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Synchronization in coupled arrays of chaotic oscillators with nonreciprocal coupling

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DRAFT

There are in general two classes of results regarding the synchronization of an array of coupled identical chaotic systems. The first class of results relies on Lyapunov's direct method and gives analytical criteria for global or local synchronization. The second class of results relies on linearization around the synchronization manifold and the computation of Lyapunov exponents. The computation of Lyapunov exponents is mainly done via numerical experiments and can only show local synchronization in the neighborhood of the synchronization manifold. On the other hand, Lyapunov's direct method is more rigorous and can give global results. The coupling topology is generally expressed in matrix form and the first class of methods mainly deals with symmetric matrices whereas the second class of methods can work with all diagonalizable matrices. The purpose of this paper is to consider the nonsymmetric case for the first class of methods.

Keywords

Synchronization, chaos, coupled oscillators, convex optimization.

I. INTRODUCTION

We consider the array of linearly coupled chaotic systems

$$\dot{x} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} + (G \otimes D)x \quad (1)$$

where $G \otimes D$ is the Kronecker product of G and D and $x = (x_1, \dots, x_n)^T$. We say the array of coupled systems (1) synchronizes if $\|x_i - x_j\| \rightarrow 0$ as $t \rightarrow \infty$ for all i, j . The trajectories of the synchronizing array approach the linear subspace $\{x : x_i = x_j, \forall i, j\}$, which we denote as the synchronization manifold. If we assume that G has zero row sums, then the trajectories of the synchronizing arrays approach the trajectories of the uncoupled systems $\dot{x}_i = f(x_i, t)$.

There are two classes of results which give criteria under which Eq. (1) synchronizes. The first class of results utilizes Lyapunov's direct method by constructing a Lyapunov function which decreases along trajectories and gives analytical criteria for local or global synchronization [1]. The second class of results linearizes around the synchronization manifold and computes numerically the Lyapunov exponents of the variational equations [2].

There are several reasons why Lyapunov functions based techniques are preferable in some situations. First of all, Lyapunov exponents based techniques give only local results near the synchronization manifold. Secondly, Lyapunov exponents are generally calculated via numerical techniques on a single numerically integrated trajectory which can cause errors in estimating the stability of the system. Thirdly, with Lyapunov exponents based techniques it is difficult to determine the robustness of the synchronization with respect to parameter variations.

On the other hand, Lyapunov exponents based methods are more generally applicable and give less conservative bounds. For instance, the synchronization criteria are usually given in terms of the eigenvalues of G . For the Lyapunov function based methods, mainly the cases of symmetric and normal matrices G are studied in detail [1], whereas for the Lyapunov exponents based methods the matrix G can be any diagonalizable

matrix [2]. When G is not symmetric, in [1] it was shown that a simple application of Lyapunov function³ based methods can give very conservative bounds. To motivate this paper, we will repeat this example here. Consider the following matrix G .

$$G = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 0 \end{pmatrix} \quad (2)$$

Definition 1: $f(x, t)$ is P -uniformly decreasing if $(x - y)^T P(f(x, t) - f(y, t)) \leq -c\|x - y\|^2$ for some $c > 0$ and all x, y .

By using the Lyapunov function $\sum_i (x_i - x_{i+1})^2$, it was shown in [1] that the array synchronizes if $f(x, t) + \frac{1}{\alpha} Dx$ is P -uniformly decreasing for some symmetric positive definite matrix P and PD is symmetric negative semidefinite where $\alpha \rightarrow 0$ as n grows where G is an $n \times n$ matrix. In fact α is equal to $1 - \cos(\frac{\pi}{n})$. This is however a very conservative bound. In [1] it was shown using other Lyapunov functions that the array with G as in Eq. (2) synchronizes if $f(x, t) + Dx$ is P -uniformly decreasing for some symmetric positive definite P .

The purpose of this paper is to reduce the gap in the applicability and generality between these two classes of methods by extending the Lyapunov function based methods to the case where G is not symmetric nor normal. In particular, we utilize optimization techniques to search for a suitable Lyapunov function. This is a commonly used approach in control systems design and analysis.

II. SYNCHRONIZATION CRITERIA VIA LYAPUNOV'S DIRECT METHOD

We will only consider real matrices in this paper.

Definition 2: Let W be the set of matrices with zero row sums and only nonpositive off-diagonal elements.

Definition 3: Let W_i be the set of irreducible matrices in W .

Definition 4: A (not necessarily symmetric) real matrix A is positive (semi)definite if $x^T Ax < 0 (\leq 0)$ for all $x \neq 0$. A matrix A is negative (semi)definite if $-A$ is positive (semi)definite. Equivalently, a real matrix A is positive (semi)definite if all eigenvalues of $A + A^T$ are positive (nonnegative).

First, let us consider the following synchronization theorem for array of coupled systems [1], [3]:

Theorem 1: Eq. (1) synchronizes if

- $f(x, t) + \alpha Dx$ is P -uniformly decreasing for some positive definite symmetric P ;
- There exists a symmetric matrix U in W_i such that

$$(U \otimes P)(G \otimes D - \alpha I \otimes D) = U(G - \alpha I) \otimes PD$$

is negative semidefinite.

Definition 5: For a zero row sum matrix G , let $L(G)$ denote the set of eigenvalues of G which do not correspond to the eigenvector $(1, \dots, 1)^T$.

In particular, if G has a zero eigenvalue of multiplicity 1, then $0 \notin L(G)$.

Definition 6: Let $\mu(G)$ be the supremum of all real numbers μ such that $U(G - \mu I)$ is positive semidefinite for some symmetric matrix U in W_i .

Note that if $U(G - \mu I) \geq 0$ for some $U \geq 0$, then $U(G - \lambda I) \geq 0$ for all $\lambda \leq \mu$. Furthermore, by Lemma 7 in [1], $\mu(G)$ is only defined if G has constant row sums.

Corollary 1: System (1) synchronizes if there exists some symmetric positive definite matrix P such that $f(x, t) + \mu(G)Dx$ is P -uniformly decreasing and PD is symmetric negative semidefinite.

Proof If $U(G - \mu(G)I) \geq 0$, then $U(G - \mu(G)I) \otimes PD$ is negative semidefinite since PD is symmetric negative semidefinite and the proof follows directly from Theorem 1. Otherwise, for small enough $\epsilon > 0$, $U(G - (\mu(G) - \epsilon)I) \geq 0$ and $f(x, t) + (\mu(G) - \epsilon)Dx$ is still P -uniformly decreasing and the proof follows from Theorem 1. \square

Thus for a fixed D , the larger $\mu(G)$ is, the easier it is to synchronize the array. It is easy to see that if PD is negative definite and $f + D$ is P -uniformly decreasing, then $f + \mu D$ is P -uniformly decreasing for $\mu \geq 1$. This implies that increasing the coupling term preserves the synchronization in this type of coupling.

III. COMPUTING $\mu(G)$

The following theorem gives an upper bound on $\mu(G)$ when G is a zero row sum matrix.

Theorem 2: Let G be a zero row sums matrix. If λ is a real eigenvalue in $L(G)$, then $\mu(G) \leq \lambda$.

Proof Let λ be a real eigenvalue of G in $L(G)$ with eigenvector v . Since v is not of the form $\gamma(1, 1, \dots, 1)^T$, v is also not in the kernel of U . $(G - \mu I)v = (\lambda - \mu)v$, and thus $v^T U(G - \mu I)v = (\lambda - \mu)v^T Uv$. $v^T Uv > 0$ since v is not in the kernel of U . This in combination of the definition of $\mu(G)$ implies that $\lambda - \mu(G) \geq 0$. \square

We now show two classes of matrices with real eigenvalues where this upper bound is also a lower bound, i.e., $\mu(G)$ is equal to the least real eigenvalue in $L(G)$. The first class is the class of triangular zero row sums matrices.

Lemma 1: For any n by n constant row sum matrix A , there exists $n - 1$ by $n - 1$ matrix $B = S(A)$ such that $CA = BC$ where C is defined as

$$C = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \dots & \\ & & & & 1 & -1 \end{pmatrix} \quad (3)$$

In particular, $B = CAQ$ where

$$Q = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ & & \ddots & & 1 \\ & & & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Furthermore, $S(A + \alpha I) = S(A) + \alpha I$.

Proof See [1]. □

Theorem 3: If G is a triangular zero row sums matrix, then $\mu(G)$ is the least eigenvalue of $L(G)$.

For instance, if G is an upper triangular zero row sums matrix, then $\mu(G)$ is equal to the least diagonal element of G , excluding the lower-right diagonal element.

Proof Without loss of generality, suppose that G is an $n \times n$ upper triangular zero sum matrix:

$$G = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & a_{n-1,n-1} & -a_{n-1,n-1} \\ 0 & 0 & \dots & & 0 \end{pmatrix} \quad (4)$$

From Theorem 2, $\mu(G) \leq \min_{1 \leq i \leq n-1} a_{i,i}$. By Lemma 1,

$$B = \begin{pmatrix} a_{1,1} & b_{1,2} & \dots & & \\ & a_{2,2} & b_{2,3} & \dots & \\ & & \ddots & & \\ & & & & a_{n-1,n-1} \end{pmatrix}$$

is an $(n-1) \times (n-1)$ upper triangular matrix which satisfies $CG = BC$ where C is defined in Eq. (3). Let $\Delta = \text{diag}(\alpha_1, \dots, \alpha_{n-1})$ where $\alpha_i > 0$ and $H = \Delta B \Delta^{-1}$. The (i, j) -th element of H is $b_{i,j} \frac{\alpha_i}{\alpha_j}$ if $j > i$ and 0 if $j < i$. Therefore, for each $\epsilon > 0$, if we choose $\alpha_j \gg \alpha_i$ for all $j > i$, then we can ensure that the (i, j) -th element of H has absolute value less than $\frac{2\epsilon}{n-2}$ for $j > i$. By Gershgorin's circle criterion the eigenvalues of $\frac{1}{2}(H + H^T)$ is larger than $\min_{1 \leq i \leq n-1} a_{i,i} - \epsilon$. Consider $U = C^T \Delta^2 C$. $U \in W_i$ by Lemma 6 in [1]. Using Lemma 1

$$U(G - \mu I) = C^T \Delta^2 C(G - \mu I) = C^T \Delta(H - \mu I) \Delta C$$

which is positive semidefinite if $H - \mu I$ is positive semidefinite. From the discussion above $H - \mu I \geq 0$ if $\mu < \min_{1 \leq i \leq n-1} a_{i,i}$. Therefore $\mu(G) \geq \min_{1 \leq i \leq n-1} a_{i,i}$. □

The second class of matrices where we can explicitly determine $\mu(G)$ is the class of symmetric matrices in W_i .

Theorem 4: If $G \in W_i$ is symmetric, then $\mu(G)$ is the least nonzero eigenvalue of G .

Proof Let α be the least nonzero eigenvalue of G . If we choose $U = G$, then $U(G - \alpha I) = U^2 - \alpha U$ is a symmetric matrix whose eigenvalues are of the form $\lambda(\lambda - \alpha)$ for λ an eigenvalue of U . Since $\lambda \geq \alpha$ for $\lambda \neq 0$, this implies that $U(G - \alpha I)$ has only nonnegative eigenvalues and is thus positive semidefinite. This implies that $\alpha \leq \mu(G)$. Combine this Theorem 2 we get $\alpha = \mu(G)$. \square

We conjecture that the bound in Theorem 2 also give the value of $\mu(G)$ for a class of nonsymmetric matrices G in W_i .

Conjecture 1: Let G be a matrix in W_i . If there exists a real eigenvalue λ of G such that the real parts of all nonzero eigenvalues of G are larger than or equal to λ , then $\mu(G) = \lambda$.

This conjecture was verified by numerical experiments on small matrices G . In particular, the following optimization problem was solved numerically:

$$F = \frac{1}{2} \max_{U \in W_i, U=U^T} \lambda_{\min}(UG - \alpha U + (UG - \alpha U)^T)$$

where $\lambda_{\min}(A)$ is the smallest eigenvalue of A and α is the least real part of the nonimaginary eigenvalues of G .

By Theorem 2, $F \leq 0$. If $F = 0$ for the set of matrices G under consideration, then Conjecture 1 is true. For numerical stability in performing this optimization, the set W_i is replaced by the subset $W_{\geq 1} \subset W_i$ which consists of matrices in W_i whose nonzero eigenvalues are larger than or equal to 1. It's clear that if $F = 0$, then using $W_{\geq 1}$ instead of W_i gives the same value for F . Numerical simulations have shown that F is close to zero for small matrices G which satisfy the properties in Conjecture 1. When G does not have a real nonzero eigenvalue whose real part is the smallest among all nonzero eigenvalues, then the value of F returned by the optimization program is strictly less than 0. It remains to be seen what the value of $\mu(G)$ is in this case.

Next we show that the calculation of F can be recast as a convex programming problem. Consider the subset of W_i where the matrix is symmetric and all the matrix elements are nonzero and denote this subset by W_n . It is clear that a symmetric matrix in W_i is uniquely determined by the superdiagonal elements¹. Note that by the Fischer-Courant Theorem [4] F can be written as:

$$-F = \frac{1}{2} \min_{U \in W_i, U=U^T} \lambda_{\max}((\alpha U - UG) + (\alpha U - UG)^T) = \min_{U \in W_i, U=U^T} \max_{\|x\|=1} x^T(\alpha U - UG)x$$

Any symmetric matrix in W_i can be approximated by a series of matrices in W_n and therefore we can calculate F as

$$-F = \min_{U \in W_n} h(U)$$

where $h(U) = \frac{1}{2} \lambda_{\max}((\alpha U - UG) + (\alpha U - UG)^T) = \max_{\|x\|=1} x^T(\alpha U - UG)x$. Note that h is a convex function of the superdiagonal elements of U and that W_n is a convex set of the superdiagonal elements of U and we can perform this minimization efficiently over the superdiagonal elements of U .

¹i.e., the matrix elements above the main diagonal.

Let us now return to the example considered earlier. For Eq. (2), $\mu(G) = 1$ by Theorem 3 and thus⁷ the array synchronizes if there exists some symmetric positive definite matrix P such that $f(x, t) + Dx$ is P -uniformly decreasing and PD is symmetric negative semidefinite.

In fact, for a triangular matrix G , the requirement that PD is symmetric negative semidefinite is not necessary.

Definition 7: $\dot{x} = f(x, t) + u(t)$ is u -asymptotically stable if $u_1 \rightarrow u_2$ as $t \rightarrow \infty$, then the trajectories of $\dot{x} = f(x, t) + u_1(t)$ and $\dot{x} = f(x, t) + u_2(t)$ converge towards each other as $t \rightarrow \infty$.

In [1] it was shown that if f is P -uniformly decreasing for some symmetric positive definite P , then $\dot{x} = f(x, t) + u(t)$ is u -asymptotically stable.

Theorem 5: Suppose $G \in W$ is upper triangular as in Eq. (4). If $f(x, t) + a_{i,i}Dx$ is P -uniformly decreasing for each $1 \leq i \leq n - 1$ and some positive definite P , then system (1) is synchronizing.

Proof We show that $x_i \rightarrow x_{i+1}$ by induction. Since $\dot{x}_n = f(x_n, t)$ and $\dot{x}_{n-1} = f(x_{n-1}, t) + a_{n-1,n-1}Dx_{n-1} - a_{n-1,n-1}Dx_n$, we can use the Lyapunov function $(x_n - x_{n-1})^T P(x_n - x_{n-1})$ and the fact that $f + a_{n-1,n-1}Dx$ is P -uniformly decreasing to show that $x_{n-1} \rightarrow x_n$ as $f(x, t) + a_{n-1,n-1}Dx$ is P -uniformly decreasing. Suppose that $x_j \rightarrow x_{j+1}$ for all $j \geq i$. Let $B_1 = \sum_{i \leq j \leq n} a_{i-1,j}Dx_j$ and $B_2 = -a_{i-1,i-1}Dx_i + \sum_{i \leq j \leq n} a_{i,j}Dx_j$. Then $\dot{x}_{i-1} = f(x_{i-1}, t) + a_{i-1,i-1}Dx_{i-1} + B_1$ and $\dot{x}_i = f(x_i, t) + a_{i-1,i-1}Dx_i + B_2$. Since $\sum_{i \leq j \leq n} a_{i-1,j} = -a_{i-1,i-1} = -a_{i-1,i-1} + \sum_{i \leq j \leq n} a_{i,j}$, it follows that $B_1 \rightarrow B_2$. Since $f(x, t) + a_{i-1,i-1}Dx$ is u -asymptotically stable, it follows that $x_{i-1} \rightarrow x_i$. \square

The reader is referred to [1] for more general reducible matrices G which are decomposed into irreducible components.

REFERENCES

- [1] Chai Wah Wu and Leon O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 430-447, Aug. 1995.
- [2] Louis M. Pecora and Thomas L. Carroll, "Master stability functions for synchronized chaos in arrays of oscillators," in *Proceedings of the 1998 IEEE Int. Symp. Circ. Syst.* IEEE, 1998, vol. 4, pp. IV-562-567.
- [3] Chai Wah Wu, "Synchronization in arrays of coupled chaotic circuits and systems: Theory and applications," in *Controlling Chaos and Bifurcations in Engineering Systems*, Guanrong Chen, Ed., chapter 23. CRC Press, Boca Raton, FL, 1999.
- [4] G. W. Stewart and Ji guang Sun, *Matrix Perturbation Theory*, Academic Press, 1990.

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Footnote

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