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Clopped Cubes

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Abstract

We study certain generalized covering polytopes that we call "clopped cubes". These polytopes generalize the clipped cubes which Coppersmith and Lee used to study the nondyadic indivisibility polytopes. Our main results are (i) a totally dual integral inequality description of the clopped cubes, and (ii) an efficient separation procedure.

Keywords: linear program, integer program, polytope, totally dual integral, ideal matrix, generalized set covering, clipped cube, clopped cube, balanced matrix.

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Introduction

We assume familiarity with the basics of polytopes (see Ziegler [5]) and integer programming (see Nemhauser and Wolsey [4]).

Let N be a finite nonempty set with $n = |N|$. Let $\mathcal{F}$ be a set of partitions (S, T, U) of N satisfying F0:

$$U \neq N,\ \forall\ (S, T, U) \in \mathcal{F}.$$

We associate with each $(S, T, U) \in \mathcal{F}$, the set of points

$$F(S, T, U) := \{x \in \{0,1\}^N\ :\ x_j = 1,\ \forall\ j \in S;\ x_j = 0,\ \forall\ j \in T\}.$$

We note that $|F(S, T, U)| = 2^{|U|}$. Let $\hat{F}(S, T, U) := \operatorname{conv}(F(S, T, U))$. Then $\hat{F}(S, T, U)$ is a face of the n-cube $[0,1]^N$ having dimension $|U|$. In particular, when $U = \emptyset$, we have $|F(S, T, U)| = 1$ and $\dim(\hat{F}(S, T, U)) = 0$ — i.e., $F(S, T, U)$ consists of a single extreme point of $[0,1]^N$.

From the set $\mathcal{F}$, we define the polytope

$$\mathcal{Q}(\mathcal{F}) := \operatorname{conv}\{x \in \{0,1\}^N\ :\ x \notin F(S, T, U),\ \forall\ (S, T, U) \in \mathcal{F}\}.$$

Obviously the *simple bound inequalities*

$$-x_j \leq 0, \tag{1}$$

$$x_j \leq 1, \tag{2}$$

$\forall\ j \in N$ are valid for $\mathcal{Q}(\mathcal{F})$. We also have the *clopping inequalities* $C(S, T, U)$:

$$\sum_{j \in S} x_j - \sum_{j \in T} x_j \leq |S| - 1, \tag{3}$$

$\forall\ (S, T, U) \in \mathcal{F}$, which are valid for $\mathcal{Q}(\mathcal{F})$. Interpreting the inequality $C(S, T, U)$ in the context of propositional logic, where x_j is a literal and $-x_j$ is a negated literal, the inequality indicates that at least one literal indexed from S is false or at least one indexed from T is true.

The polytope $\mathcal{Q}(\mathcal{F})$ is a *generalized covering polytope* (see Cornuéjols [2], for example), since each inequality (3), multiplied by -1, looks like a covering inequality when $S = \emptyset$. If $\mathcal{Q}(\mathcal{F})$ is the real solution set of (1–2) $\forall\ j \in N$, (3) $\forall\ (S, T, U) \in \mathcal{F}$, then the constraint matrix for the system (3) $\forall\ (S, T, U) \in \mathcal{F}$ is called *ideal*. For example, if the constraint matrix of (3) $\forall\ (S, T, U) \in \mathcal{F}$ is *balanced* (i.e., each square nonsingular submatrix with exactly two ± 1's in each row and column has the sum of its entries divisible by 4), then it is ideal as well (see Cornuéjols [2]).

Another situation for which the system (3) $\forall\ (S, T, U) \in \mathcal{F}$ is ideal is described in Coppersmith and Lee [1]. In the remainder of this paper, we describe a generalization of their property on $\mathcal{F}$ which yields idealness.

1 Clopped Cubes

Let $\mathcal{F}$ be a set of partitions (S,T,U) of N satisfying F0. If $\mathcal{F}$ satisfies F1:

$$\begin{aligned}&|S_1 \cap T_2| + |S_2 \cap T_1| \geq 2,\\ &\forall \text{ distinct } (S_1,T_1,U_1),\ (S_2,T_2,U_2)\ \in \mathcal{F},\end{aligned}$$

then $\mathcal{F}$ is *cloppable*. If $\mathcal{F}$ is cloppable, then we call $\mathcal{Q}(\mathcal{F})$ a *clopped cube*.

If $U_1 = U_2 = \emptyset$, then F1 amounts to $|S_1 \triangle S_2| \neq 1$, so this generalizes the central idea of "clippable" from Coppersmith and Lee [1]. Without specific reference, all of the results below generalize corresponding results from Coppersmith and Lee [1].

The first result explains the property F1.

Proposition 1 *Let $\mathcal{F}$ be cloppable. Then no point of $[0,1]^N$ violates both $C(S_1,T_1,U_1)$ and $C(S_2,T_2,U_2)$, for distinct (S_1,T_1,U_1), $(S_2,T_2,U_2) \in \mathcal{F}$.*

Proof: Let (S_1,T_1,U_1), $(S_2,T_2,U_2) \in \mathcal{F}$ be distinct. Suppose that $x \in [0,1]^N$ satisfies

$$-\sum_{j \in S_i} x_j + \sum_{j \in T_i} x_j < 1 - |S_i|,$$

for $i = 1, 2$. Adding these together, we get

$$\begin{aligned}&-2\sum_{j \in S_1 \cap S_2} x_j - \sum_{j \in S_1 \cap U_2} x_j + 2\sum_{j \in T_1 \cap T_2} x_j + \sum_{j \in T_1 \cap U_2} x_j\\ &-\sum_{j \in U_1 \cap S_2} x_j + \sum_{j \in U_1 \cap T_2} x_j < 2 - |S_1| - |S_2|.\end{aligned}$$

Adding in appropriate positive multiples of the simple lower bound inequalities (1) to kill off the variables with positive coefficients, and adding in appropriate positive multiples of the simple upper bound (2) inequalities to kill off the variables with negative coefficients, we get

$$0 < 2 - |S_1| - |S_2| + 2|S_1 \cap S_2| + |S_1 \cap U_2| + |U_1 \cap S_2|,$$

which is just

$$|S_1| - |S_1 \cap S_2| - |S_1 \cap U_2| + |S_2| - |S_1 \cap S_2| - |U_1 \cap S_2| < 2,$$

or

$$|S_1 \cap T_2| + |S_2 \cap T_1| < 2,$$

in direct contradiction to F1. □

Proposition 2 *Let $\mathcal{F}$ be cloppable. Then*

$$\mathrm{vol}_n(\mathcal{Q}(\mathcal{F})) = 1 - \sum_{(S,T,U)\in\mathcal{F}} \frac{1}{(|S|+|T|)!}.$$

Proof: The clopping inequality excludes from $[0,1]^N$ the set that is the cross product of the $|U|$-cube $[0,1]^U$ with the simplex of points in $[0,1]^{S\cup T}$ violating $C(S,T,\emptyset)$. As was indicated in Coppersmith and Lee [1], this simplex has volume $\frac{1}{(|S|+|T|)!}$. The result follows from Proposition 1. $\square$

As an alternative to the clopping inequality $C(S,T,U)$, we could exclude the points of $F(S,T,U)$, one by one, with the $2^{|U|}$ "clipping inequalities"

$$\sum_{j\in S\cup K} x_j - \sum_{j\in N\setminus(S\cup K)} x_j \leq |S\cup K|,\ \forall\ K\subset U.$$

Each of these clipping inequalities excludes volume $\frac{1}{n!}$, so the maximum volume excluded by them as a group is $\frac{2^{|U|}}{n!}$ (whether or not $\mathcal{F}$ is cloppable). For large $|U|$, this is much less than the volume of $\frac{1}{(|S|+|T|)!}$ that is excluded by $C(S,T,U)$. This is, then, a simple situation which analytically supports the general empirical observation that relatively simple valid inequalities (here "simple" manifests itself as "sparser") like $C(S,T,U)$ with $|U|$ large, can be relatively strong.

The next result will prove useful in establishing our main result.

Proposition 3 *Let $\mathcal{F}$ be cloppable, let (S,T,U) be in $\mathcal{F}$, and let $x(W)$ be in $F(S,T,U)$. Then*

(i) $S\subset W\subset S\cup U$;

(ii) $x(W+k)\in\mathcal{Q}(\mathcal{F}),\ \forall\ k\in T$;

(iii) $x(W-l)\in\mathcal{Q}(\mathcal{F}),\ \forall\ l\in S$.

Proof: (i) is just a restatement of the definition of $F(S,T,U)$. We also note that (i) is equivalent to

$$S\subset W,\ W\cap T=\emptyset.$$

From (i), we see that for $k\in T$, $x(W+k)$ satisfies $C(S,T,U)$. If $x(W+k)$ violates $C(S',T',U')$, then $x(W+k)\in F(S',T',U')$. So, again by (i), $S'\subset W+k\subset S'\cup U'$, or

$$S'\subset W+k,\ (W+k)\cap T'=\emptyset.$$

So we conclude that $S \cap T' = \emptyset$ and $|S' \cap T| \leq 1$ (the only possible element in $S' \cap T$ is k). Therefore, by F1, $(S', T', U') \notin \mathcal{F}$ and (ii) holds.

Applying (i) again, we see that for $l \in S$, $x(W - l)$ satisfies $C(S, T, U)$. If $x(W - l)$ violates $C(S', T', U')$, then $x(W - l) \in F(S', T', U')$. So, again by (i), $S' \subset W - l \subset S' \cup U'$, or

$$S' \subset W - l, \ (W - l) \cap T' = \emptyset.$$

So we conclude that $|S \cap T'| \leq 1$ (the only possible element in $S \cap T'$ is l), and $S' \cap T = \emptyset$. Therefore, by F1, $(S', T', U') \notin \mathcal{F}$ and (iii) holds. $\square$

And now we get to our main result.

Proposition 4 *Let $\mathcal{F}$ be cloppable. Then (1–2) $\forall\ j \in N$, (3) $\forall\ (S, T, U) \in \mathcal{F}$, is a totally dual integral system describing $\mathcal{Q}(\mathcal{F})$.*

Proof: Let c be in $\mathbb{Z}^N$. Consider the primal linear program P:

$$\begin{aligned}
&\max \sum_{j \in N} c_j x_j \\
&\text{subject to} \sum_{j \in S} x_j - \sum_{j \in T} x_j \leq |S| - 1, \ \forall\ (S, T, U) \in \mathcal{F}; \\
&x_j \leq 1, \ \forall\ j \in N; \\
&x_j \geq 0, \ \forall\ j \in N,
\end{aligned}$$

and its dual linear program D:

$$\begin{aligned}
&\min \sum_{j \in N} y_j + \sum_{(S,T,U) \in \mathcal{F}} (|S| - 1) z_{S,T,U} \\
&\text{subject to } y_j + \sum_{\substack{(S,T,U) \in \mathcal{F}: \\ j \in S}} z_{S,T,U} - \sum_{\substack{(S,T,U) \in \mathcal{F}: \\ j \in T}} z_{S,T,U} \geq c_j, \ \forall\ j \in N; \\
&y_j \geq 0, \ \forall\ j \in N; \\
&z_{S,T,U} \geq 0, \ \forall\ (S, T, U) \in \mathcal{F}.
\end{aligned}$$

We will give a recipe for constructing *integral* optimal solutions to the *linear* programs P and D. Let

$$S^+ := \{j \in N \ : \ c_j > 0\}.$$

Case 1: If $x(S^+)$ satisfies (3) $\forall\ (S, T, U) \in \mathcal{F}$, then we let $y_j := c_j$ for $j \in S^+$, and we let all other dual variables equal 0. It is easy to see that (y, z) is dual feasible with objective value equal to $\sum_{j \in S^+} c_j$; therefore, by the weak duality theorem of linear programming, $x(S^+)$ is primal optimal and (y, z) is dual optimal. Moreover, our construction provides *integer* optimal solutions to P and D.

Case 2: Suppose then that $x(S^+)$ violates a clopping inequality. By Proposition 1, there is a unique $(\tilde{S}, \tilde{T}, \tilde{U}) \in \mathcal{F}$ such that $x(S^+) \in F(\tilde{S}, \tilde{T}, \tilde{U})$. By part (i) of Proposition 3, we have

$$\tilde{S} \subset S^+ \subset \tilde{S} \cup \tilde{U}.$$

Let $k := \text{argmax}\{c_j \ : \ j \in \tilde{T}\}$ (so $c_k \leq 0$; note that $c_k = -\infty$ is possible), and let $l := \text{argmin}\{c_j \ : \ j \in \tilde{S}\}$ (so $c_l > 0$; note that $c_l = +\infty$ is possible).

Subcase 2a: Suppose that $-c_k \leq c_l$ (including the case of finite c_k with $c_l = +\infty$). By part (ii) of Proposition 3, we have $x(S^+ + k) \in \mathcal{Q}(\mathcal{F})$. The primal objective value of $x(S^+ + k)$ is $c_k + \sum_{j \in S^+} c_j$. Let

$$y_j := \begin{cases} c_j + c_k \ , & \text{for } j \in \tilde{S}; \\ c_j \ , & \text{for } j \in S^+ \setminus \tilde{S}, \end{cases}$$

let $z_{\tilde{S},\tilde{T},\tilde{U}} := -c_k$, and let all other dual variables equal 0.

We begin to check dual feasibility, by first checking nonnegativity. For $j \in \tilde{S}$, we have

$$\begin{aligned} y_j &= c_j + c_k \text{ (by the definition of } y_j) \\ &\geq c_j - c_l \text{ (by the hypothesis of Subcase 2a)} \\ &\geq 0 \text{ (by the definition of } l). \end{aligned}$$

For $j \in S^+ \setminus \tilde{S}$, we have

$$\begin{aligned} y_j &= c_j \text{ (by the definition of } y_j) \\ &> 0 \text{ (by the definition of } S^+). \end{aligned}$$

Also, we have

$$\begin{aligned} z_{\tilde{S},\tilde{T},\tilde{U}} &= -c_k \text{ (by the definition of } z_{\tilde{S},\tilde{T},\tilde{U}}) \\ &\geq 0 \text{ (by the definition of } k \text{ and the selection of } (\tilde{S}, \tilde{T}, \tilde{U})). \end{aligned}$$

So the dual solution is nonnegative.

We continue our check of dual feasibility, by checking the structural constraints of D. For $j \in \tilde{S}$, we have

$$\begin{aligned} y_j + \sum_{\substack{(S,T,U) \in \mathcal{F}: \\ j \in S}} z_{S,T,U} - \sum_{\substack{(S,T,U) \in \mathcal{F}: \\ j \in T}} z_{S,T,U} &= y_j + z_{\tilde{S},\tilde{T},\tilde{U}} \\ &= (c_j + c_k) + (-c_k) = c_j \ . \end{aligned}$$

For $j \in S^+ \setminus \tilde{S}$, we have

$$y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in T}} z_{S,T,U} \;=\; y_j = c_j \;.$$

For $j \in \tilde{U} \setminus S^+$, we have

$$y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in T}} z_{S,T,U} \;=\; 0 \geq c_j \;,$$

since $c_j \leq 0 \;\forall\; j \in N \setminus S^+$. For $j \in \tilde{T}$, we have

$$\begin{aligned} y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}:\\ j\in T}} z_{S,T,U} \;&=\; y_j - z_{\tilde{S},\tilde{T},\tilde{U}} \\ &=\; 0 - (-c_k) = c_k \\ &\geq\; c_j \text{ (by the definition of } k). \end{aligned}$$

Therefore, (y, z) is dual feasible.

The dual objective value is

$$\begin{aligned} &y_j + \sum_{(S,T,U)\in\mathcal{F}} (|S| - 1) z_{S,T,U} \\ &= \sum_{j\in\tilde{S}} (c_j + c_k) + \sum_{j\in S^+\setminus\tilde{S}} c_j + (|\tilde{S}| - 1)(-c_k) \\ &= c_k + \sum_{j\in S^+} c_j \;, \end{aligned}$$

which is precisely the primal objective value of $x(S^+ + k)$. Therefore, by the weak duality theorem of linear programming, $x(S^+ + k)$ is primal optimal and (y, z) is dual optimal. Moreover, our construction provides *integer* optimal solutions to P and D.

Subcase 2b: Suppose that $c_l \leq -c_k$ (including the case of finite c_l with $c_k = -\infty$). By part (iii) of Proposition 3, we have $x(S^+ - l) \in \mathcal{Q}(\mathcal{F})$. The primal objective value of $x(S^+ - l)$ is $-c_l + \sum_{j\in S^+} c_j$. Let

$$y_j := \begin{cases} c_j - c_l \;, & \text{for } j \in \tilde{S}; \\ c_j \;, & \text{for } j \in S^+ \setminus \tilde{S}, \end{cases}$$

let $z_{\tilde{S},\tilde{T},\tilde{U}} := c_l$, and let all other dual variables equal 0.

We begin to check dual feasibility, by first checking nonnegativity. For $j \in \tilde{S}$, we have

$$\begin{aligned} y_j &= c_j - c_l \text{ (by the definition of } y_j) \\ &> 0 \text{ (by the definition of } l). \end{aligned}$$

For $j \in S^+ \setminus \tilde{S}$, we have

$$\begin{aligned} y_j &= c_j \text{ (by the definition of } y_j) \\ &> 0 \text{ (by the definition of } S^+). \end{aligned}$$

Also, we have

$$\begin{aligned} z_{\tilde{S},\tilde{T},\tilde{U}} &= c_l \text{ (by the definition of } z_{\tilde{S},\tilde{T},\tilde{U}}) \\ &> 0 \text{ (by the definition of } l). \end{aligned}$$

So the dual solution is nonnegative.

We continue our check of dual feasibility, by checking the structural constraints of D. For $j \in \tilde{S}$, we have

$$\begin{aligned} y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in T}} z_{S,T,U} &= y_j + z_{\tilde{S},\tilde{T},\tilde{U}} \\ &= (c_j - c_l) + (c_l) = c_j \ . \end{aligned}$$

For $j \in S^+ \setminus \tilde{S}$, we have

$$y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in T}} z_{S,T,U} = y_j = c_j \ .$$

For $j \in \tilde{U} \setminus S^+$, we have

$$y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in T}} z_{S,T,U} = 0 \geq c_j \ ,$$

since $c_j \leq 0 \ \forall \ j \in N \setminus S^+$. For $j \in \tilde{T}$, we have

$$\begin{aligned} y_j + \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in S}} z_{S,T,U} - \sum_{\substack{(S,T,U)\in\mathcal{F}: \\ j\in T}} z_{S,T,U} &= y_j - z_{\tilde{S},\tilde{T},\tilde{U}} \\ &= 0 - c_l = -c_l \\ &\geq c_k \text{ (by the hypothesis of Subcase 2b)} \\ &\geq c_j \text{ (by the definition of } k). \end{aligned}$$

Therefore, (y,z) is dual feasible.

The dual objective value is

$$\begin{aligned} & y_j + \sum_{(S,T,U)\in\mathcal{F}} (|S|-1) z_{S,T,U} \\ &= \sum_{j\in\tilde{S}} (c_j - c_l) + \sum_{j\in S^+\setminus\tilde{S}} c_j + (|\tilde{S}|-1)c_l \\ &= -c_l + \sum_{j\in S^+} c_j \ , \end{aligned}$$

which is precisely the primal objective value of $x(S^+ - l)$. Therefore, by the weak duality theorem of linear programming, $x(S^+ - l)$ is primal optimal and (y,z) is dual optimal. Moreover, our construction provides *integer* optimal solutions to P and D.

Subcase 2c: Suppose that $c_k = -\infty$ and $c_l = +\infty$. That is, $\tilde{T} = \tilde{S} = \emptyset$. Then $\tilde{U} = N$ which contradicts F0. □

Suppose that $\mathcal{F}$ is cloppable. Given a set $W \subset N$, a *face exclusion oracle* for $\mathcal{F}$ either asserts that $x(W)$ is in $\mathcal{Q}(\mathcal{F})$ or it delivers the $(S,T,U) \in \mathcal{F}$ with $x(W) \in F(S,T,U)$.

Proposition 5 *Let c be in $\mathbb{Q}^N$. Given a face exclusion oracle for a cloppable $\mathcal{F}$, there is an efficient algorithm for maximizing $\langle c,x\rangle$ on $\mathcal{Q}(\mathcal{F})$ which requires only one call to the oracle.*

Proof: The result follows from the construction in the proof of Proposition 4. The only call to the oracle is with $x(S^+)$ to determine whether we are in Case 1 or 2. □

Proposition 6 *Let x be a point in $[0,1]^N$. Given a face exclusion oracle for a cloppable $\mathcal{F}$, there is an efficient algorithm requiring at most $n+1$ calls to the oracle, that determines whether or not x is in $\mathcal{Q}(\mathcal{F})$, and, if not, it identifies an $(S,T,U) \in \mathcal{F}$ for which the clopping inequality $C(S,T,U)$ is violated by x.*

Proof: Using the well-known results connecting optimization and separation (see Grötschel, Lovász and Schrijver [3]), the existence of an efficient algorithm follows from Proposition 5. But this is not enough to deduce that we can solve this separation problem with at most $n+1$ calls to the oracle. To do this, we provide an efficient direct algorithm, which is practical for computation.

Let $W := \{j \in N : x_j > \frac{1}{2}\}$, and let $\overline{W} := \{j \in N : x_j \leq \frac{1}{2}\}$. If x violates $C(S,T,U)$, then

$$\sum_{j \in S}(1 - x_j) + \sum_{j \in T} x_j < 1.$$

So, by the definition of W and $\overline{W}$, we have

$$|S \cap \overline{W}| + |T \cap W| \leq 1. \tag{4}$$

The idea, then, is to consider the possibilities for $(S,T,U) \in \mathcal{F}$ that can satisfy (4), by fixing (0 or 1) elements of $(S \cap \overline{W}) \cup (T \cap W)$. So we consider each of the following $n+1$ *preliminary* choices of (S,T,U):

(i) $\underline{|S \cap \overline{W}| + |T \cap W| = 0}$: $S = \emptyset$, $T = \emptyset$, $U = N$;

(ii) $\underline{|S \cap \overline{W}| = 1}$: $S = \{l\}$ for some $l \in \overline{W}$, $T = \emptyset$, $U = N - l$;

(iii) $\underline{|T \cap W| = 1}$: $S = \emptyset$, $T = \{k\}$ for some $k \in W$, $U = N - k$,

For each of these preliminary choices, the only flexibility that we allow, is that some elements from $U \cap \overline{W}$ might be moved from U to T, and some elements from $U \cap W$ might be moved from U to S. Now here is the important point: *For each preliminary choice, regardless of such subsequent allowed moves, the inequality $C(S,T,U)$ will be violated by a particular choice of extreme point of* $[0,1]^N$. Namely, respective to the three cases above,

(i) $x(W)$;

(ii) $x(W + l)$ for $l \in \overline{W}$;

(iii) $x(W - k)$ for $k \in W$.

So we call the face exclusion oracle for each of these $n+1$ points. Whenever the oracle returns a set $(S,T,U) \in \mathcal{F}$, we check whether the clopping inequality $C(S,T,U)$ is violated. □

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