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Fractional packing of T-joins

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FRACTIONAL PACKING OF T-JOINS

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Abstract. Given a graph with nonnegative capacities on its edges, it is well known that the weight of a minimum T-cut is equal to the value of a maximum packing of T-joins. Padberg-Rao's algorithm finds a minimum weight T-cut but it does not produce a T-join packing, we present a polynomial combinatorial algorithm for finding an optimal T-join packing.

1. Introduction

We present a polynomial combinatorial algorithm for packing T-joins in a capacitated graph. Given a graph $G = (V, E)$ and $S \subseteq V$, the set of all edges with exactly one endnode in S is called a *cut* and denoted by $\delta(S)$. We say that S *defines* the cut $\delta(S)$. Given a set $T \subseteq V$ of even cardinality, we say that a cut $\delta(S)$ is a T-cut if $|S \cap T|$ is odd. A set of edges J is called a *T-join* if in the subgraph $G' = (V, J)$ the nodes in T have odd degree and the nodes in $V \setminus T$ have even degree. T-joins appear in the solution of the Chinese postman problem by Edmonds & Johnson [5]. Here the nodes in T are the nodes of odd degree and a T-join is a set of edges that have to be duplicated to obtain an Eulerian graph.

Edmonds & Johnson [5] proved that if A is a matrix whose rows are the incidence vectors of T-cuts, then for any nonnegative objective function the linear program below has an optimal integer solution that is the incidence vector of a T-join.

$$\min wx \tag{1}$$
$$Ax \geq 1 \tag{2}$$
$$x \geq 0. \tag{3}$$

Edmonds & Johnson gave a combinatorial polynomial algorithm to solve the linear program above and its dual

$$\max y1 \tag{4}$$
$$yA \leq w \tag{5}$$
$$y \geq 0. \tag{6}$$

This gives a packing of T-cuts. Seymour [15] proved that if the coefficients of w are integer, and their sum over every cycle is an even number, then (4)-(6) has an optimal integer solution. The algorithm of Edmonds & Johnson can be modified to produce this integer dual optimal solution, see [2].

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It follows from the theory of Blocking Polyhedra [6] that if B is a matrix whose rows are all incidence vectors of T-joins then for any nonnegative objective function c the linear program below has also an optimal integer solution that is the incidence vector of a T-cut.

$$\min cx \tag{7}$$
$$Bx \geq 1 \tag{8}$$
$$x \geq 0. \tag{9}$$

The dual problem is

$$\max y1 \tag{10}$$
$$yB \leq c \tag{11}$$
$$y \geq 0. \tag{12}$$

A solution of (10)-(12) is a maximum packing of T-joins. So from linear programming duality we have that the value of a maximum packing of T-joins is equal to the value of a minimum T-cut. Padberg & Rao [13] gave a polynomial combinatorial algorithm that finds a minimum T-cut. However this algorithm does not give a maximum packing of T-joins, and this has remained unsolved. Due to the equivalence between separation and optimization, one could solve this in polynomial time with the ellipsoid method, see [10]. The purpose of this paper is to give a polynomial combinatorial algorithm for finding a maximum (fractional) packing. To the best of our knowledge the only case that is well solved is when $|T| = 2$, this the well known maximum flow problem. Our algorithm has many similarities with an algorithm for packing arborescences given by Gabow and Manu [8].

There are several conjectures and questions related to the case when the linear program (10)-(12) has an integer solution. We discuss them below.

A graph is called *r-regular* if all its vertices have degree r. A graph is called an *r-graph* if it is r-regular and every V-cut has cardinality greater than or equal to r. A *perfect matching* is a set of non-adjacent edges that includes every vertex of the graph. Fulkerson made the following conjecture.

Conjecture 1. *Every 3-graph has six perfect matchings that include each edge at most twice.*

Notice that for a 3-graph, when $T = V$ every vertex defines a minimum T-cut. Also every T-join with positive weight in a maximum packing should intersect a minimum T-cut in exactly one edge, so the T-join should be a perfect matching. Thus in our terminology the conjecture above is equivalent to say that for a 3-graph when $T = V$ and c is a vector of all twos, then (10)-(12) has an optimal solution that is integer.

Seymour [14] generalized Fulkerson's conjecture as below.

Conjecture 2. *Every r-graph has $2r$ perfect matchings that include each edge at most twice.*

Seymour [14] also made the following two conjectures and proved that they are implied by Conjecture 2. A family of T-joins is called *k-disjoint* if every edge is included in at most k of them.

Conjecture 3. *If every vertex has an even degree then the size of a maximum 2-disjoint family of T-joins equals the double of the size of a minimum T-cut.*

Conjecture 4. *The size of a 4-disjoint family of T-joins equals four times the size of a minimum T-cut.*

Cohen & Lucchesi [3] made the conjecture below and proved that it is equivalent to Conjecture 2.

Conjecture 5. *If all T-cuts have the same parity then the size of a maximum 2-disjoint family of T-joins equals the double of the size of a minimum T-cut.*

They also proved the following.

Theorem 6. *If $|T| \leq 8$ and every T-cut has the same parity then the size of a maximum disjoint family of T-joins equals the size of a minimum T-cut.*

Conforti & Johnson [4] made the following conjecture. They proved their conjecture for graphs without a 4-wheel minor.

Conjecture 7. *If T is the set of nodes of odd degree, and the graph is not contractible to the Petersen graph, then the size of a maximum disjoint family of T-joins equals the size of a minimum T-cut.*

Holyer [11] proved that deciding whether a 3-regular simple graph has 3 disjoint perfect matchings is NP-complete. So finding an optimal integer solution of (10)-(12) is NP-hard. Tait [16] proved that the Four Color Theorem is equivalent to the statement that every 2-connected planar 3-regular graph has 3 disjoint perfect matchings. This is equivalent to say that for every 2-connected planar 3-regular graph, when $T = V$ and c is the vector of all ones, the linear program (10)-(12) has an optimal solution that is integer.

Now we give some extra notation and definitions. Let $n = |V|$ and $m = |E|$. We assume that every edge e has a non-negative *capacity* $c(e)$. If $c(e)$ is zero then the edge e is removed from the graph. For $S \subseteq V$ we use $\theta(S)$ to denote the *cut function*

$$\theta(S) = \sum\{c(e) \,:\, e \in \delta(S)\}.$$

Given $A, B \subseteq V$, we say that they *cross* if the sets $A \setminus B$, $B \setminus A$, and $A \cap B$ are nonempty. A family of sets such that no two of them cross is called *laminar*. A laminar family of subsets of V can have at most $2n - 1$ nonempty sets. It is well known that θ is a *submodular* function, i. e., for any two sets $A, B \subseteq V$ that cross

$$\theta(A \cup B) + \theta(A \cap B) = \theta(A) + \theta(B) + 2\beta(A, B),$$

where $\beta(A, B)$ is the sum of the capacities of the edges with one endnode in $A \setminus B$ and the other in $B \setminus A$. We use $\lambda(G)$ to denote the value of a minimum T-cut in G, i. e.,

$$\lambda(G) = \min\{\theta(S) \,:\, S \subset V,\ |S \cap T| \text{ is odd}\}.$$

For $U \subseteq E$ we use $\mu(U)$ to denote the *capacity* of U defined as

$$\mu(U) = \min\{c(e) \,:\, e \in U\}.$$

If J is a T-join and $\delta(S)$ is a cut, then $|J \cap \delta(S)|$ is odd if and only if $\delta(S)$ is a T-cut. If $U \subseteq E$ and $0 \leq \alpha \leq \mu(U)$, we denote by $G - \alpha U$ the graph obtained by replacing the capacity $c(e)$ of every edge $e \in U$, by $c(e) - \alpha$. A minimum cut separating nodes s and t is called a *minimum st-cut*. The nodes in the set T are called *T-nodes*.

This paper is organized as follows. In Section 2 we give a short description of Padberg-Rao's algorithm for finding a minimum T-cut. In Section 3 we present an initial description of the algorithm for packing T-joins. Sections 4 and 5 are devoted to more technical aspects required to complete the description of our algorithm. Section 6 contains a final analysis of our algorithm.

2. Padberg-Rao's algorithm

For the sake of completeness we give a short description of Padberg-Rao's algorithm for finding a minimum T-cut. It is based on the following lemma.

Lemma 1. *Let S define a minimum cut separating at least two nodes in T. If $|S \cap T|$ is odd then S defines a minimum T-cut. Otherwise, there is a set $S' \subseteq S$ or $S' \subseteq V \setminus S$ that defines a minimum T-cut.*

Proof. Assume that $|S \cap T|$ is even and consider a set A that defines a minimum T-cut. Suppose that A and S cross.

Case 1: $|A \cap S \cap T|$ is odd. Then $|(A \cup S) \cap T|$ is even. We have

$$\theta(A \cap S) + \theta(A \cup S) \leq \theta(A) + \theta(S).$$

Therefore $\theta(A \cap S) = \theta(A)$ and $\theta(A \cup S) = \theta(S)$. Thus $A \cap S$ defines a minimum T-cut.

Case 2: $|A \cap S \cap T|$ is even. Let $\bar{S} = V \setminus S$. Then $|A \cap \bar{S} \cap T|$ is odd and $|(A \cup \bar{S}) \cap T|$ is even. We have

$$\theta(A \cap \bar{S}) + \theta(A \cup \bar{S}) \leq \theta(A) + \theta(\bar{S}).$$

Therefore $\theta(A \cap \bar{S}) = \theta(A)$ and $\theta(A \cup \bar{S}) = \theta(\bar{S})$. Thus $A \cap \bar{S}$ defines a minimum T-cut. □

This lemma suggests a very simple algorithm, namely if S defines a minimum cut separating at least two nodes in T, then either S defines a minimum T-cut or one should continue working recursively with the graph G_1 obtained by contracting S and with the graph G_2 obtained by contracting $V \setminus S$.

Padberg & Rao also pointed out that one should first compute a Gomory-Hu (GH) tree [9], and then carry out the algorithm above on the GH-tree. This is because any minimum *st*-cut in the graph is given by a minimum *st*-cut in the GH-tree. Because of the tree structure, the algorithm becomes extremely simple: among all edges in the tree that are a T-cut, we should pick one of minimum capacity.

Thus the complexity of this procedure is the complexity of computing a GH-tree, i. e., computing $(n-1)$ minimum *st*-cuts.

3. The Algorithm

We start this section with an initial description of the algorithm. Clearly the capacity of any T-cut is an upper bound for the value of a T-join packing, and a minimum T-cut gives the value of an optimal packing. For the bound to be tight, any T-join with a positive weight in an optimal packing must intersect any minimum T-cut in exactly one edge, the algorithm works based on this property.

Using $\lambda(G)$ as the target value, the problem is solved recursively in a *greedy* way as follows. For a T-join U, let α_U be the largest value of α such that $\lambda(G - \alpha U) = \lambda(G) - \alpha$

and $0 \le \alpha \le \mu(U)$. Then the weight α_U is assigned to U. If $\lambda(G - \alpha_U U) > 0$ one should continue working recursively with $G - \alpha_U U$. In the remainder of this paper we show that a refinement of this algorithm runs in polynomial time. We need first a simple lemma.

Lemma 1. *If U is a T-join and $\alpha_U = 0$ then there is a minimum T-cut $\delta(S)$ such that $|\delta(S) \cap U| > 1$.*

Proof. First notice that $\lambda(G - \alpha U) \le \lambda(G) - \alpha$, for $0 \le \alpha \le \mu(U)$. This is because in $G - \alpha U$ the capacity of every T-cut $\delta(S)$ is $\theta(S) - k\alpha$, where $k = |\delta(S) \cap U|$.

So if $|\delta(S) \cap U| = 1$ for every minimum T-cut $\delta(S)$ then there is a small value of $\alpha > 0$, such that $\lambda(G - \alpha U) = \lambda(G) - \alpha$ and $\alpha \le \mu(U)$. □

From the lemma above we can see that one should concentrate on T-joins that intersect *every* minimum T-cut in exactly *one* edge. When we impose this condition for a minimum T-cut $\delta(S)$, we say that it is *tight*, we also say that S is a *tight set*. The two lemmas below show that we only need to impose this for a laminar family of tight sets.

Lemma 2. *Assume that A and B define minimum T-cuts, they cross, and $|A \cap B \cap T|$ is odd. Then the tightness of $A \cap B$ and $A \cup B$ imply the tightness of A and B.*

Proof. We have that

$$\theta(A \cap B) + \theta(A \cup B) \le \theta(A) + \theta(B).$$

Since A and B define minimum T-cuts, then $A \cap B$ and $A \cup B$ also define minimum T-cuts. Therefore this inequality must hold as equation. This implies that there is no edge between $A \setminus B$ and $B \setminus A$. Moreover for a T-join U and any cut $\delta(S)$ the cardinality of $\delta(S) \cap U$ is odd if S defines a T-cut and even otherwise. Then by a counting argument it is easy to see that any T-join that has exactly one edge entering $A \cap B$ and exactly one edge entering $A \cup B$ must have exactly one edge entering A and exactly one edge entering B. Figure 1 displays all possible configurations. □

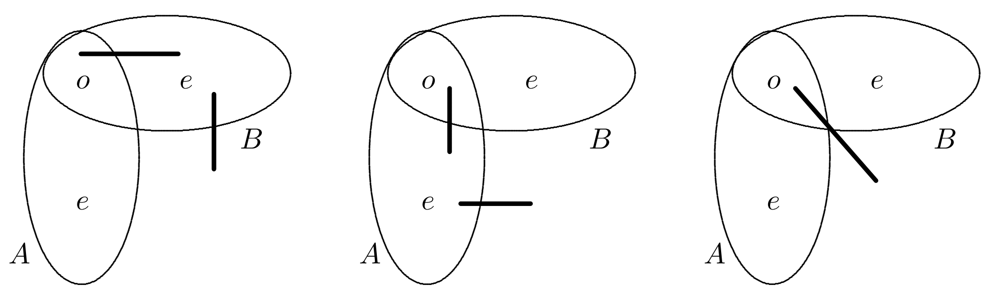

FIGURE 1. The labels e (even) and o (odd) refer to the parity of $|(A \setminus B) \cap T|$, $|A \cap B \cap T|$ and $|(B \setminus A) \cap T|$.

Lemma 3. *Assume that A and B define minimum T-cuts, they cross, and $|A \cap B \cap T|$ is even. Then the tightness of $A \setminus B$ and $A \setminus B$ imply the tightness of A and B.*

Proof. Apply Lemma 2 to A and $\bar{B} = V \setminus B$. □

So when we keep a family of tight sets, we can apply the last two lemmas to convert it into a laminar family. Denote by Φ this family, it can contain at most $2n - 1$ tight sets. We are going to find a T-join that intersects every T-cut given by Φ in exactly one edge. Let U be this T-join. There are two possible cases:

1. If $\alpha_U = \mu(U)$ then the number of edges in $G - \alpha_U U$ is at least one less than the number of edges in G.

2. If $\alpha_U < \mu(U)$ then in $G - \alpha_U U$ there is a minimum T-cut $\delta(S)$, $S \notin \Phi$, such that $|U \cap \delta(S)| > 1$. In this case we should add S to Φ and uncross it using Lemmas 2 and 3 as in the procedure below.

Uncross Φ

While there are two sets A and B in Φ that cross
do
  if $|A \cap B \cap T|$ is odd set $\Phi \leftarrow (\Phi \setminus \{A, B\}) \cup \{A \cap B, A \cup B\}$,
  otherwise set $\Phi \leftarrow (\Phi \setminus \{A, B\}) \cup \{A \setminus B, B \setminus A\}$
end

It is easy to see that at each uncrossing step the number of crossing pairs decreases by one.

Now we can give a formal description of the algorithm.

Pack T-joins

Step 0. Set $\Phi \leftarrow \emptyset$.
Step 1. Find a T-join U such that $|U \cap \delta(S)| = 1$, for all $S \in \Phi$.
Step 2. Compute α_U as the maximum of α such that
$\lambda(G - \alpha U) = \lambda(G) - \alpha$, and $0 \leq \alpha \leq \mu(U)$.
Step 3. If $\alpha_U < \mu(U)$, a new tight T-cut $\delta(S)$ has been found. Set $\Phi \leftarrow \Phi \cup \{S\}$ and uncross Φ.
Step 4. Set $G \leftarrow G - \alpha_U U$. If $\lambda(G) = 0$ stop, otherwise go to Step 1.

Since at each iteration either the cardinality of Φ increases or one edge is deleted, the total number of iterations is at most $2n - 1 + m$. It remains to describe how to perform Steps 1 and 2. This is the subject of the next two sections.

4. Finding a T-join in Step 1

Given the family Φ of tight sets we need to find a T-join U such that $|U \cap \delta(S)| = 1$, for all $S \in \Phi$. This will be done recursively.

The first time we start with $S = V$ and define G_S as the subgraph induced by S, with every maximal set of Φ that is properly contained in S contracted, labeled as a T-node and marked as *tight*. Let T_S be the set of T-nodes in G_S. We define an auxiliary graph whose node set is T_S, this is a complete graph. For any two nodes in T_S we find a path in G_S between them of minimum cardinality. Tight nodes can be the beginning or the end of a path, but not an intermediate node. This is to ensure that the resulting T-join intersects exactly once every tight T-cut. The cardinality of this path becomes the weight of the corresponding edge in the auxiliary graph. We find a minimum weight perfect matching in the auxiliary graph. This is to ensure that the resulting T-join is minimal. In G_S we take the union of all paths whose corresponding edges are in the matching. This gives a T-join U_S in G_S. Every tight node has exactly one edge of U_S incident to it.

Then we have to deal with each set W that has been contracted. In the T-join above, there is exactly one edge $e = \{i, j\}$, with $j \in W$. This time G_S is the subgraph induced by W plus the edge e, and the node i labeled as a T-node. Again every every maximal set of Φ that is properly contained in S is contracted and we proceed as above.

The complexity of finding a minimum weight perfect matching in a complete graph with t nodes is $O(t^3)$, see [7, 12]. Also the complexity of finding all shortest paths in G_S is $O(t^3)$. Therefore the complexity of Step 1 is $O(n^3)$.

5. Finding α_U in Step 2

Given a T-join U we are going to compute the maximum value of α such that

$$\lambda(G - \alpha U) = \lambda(G) - \alpha, \quad \text{and} \quad 0 \leq \alpha \leq \mu(U).$$

Let us define $f(\alpha) = \lambda(G - \alpha U)$. The function f is the minimum of a set of affine linear functions, so it is concave and piecewise linear. We have to find its first breakpoint. For this we start with a tentative value $\alpha_U = \mu(U)$. We compute $f(\alpha_U)$, if $f(\alpha_U) = \lambda(G) - \alpha_U$ we are done, otherwise let $\delta(S)$ be a minimum T-cut in $G - \alpha_U U$. Let $k = |U \cap \delta(S)|$. Let $\bar{\alpha}$ be the solution of $\lambda(G) - \alpha = \theta(S) - k\alpha$. We set $\alpha_U \leftarrow \bar{\alpha}$ and continue. See Figure 2.

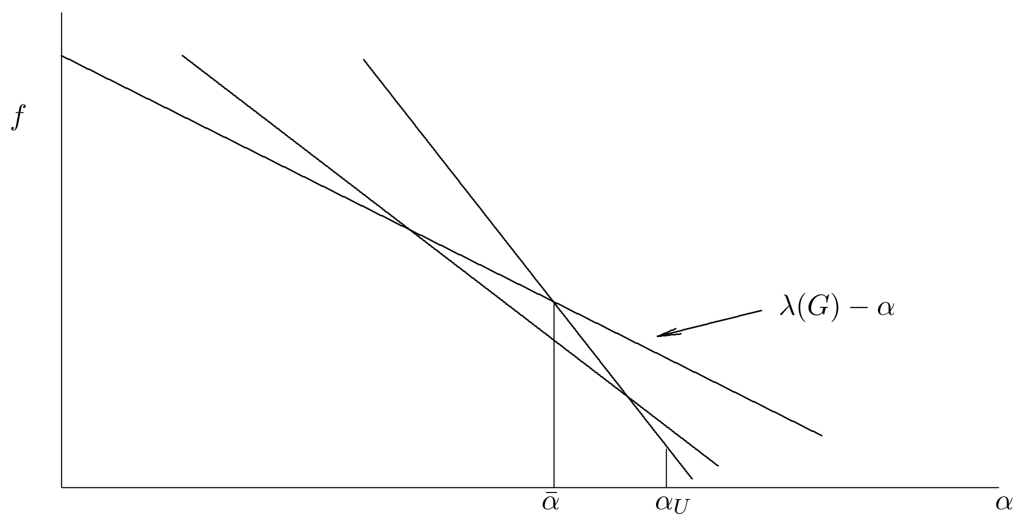

Figure 2

A formal description of this algorithm is below.

Find α_U

Step 0. Set $\alpha_U \leftarrow \mu(U)$.
Step 1. Find a minimum T-cut $\delta(S)$ in $G - \alpha_U U$. If $\lambda(G - \alpha_U U) = \lambda(G) - \alpha_U$ stop. Otherwise continue.
Step 2. Compute $\bar{\alpha}$ as the solution of of $\lambda(G) - \alpha = \theta(S) - k\alpha$. Where $k = |U \cap \delta(S)|$.
Step 3. Set $\alpha_U \leftarrow \bar{\alpha}$ and go to Step 1.

The complexity of this algorithm is given below.

Lemma 1. *If $\alpha_U = \mu(U)$ this algorithm requires $O(n)$ minimum st-cut computations, otherwise it requires $O(n^2)$ minimum st-cut computations.*

Proof. If $\alpha_U = \mu(U)$ only one iteration is performed. Otherwise at each iteration the value of $k = |U \cap \delta(S)|$ decreases. Since $|U| \leq n-1$, the above algorithm takes at most $n-1$ iterations. At each iteration one has to find a minimum T-cut with Padberg-Rao's algorithm, this requires $n-1$ minimum st-cut computations, then the result follows. $\square$

6. Final Analysis

Clearly the running time of the algorithm in Section 3 is dominated by the running time of Steps 1 and 2. Also notice that at most $2n-1+m$ iterations are performed. Thus the total running time of Step 1 is $O((n+m)n^3)$. For Step 2 there are at most m iterations where $\alpha_U = \mu(U)$ that require $n-1$ minimum st-cuts, and at most $2n-1$ iterations that require at most $(n-1)^2$ minimum st-cuts. The complexity of finding a minimum st-cut is $O(n^3)$, see [1]. Thus the total running time of Step 2 is $O((mn+n^3)n^3)$. Therefore the complexity of this algorithm is $O(n^6)$.

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