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Simple Three Oscillator Universal Probe for Determining Synchronization Stability in Coupled Arrays of Oscillators

Chai Wah Wu

IBM T. J. Watson Research Center
P. O. Box 218
Yorktown Heights, NY 10598

Research Division
Almaden · Austin · Beijing · Haifa · T.J. Watson · Tokyo · Zurich

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Chai Wah Wu†

IBM Research Division, Thomas J. Watson Research Center,
P. O. Box 218, Yorktown Heights, NY 10598, U. S. A.

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Abstract

In [1], Fink et al. describe how a coupled array of three oscillators can serve as a universal probe of the synchronization properties in general coupled arrays of oscillators. However, the coupling topology in this probe can cause difficulties in experimental implementations. For instance, every oscillator is coupled to every other oscillator resulting in many coupling terms. Furthermore, the variable coupling weights at several coupling terms need to be set equal to each other. This means that some component values need to be varied over a range and be perfectly matched to each other in the experimental setup. We present simpler universal probes by utilizing the minimal number of coupling terms and by minimizing the constraint of having the coupling weights matched.

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INTRODUCTION

In [1], Fink et al. describe how a coupled array of three oscillators can serve as a universal probe of the synchronization properties in general coupled arrays of oscillators. However, the coupling topology in this probe can cause difficulties in experimental implementations. For instance, every oscillator is coupled to every other oscillator. As more coupling terms generally imply increased complexity, a minimal number of coupling terms is preferred. Furthermore, the coupling weights at several coupling terms need to be set equal to each other. This means that some component values need to be varied over a range and at the same time be perfectly matched to each other in the experimental setup. We present simpler universal probes by utilizing the minimal number of coupling terms and by minimizing the constraint of having the coupling weights matched.

MASTER STABILITY FUNCTION

Consider the coupled array of oscillators:

$$\dot{x} = I \otimes f(x_i) + G \otimes H(x_i) \tag{1}$$

where G is an irreducible matrix with zero row sums, $x = (x_1, x_2, \ldots)^T$ and we use the notation

$$A \otimes B(x_i) = \begin{bmatrix} A_{11}B(x_1) & A_{12}B(x_2) & \cdots & A_{1n}B(x_n) \\ \vdots & & & \vdots \\ A_{m1}B(x_1) & & \cdots & A_{mn}B(x_n) \end{bmatrix}$$

As noted in [2, 3], the case where G has a constant row sum γ can be reduced to the case of G having zero row sums by replacing $f(x_i)$ with $f(x_i) + \gamma H(x_i)$.

In [4], it is shown that the Lyapunov exponents on the synchronization manifold of the coupled array (1) can be deduced from the Lyapunov exponents of the variational equations:

$$\dot{\eta} = (Df(x) + \mu DH(x))\eta \tag{2}$$

for each μ a nonzero eigenvalue of G. This leads to the study of a three oscillator probe in [1], which contains variational equations of the form (2) for some μ. By varying two real parameters, μ can be set to any complex number. The three oscillator probe has the state equations (1) with coupling matrix

$$G = \begin{pmatrix} -2\frac{\epsilon}{3} & \frac{\epsilon}{3} + \frac{\delta}{\sqrt{3}} & \frac{\epsilon}{3} - \frac{\delta}{\sqrt{3}} \\ \frac{\epsilon}{3} - \frac{\delta}{\sqrt{3}} & -2\frac{\epsilon}{3} & \frac{\epsilon}{3} + \frac{\delta}{\sqrt{3}} \\ \frac{\epsilon}{3} + \frac{\delta}{\sqrt{3}} & \frac{\epsilon}{3} - \frac{\delta}{\sqrt{3}} & -2\frac{\epsilon}{3} \end{pmatrix} \tag{3}$$

G describes how the three oscillators are coupled to each other. The variational equation contains Eq. (2) with $\mu = \epsilon + i\delta$.

In practical implementations, the probe (1) with G defined as (3) can cause some difficulties. First of all, the form of G implies that all three oscillators are connected to each other. There are 6 nonzero off-diagonal terms in G, implying 6 coupling terms between the three oscillators. In general, the higher the number of coupling terms, the more complicated the coupled array. Furthermore, the variable parameters ϵ and δ appear in several entries of G, meaning that they need to be varied over some range of parameters and at the same time be perfectly matched to each other in the corresponding coupling terms. In particular, in [1] the exactly same value for ϵ (and for δ) needs to be sent to the inputs of several matching analog multipliers. This matching problem becomes more problematic when the analog multipliers are replaced by physical resistors as is common in several circuit implementations of coupled arrays.

A SIMPLER UNIVERSAL PROBE FOR SYNCHRONIZATION STABILITY

The purpose of this letter is to propose other universal probes for synchronization stability which use the minimum number of coupling terms and minimize the occurrence of the same variable parameter in several coupling terms. The main idea is that other matrices G exist which also allow Eq. (1) to be a universal probe. The only requirement on G is that G is a real-valued diagonalizable 3×3 zero row sum matrix with two real parameters ϵ and δ such that the eigenvalues of G are $\{0, \epsilon + i\delta, \epsilon - i\delta\}$. We will call this requirement A. Consider

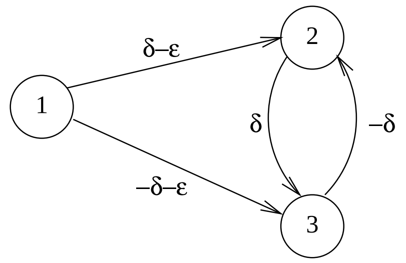

FIG. 1: Schematic of coupling scheme for a simplified universal probe. The circles indicate the three oscillators and the labels on the edges are the weights of the corresponding coupling terms.

the following G:

$$G = \begin{pmatrix} 0 & 0 & 0 \\ \delta - \epsilon & \epsilon & -\delta \\ -\delta - \epsilon & \delta & \epsilon \end{pmatrix} \tag{4}$$

Note that this matrix G satisfies requirement A and has 4 nonzero off-diagonal terms. Furthermore, there is no coupling into the first oscillator. Its trajectories are those of an uncoupled oscillator. One possible implementation of the universal probe is to have the first oscillator replaced by a "memory" device which plays back the trajectory of an uncoupled oscillator. A schematic of this coupling scheme is shown in Fig. 1.

What is the minimum number of nonzero off-diagonal elements for matrices G which satisfy requirement A? If a zero row sum G has two nonzero off-diagonal elements, they must be on different rows, otherwise G has two zero eigenvalues. In this case G takes on one of the following three forms:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \beta & -\beta \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & -\alpha \\ -\beta & 0 & \beta \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -\alpha & \alpha & 0 \\ -\beta & 0 & \beta \end{pmatrix} \tag{5}$$

The first matrix has two zero eigenvalues, while the second and third matrix have as eigenvalues $\{0, \alpha, \beta\}$ which are all real. Therefore G must have at least nonzero 3 off-diagonal elements to satisfy requirement A.

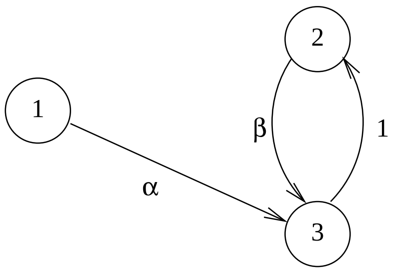

FIG. 2: Schematic of coupling scheme for another simplified universal probe for the case when the eigenvalues are not real. In this scheme there are only three coupling elements and no coupling elements which need to be matched.

If $\delta = 0$, i.e. the eigenvalue $\mu = \epsilon + i\delta$ is real, then a two oscillator probe with coupling matrix

$$G = \begin{pmatrix} 0 & 0 \\ -\epsilon & \epsilon \end{pmatrix} \tag{6}$$

suffices. Therefore we will assume that $\delta \neq 0$. In this case, consider the three oscillator probe (1) with coupling matrix:

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ \epsilon^2 + \delta^2 & -(\epsilon+1)^2 - \delta^2 & 2\epsilon + 1 \end{pmatrix} \tag{7}$$

Since $\delta \neq 0$, G is diagonalizable and satisfies requirement A. G also contains the minimal number of nonzero off-diagonal elements. By using the map $(\alpha, \beta) = (\epsilon^2 + \delta^2, -(\epsilon+1)^2 - \delta^2) = V(\epsilon, \delta)$, G can be written as:

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ \alpha & \beta & -\alpha - \beta \end{pmatrix} \tag{8}$$

which minimizes the problem of having to match parameter values [6]. Again, the first oscillator operates in the uncoupled mode. A schematic of this coupling scheme is shown in Fig. 2.

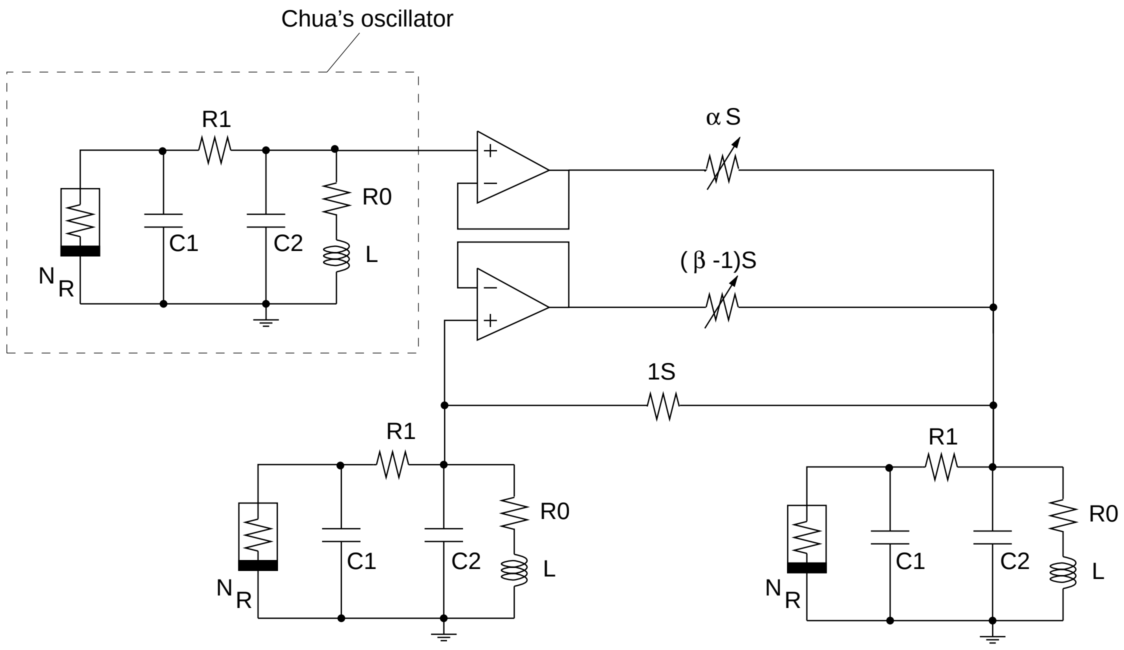

FIG. 3: Universal probe consisting of 3 coupled Chua's oscillators. The conductances of the coupling resistors are shown.

Written explicitly, the state equations of this universal probe are:

$$
\begin{aligned}
\dot{x}_1 &= f(x_1) \\
\dot{x}_2 &= f(x_2) + H(x_3) - H(x_2) \\
\dot{x}_3 &= f(x_3) + \alpha(H(x_1) - H(x_3)) + \beta(H(x_2) - H(x_3))
\end{aligned}
$$

To illustrate the simplicity of the coupling, consider a coupled array of three Chua's oscillators [5] where the coupling occurs between the state variable V_{C2} of each oscillator. The circuit diagram of the array is given in Fig. 3 with the coupling conductances shown.

CONCLUSIONS

We have presented a three oscillator universal probe for synchronization stability in coupled arrays which is simpler than the one proposed in [1]. It contains the minimal number of coupling terms and matching of coupling terms is also minimized.

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† chaiwah@watson.ibm.com

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[6] Note that G in (7) has 5 nonzero entries. If $\epsilon \neq 0$ and $\delta \neq 0$, then $G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\epsilon & -2\epsilon \\ -\frac{\epsilon^2+\delta^2}{2\epsilon} & \frac{\epsilon^2+\delta^2}{2\epsilon} & 0 \end{pmatrix}$ satisfies requirement A and has only 4 nonzero entries.