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Synchronization in Arrays of Coupled Nonlinear Systems: Passivity, Circle Criterion and Observer Design

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Abstract

It has been shown that synchronization between two nonlinear systems can be studied as a control theory problem. We show in this letter that this relation can be extended to synchronization in coupled arrays of nonlinear systems. In particular, we extend several stability conditions to synchronization criteria in arrays: the passivity criterion, the circle criterion and a result on observer design of Lipschitz nonlinear systems.

Keywords

Synchronization of chaos, observer, stability, passivity, circle criterion, arrays, nonlinear control.

I. Introduction

Since the seminal papers by Pecora and Carroll [1], [2] show that chaotic systems can be synchronized, it has been observed that synchronization between two identical systems can be studied as a control theory problem. For instance, in [3], [4], synchronization is related to asymptotical stability and in [5], [6], synchronization in two systems coupled via unidirectional coupling is related to the stability of observers. In fact, for additive bidirectional coupling, a similar connection between synchronization and observer design can be seen. For example, consider the following two identical systems coupled via additive bidirectional coupling

$$\begin{aligned}\dot{x} &= f(x,t) + c(x,\tilde{x},t)\\ \dot{\tilde{x}} &= f(\tilde{x},t) + \tilde{c}(x,\tilde{x},t)\end{aligned}$$

The synchronization problem asks whether $e(t) = x - \tilde{x}$ goes to zero. Consider the corresponding observer problem:

$$\begin{aligned}\dot{y} &= f(y,t)\\ \dot{\tilde{y}} &= f(\tilde{y},t) + \tilde{c}(y,\tilde{y},t) - c(y,\tilde{y},t)\end{aligned}$$

If $e_2(t) = y - \tilde{y}$ goes to zero, then $\tilde{y}$ is a valid observer for y. Since the dynamics of $e(t)$ and $e_2(t)$ are identical, it is clear that a valid observer implies that the two coupled systems x and $\tilde{x}$ are synchronized.

The purpose of this letter is to illustrate how such results from control theory can also be used to derive conditions for synchronization in arrays of coupled systems. In particular, we extend the passivity criterion, the circle criterion and a result on observers for Lipschitz nonlinear systems to synchronization criteria on coupled arrays of identical nonlinear systems.

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II. Synchronization in coupled arrays of identical nonlinear systems

Consider the coupled array of identical systems with uniform linear coupling [7]:

$$\dot{x} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_m, t) \end{pmatrix} + (G \otimes D)x \tag{1}$$

where $x = (x_1, \ldots, x_m)^T$. The matrix G describes the coupling topology between the systems, while the matrix D is the individual coupling between two systems. In many cases, G is the Laplacian matrix of the underlying graph. The goal of this letter is to derive conditions under which the coupled array synchronizes, i.e. $\|x_i - x_j\| \to 0$ as $t \to \infty$ for all i,j.

Definition 1: We say $V(x) = x^T P x$ is a synchronizing Lyapunov function for $\dot{x} = f(x,t)$ if $V(x-y) \geq 0$ and $\dot{V}(x-y) \leq 0$ along the trajectories of $\dot{x} = f(x,t)$, $\dot{y} = f(y,t)$.

Existence of a synchronizing Lyapunov function implies that all trajectories of $\dot{x} = f(x,t)$ converge towards another. This property is also known as the unique steady state property [8] or asymptotical stability [9].

In [10], [11] the following result is proved:

Theorem 1: Let P be a symmetric positive definite matrix such that $V(x) = x^T P x$ is a synchronizing Lyapunov function for $\dot{x} = f(x,t) + Dx$. Let G be an irreducible symmetric matrix with zero row sums and only nonpositive off-diagonal elements. If $PD + D^T P$ is negative definite and the nonzero eigenvalues of G are larger than or equal to 1, then the coupled system (1) synchronizes.

In the remainder of this letter, we will assume that G is an irreducible symmetric matrix with zero row sums and only nonpositive off-diagonal elements.

Application of Theorem 1 includes solving the following problem: does there exist a symmetric positive definite P such that $V(x) = x^T P x$ is a synchronizing Lyapunov function for $\dot{x} = f(x,t) + Dx$ and $PD + D^T P$ is negative definite. There are several stability results in the literature that give conditions under which such a P exists. We will illustrate a few of these results: The passivity criterion, the circle criterion and a theorem by Rajamani. By applying them to Theorem 1, we extend these results to synchronization criteria for an array of coupled systems.

A. Passivity and Circle Criterion

Definition 2 ([12]) A function $\phi : R^n \to R^n$ is increasing if $(x-y)^T(\phi(x,t) - \phi(y,t)) \geq 0$ for all x,y, t.

Consider the passivity theorem [13]:

Theorem 2: Let ϕ be an increasing function and A be Hurwitz. Let $H(s)$ be defined as $C(sI - A)^{-1}B$. Suppose that (A, B, C) form a minimal realization of $H(s)$ and $H(s)$ is strictly positive real[1], then there exists a symmetric positive definite matrix P such that $V(x) = x^T P x$ is a synchronizing Lyapunov function of $\dot{x} = Ax - B\phi(C^T x, t)$ and $PA + A^T P$ is negative definite.

[1] $H(s)$ is strictly positive real if $\inf_{\omega \in R} \lambda_{\min}(H(j\omega) + H^*(j\omega)) > 0$.

For Single-Input-Single-Output systems, the above result reduces to the circle criterion [13]:

Theorem 3: Let ϕ be a function such that $0 \leq (y-z)(\phi(y,t)-\phi(z,t)) \leq k(y-z)^2$. Let $h(s)$ be the scalar transfer function defined as $c(sI-A)^{-1}b$. Suppose that A is Hurwitz and (A,b,c) form a minimal realization of $h(s)$ and $\Re(1+kh(j\omega)) > 0$ for all ω, then there exists a symmetric positive definite matrix P such that $V(x) = x^T Px$ is a synchronizing Lyapunov function of $\dot{x} = Ax - b\phi(c^T x, t)$ and $PA + A^T P$ is negative definite.

Combining Theorem 1 and 2 we obtain the following synchronization criteria for arrays of coupled systems based on passivity:

Theorem 4: Let $f(x,t)$ be of the form $\mu D - B\phi(C^T x, t)$ where ϕ is an increasing function. Let D be Hurwitz and let $H(s)$ be defined as $C(sI-D)^{-1}B$. Suppose that (D,B,C) form a minimal realization of $H(s)$. If $H(s)$ is strictly positive real and all nonzero eigenvalues of G is larger than $-\mu$, then the coupled system (1) is synchronizing.

Proof: Let $\alpha > 0$ be the smallest nonzero eigenvalue of G. Note that $G \otimes D = \frac{1}{\delta} G \otimes \delta D$ for a nonzero scalar δ. This trivial observation will be used throughout this letter. It's clear that if $C(sI-D)^{-1}B$ is strictly positive real, then $C(sI-(\delta+\mu)D)^{-1}B$ is strictly positive real for all $\delta \neq 0$ if $\delta + \mu > 0$. By setting $\delta = \alpha$, the nonzero eigenvalues of $\frac{1}{\delta}G$ are all larger than or equal to 1. ∎

Corollary 1: Let $f(x,t)$ be of the form $-B\phi(C^T x, t)$ where ϕ is an increasing function. Let D be Hurwitz and let $H(s)$ be defined as $C(sI-D)^{-1}B$. Suppose that (D,B,C) form a minimal realization of $H(s)$. If $H(s)$ is strictly positive real, then the coupled system (1) is synchronizing.

Combining Theorem 1 and 3 we obtain the following synchronization criteria for arrays of coupled systems:

Theorem 5: Let $f(x,t)$ be of the form $\mu D - b\phi(c^T x, t)$ where ϕ is a function such that $0 \leq (y-z)(\phi(y,t)-\phi(z,t)) \leq k(y-z)^2$ for some $k > 0$. Let $h(s)$ be defined as $c(sI-D)^{-1}b$ and suppose that (D,b,c) form a minimal realization of $h(s)$. Let $\beta = \inf_{\omega \in R} \Re(h(j\omega))$. Either of the following two conditions implies that the coupled system (1) is synchronizing.

- $\mu < \min(0, k\beta)$ and $-D$ is Hurwitz,
- All nonzero eigenvalues of G are larger than $\max(-\mu, -\mu - k\beta)$ and D is Hurwitz.

Proof: Again note that $G \otimes D = \frac{1}{\delta} G \otimes \delta D$. Let $\alpha > 0$ be the smallest nonzero eigenvalue of G. If the first condition is true, then $-\mu > 0$ and $-\mu + k\beta > 0$ and we can choose $\delta > 0$ to be less than $\min(\alpha, -\mu, -\mu + k\beta)$. It can be shown that $\frac{k\beta}{|\delta+\mu|} > -1$.

If all nonzero eigenvalues of G are larger than $\max(-\mu, -\mu - k\beta)$, choose $\delta = \alpha$. Again it can be shown that $\frac{k\beta}{|\delta+\mu|} > -1$.

Since $\delta \leq \alpha$, all the nonzero eigenvalues of $\frac{1}{\delta}G$ are larger than or equal to 1 for either choice of δ. Furthermore, $\Re(1 + kc(jwI - (\delta+\mu)D)^{-1}b) \geq 1 + \frac{k\beta}{|\delta+\mu|} > 0$ for all $w \in R$. ∎

B. Stability of observers

Definition 3: $f(x,t)$ is Lipschitz with Lipschitz constant γ if $\|f(x,t) - f(\tilde{x},t)\| \leq \gamma \|x - \tilde{x}\|$ for all x, $\tilde{x}$.

Consider the following nonlinear system:

$$\dot{x} = f(x,t)$$

and the observer

$$\dot{\tilde{x}} = f(\tilde{x},t) + D(x - \tilde{x}) \quad (2)$$

In [14], sufficient and necessary conditions are given for the stability of such observers for Lipschitz nonlinear systems. The following theorem can be derived from [14].

Theorem 6: Let f be Lipschitz with Lipschitz constant γ. If D has all its eigenvalues in the open left half plane and all singular values of $D - j\omega I$ are larger than γ for all $\omega \geq 0$, then there exists a symmetric positive definite matrix P such that $V(x) = x^T P x$ is a synchronizing Lyapunov function of $\dot{x} = f(x,t) + Dx$ and $PD + D^T P$ is negative definite.

Using Theorem 6 and Theorem 1, we have the following theorem which describes a synchronization criteria based solely on properties of G, D and the Lipschitz constant γ.

Theorem 7: Define $\beta = \inf_\omega \sigma_{\min}(D - j\omega I)$. Assume $\beta > 0$. System (1) synchronizes if

- $f(x,t)$ is Lipschitz with Lipschitz constant γ,
- D has all its eigenvalues in the open left half plane,
- all nonzero eigenvalues of G are larger than $\frac{\gamma}{\beta}$.

Proof: The proof is similar to that of Theorem 5. Let α be the smallest nonzero eigenvalue of G. Choose $\delta = \alpha > 0$. All singular values of $\delta D - j\omega I$ are greater than $\delta\beta > \gamma$ while the nonzero eigenvalues of $\frac{1}{\delta}G$ are larger than 1. ■

The following result can also be derived from [14]:

Theorem 8: Let $g(x,t)$ be Lipschitz with Lipschitz constant γ. If the following conditions are satisfied:

- (A, D) is observable,
- $A + D$ has all its eigenvalues in the open left half plane,
- all singular values of $A + D - j\omega I$ are larger than γ for all $\omega \geq 0$,
- all singular values of A are smaller than or equal to γ,

then there exists a symmetric positive definite matrix P such that $V(x) = x^T P x$ is a synchronizing Lyapunov function of $\dot{x} = Ax + g(x,t) + Dx$ and $PD + D^T P$ is negative definite.

Proof: In [14] a symmetric positive definite matrix P is constructed such that $(A + D)^T P + P(A + D) + \gamma^2 PP + \nu I = 0$, where $\nu > 1$. It then follows that $-(D^T P + PD) = A^T P + PA + \gamma^2 PP + \nu I = (\frac{1}{\gamma}A + \gamma P)^T(\frac{1}{\gamma}A + \gamma P) + (\nu I - \frac{1}{\gamma^2}A^T A)$ which is positive definite if $\sigma_{\max}(A) \leq \gamma$. ■

This translates into the following synchronization criteria:

Theorem 9: System (1) synchronizes if

- $f(x,t)$ is of the form $Ax + g(x,t)$ where g is Lipschitz with Lipschitz constant γ,
- (A, D) is observable,
- all singular values of A are smaller than or equal to γ,
- $A + D$ has all its eigenvalues in the open left half plane,
- all singular values of $A + D - j\omega I$ are larger than γ for all $\omega \geq 0$,
- all nonzero eigenvalues of G are larger than or equal to 1.

In [14], the test on the singular values of $A+D-j\omega I$ is replaced by a test on the eigenvalues
of $A+D$ and the conditioning of its eigenvectors.

As G describes the coupling topology of the array, for fixed D and f the results in this letter give synchronization criteria that depends on the least nonzero eigenvalue of G, a parameter that describes the connectedness of the underlying graph[15].

III. Conclusions

Synchronization between two nonlinear systems is related to stability and observer design. We show that this relation can be extended to arrays of coupled identical nonlinear systems. In particular, we show how several results in nonlinear control can be used to derive synchronization criteria for arrays of nonlinear systems.

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