\frac{1}{2})$ is also.

Example 7. The function

$$(6) \qquad G_7(z) = \prod_{n=1}^{\infty}(1-z^{f(n)}) = 1-z-z^2+z^4+z^7-z^8+z^{11}-\dots,$$

where $f(n)$ is the n-th Fibonacci number, does not appear to be acceptable. However, $G_7(z) = G_7'(z, z)$, where

$$(7) \qquad G_7'(z_1, z_2) = \prod_{n=0}^{\infty}(1 - z_1^{f(n+1)} z_2^{f(n)}),$$

and this function is acceptable since

$$(8) \qquad G_7'(z_1 z_2, z_1) = \frac{1}{1-z_1}\, G_7'(z_1, z_2).$$

Thus the values of $G_7(z)$ for rational z in the range $0 < |z| < 1$ are transcendental.

Example 8. The function

$$(9) \qquad G_8(z_1, z_2) = \Sigma_{n=0}^{\infty}\, z_1^n z_2^{n^2}$$

satisfies the functional relation

$$(10) \qquad G_8(z_1 z_2^2, z_2) = \frac{1}{z_1 z_2}\, G_8(z_1, z_2) - \frac{1}{z_1 z_2}.$$

However, the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ of this relation is not irreducible nor does it have an eigenvalue of modulus greater than unity. Thus, condition (d) fails, and we can assert nothing concerning the nature of the values of $G_8(z_1, z_2)$. The transcendence of $G_8(1, \frac{1}{2}) = \Sigma_{n=0}^{\infty}\, 2^{-n^2}$ remains an open problem.

NOTES

1. My guide in this work has been A. O. Gelfond's, Transcendental and Algebraic Numbers, Dover, 1960, principally the developments in Chapter II. On at least two points Gelfond has achieved far greater generality than I have attempted; namely, his treatment covers functions whose series have coefficients in an arbitrary algebraic number field, and also covers functions defined by systems of equations as well as by single equations. Having worked out several simple examples, I believe the functional equation approach to transcendence proofs will prove generalizable in either of these directions, though to what extent I am not sure.

2. I hope to treat the topic of functional equations and real-time computable numbers in a later report. The way in which the example discussed in Section II arises from the analysis of a real-time computation is indicated in the summary of my paper, "Functional Equations for Register Machines", Proceedings of the Hawaii International Conference on System Sciences, Honolulu, Jan. 1968.

3. For a proof of Lemma 0 see §5.3 of Volume II of W. J. LeVeque's Topics in Number Theory, Addison-Wesley, 1961.

4. See, for example, E. C. Titchmarsh's The Theory of Functions, 2nd edition, Oxford, 1958, §4.7.

5. See D. Ridout, "Rational Approximations to Algebraic Numbers", Mathematika 4 (1957), pp. 125-131.

6. Nonnegative matrices are treated in Chapter XIII of F. R. Gantmacher's Matrix Theory, 2 vols., Chelsea, 1960. The existence and properties of the matrices Φ_ℓ can be deduced from formula (23) of Chapter V.

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