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Uniqueness of Optimal Production Schedule with Convex Costs and Capacity Constrains

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1. Introduction and Literature Review

This paper revisits the classical production-planning problem studied in Modigliani and Hohn (1955). In the seminal paper, Modigliani and Hohn solved the problem of satisfying deterministic demands over a finite-planning horizon at the lowest possible cost. The objective was to minimize the sum of production costs and inventory holding costs. They assumed unlimited capacity and strictly increasing continuous marginal cost functions.

The main contribution of this paper is the uniqueness of optimal production schedule. The uniqueness theorem can not only simplify the proof of optimality in Modigliani and Hohn (1955), but also allow extension to a broader set of problems (e.g., general convex production cost functions with capacity constraints in this paper). Moreover, the permutation method used in this paper, to our best knowledge, is the first of its kind in the relevant literature. Permutation method is commonly applied in mathematics. When applied it to solve problems of production planning, the method is a powerful tool in proving optimality results with a variety of problem settings (e.g., see Duan 2002).

The existing literature can be classified according to the type of production cost functions considered. Production cost functions can be either convex or concave. When a firm operates close to its capacity, cost for capacity expansion, overtime payment, expediting procurement etc is inevitable. The unit marginal cost thus increases as the production outputs increase; and the production cost function is convex. Cecchetti, Kashyap and Wilcox (1997) provided empirical evidence of five industries whose production costs are convex. When the production outputs are far below capacity, the marginal costs decline due to economies of scale, and the production costs are then concave (Hall 1991 and Ramey 1991).

Following Modigliani and Hohn (1955), rich literature appeared on deterministic production planning problems. Morin (1955) and Johnson (1957) extended the work of Modigliani and Hohn to general convex production cost functions. Veinott (1964) solved the problem with convex production cost functions and convex inventory holding costs. Based on complementary slackness results of Brooks and Geoffrion (1966), Eppen and Gould (1968) applied Lagrangian relaxation techniques to solve a similar problem as in this paper with penalty cost for product shortage. Carlson (1976) studied a problem with convex quadratic production cost functions and non-decreasing demands. Representative papers for concave production cost functions include Wagner and Whitin (1958), Manne (1958), Wagner (1960), Zabel (1964), Zangwill (1966), Manne and Veinott (1967), Florian and Klein (1971), Ltheirove (1973), Jagannathan and Rao (1973).

The remainder of the paper is organized as follows. Section 2 formally states the problem. Section 3 analyzes the local properties of the optimal production schedule, and establishes the key result of uniqueness theorem. Section 4 presents an algorithm for finding the optimal production schedule. Section 5 treats the special case of quadratic production cost functions, and Section 6 ends the paper with discussions on extensions

2. The Problem Formulation

Let d_j be the deterministic demands for a single product in the j^{th} period, z_j be the production level in the j^{th} period, I_j be the inventory at the end of period j, and h_j be positive per unit cost of holding inventory in the j^{th} period (j=1, ..., n). X is the production capacity for each period. F(z) denotes the cost of producing z units in any period (defined for z in [0, X]). The definition of notions implies that

$$I_j = I_0 + \sum_{i=1}^{j}(z_i - d_i) = I_{j-1} + (z_j - d_j) \tag{1}$$

Without loss of generality, we assume boundary inventory conditions

$$I_0 = I_n = 0 \tag{2}$$

Otherwise $I_0>0$ can be induced to the case of $I_0=0$ by the initialization procedure in Eppen and Gould (1968); and $I_n>0$ adds a constant term to the objective function of the case $I_n=0$. Both scenarios thus don't change any results in this paper.

The objective is to satisfy demands (d_1, ..., d_n) with the lowest total costs C(z), which is the sum of production cost and inventory holding cost. No backlog is allowed (thus inventory can't be negative), and production outputs are constrained by capacity. Therefore, the problem is formulated as the following:

$$\text{Minimize} \quad C(z) = \sum_{i=1}^{n} F(z_i) + \sum_{i=1}^{n} h_i I_i \tag{3}$$

Subject to:

$$I_j = \sum_{i=1}^{j}(z_i - d_i) \geq 0; \qquad for\ j = 1, \cdots, n \tag{4}$$

$$0 \leq z_j \leq X; \qquad for\ j = 1, \cdots, n \tag{5}$$

For convenience, we assume $d_j > 0$ for all j, and the production cost function F(z) is continuously differentiable strictly convex function, i.e.,

$$F'(z_1) < F'(z_2) \quad \textit{if and only if} \ z_1 < z_2 \tag{6}$$

3. Properties of The Optimal Production Schedule

This section first studies the local structures of the optimal solutions (i.e., the relationship between consecutive planning periods). Then we show that there is a unique feasible production schedule satisfying the local optimal structures, thus establish the uniqueness of the global optimal solution to the problem defined in (3)-(5).

Intuitively, two driving forces jointly determine the total cost. The convexity of production cost requires even production levels across different periods; while inventory holding costs push production in the planning horizon to as late as possible. The optimal production schedule is thus the outcome of balance between the two forces.

A vector of production quantities Z = (z_1,...,z_n) is feasible if constraints (4) and (5) are met. An optimal schedule is a feasible schedule that minimizes the total production cost defined in (3). We denote $Z^* = (z_1^*, z_2^*, \cdots, z_n^*)$ for optimal production schedule, and $I^* = (I_1^*, I_2^*, \cdots, I_n^*)$ for the corresponding optimal inventory level.

Given a feasible schedule Z = (z_1,...,z_n), its i[th] permutation schedule $Z_i^{'} = (z_1^{'}, \cdots, z_n^{'})$ is defined by reducing z_i a small amount θ and increasing z_{i+1} by θ, i.e., $z_i^{'} = z_i - \theta, z_{i+1}^{'} = z_{i+1} + \theta$ and $z_j^{'} = z_j$ for j≠i,i+1. When z_i>0, I_i>0 and z_{i+1}<X, it is easy to check that $Z_i^{'}$ is also a feasible schedule for sufficiently small number θ. For such θ, define a new function:

$$f(\theta, Z, i) = C(Z_i^{'}) - C(Z) = F(z_i - \theta) + F(z_{i+1} + \theta) - F(z_i) - F(z_{i+1}) - h_i\theta \qquad (7)$$

We then have the following properties for the optimal production schedule.

Lemma 1 The optimal production schedule has the following properties:

i) Production quantity in any period cannot be zero, i.e., $z_j^* > 0$ for $1 \le j \le n$.

ii) If $I_j^* > 0$, then $z_{j+1}^* \ge z_j^*$ (8)

iii) $F'(z_{j+1}^*) \le F'(z_j^*) + h_j \ \forall j = 1, ..., n$ (9)

iv) If $I_j^* > 0$ and production levels in periods j and j+1 are strictly less than the capacity X (i.e., $z_j^* < X$, and $z_{j+1}^* < X$), then inequality (9) becomes equality, i.e.,

v) $F'(z_{j+1}^*) = F'(z_j^*) + h_j$ (10)

Proof i) Otherwise, suppose k is the largest integer such that $z_k^* = 0$ and $z_j^* > 0$ for j<k. Because I_0=0 and d_1>0, the inventory constraint $I_1^* = I_0 + z_1^* - d_1 = z_1^* - d_1 \ge 0$ induces $z_1^* \ge d_1 > 0$. Therefore, k>1 and (k-1) is well defined. Moreover, d_k>0 and $I_k^* = I_{k-1}^* + z_k^* - d_k = I_{k-1}^* - d_k \ge 0$ imply that $I_{k-1}^* \ge d_k > 0$. Therefore the (k-1)[th] permutation schedule $Z_{k-1}^{*'}$ is feasible and the function $f(\theta, Z^*, k-1)$ is well defined for sufficiently small θ. The optimality of Z* then says $f(\theta, Z^*, k-1)$ is non-negative for all sufficiently small θ>0. Thus:

$$\partial f(\theta, Z^*, k-1) \Big/ \partial\theta = -F^{'}(z_{k-1}) + F'(0) - h_{k-1} \ge 0 \Rightarrow F^{'}(0) - F^{'}(z_{k-1}) \ge h_{k-1} > 0$$

This is in contradiction with the convexity of F(z) (i.e., $F^{'}(0) \le F^{'}(z_{k-1})$).

ii) The proof for ii) is similar to the proof of i).

iii) The optimality of Z* implies minimum value of $f(\theta, Z^*, j)$ at θ=0 whenever the function in (7) is defined, which derives (9).

iv) Because of i), $0 < z_j^* \pm \theta, z_{j+1}^* \mp \theta < X$ for sufficiently small θ, and thus $f(\theta, Z^*, j)$ is differentiable at θ=0. The optimality of Z* implies minimum value of $f(\theta, Z^*, j)$ at θ=0. Therefore $\partial f(\theta, Z^*, j) \Big/ \partial\theta = 0$, which gives (10). ■

The term $F'(z_{j+1}^*)$ is the unit marginal cost of producing in period (j+1), while $F'(z_j^*) + h_j$ is the marginal cost of producing products in period j and storing them for one period (incurring a storage cost of h_j). (9) and (10) simply state that the optimal schedule should produce one additional unit in any period only if it is more expensive to produce an extra unit in the previous period plus the inventory holding cost.

It is useful to distinguish zero-inventory periods (those with I_j=0) and positive-inventory periods (those with I_j>0). A direct corollary of Lemma 1 is the following local property theorem for the optimal production schedule(s).

Local Property Theorem The optimal production quantities in two consecutive periods satisfy the following:

i) If I_j =0, then z_j and z_{j+1} satisfy (9).

ii) If $I_j^* > 0$, then $z_{j+1} \geq z_j$ and

(a) Either z_{j+1} = X (capacity limit), and z_j and z_{j+1} satisfy (9).

(b) Or z_{j+1} < X, and then z_j and z_{j+1} satisfy (10).

The local property theorem characterizes the local structure of the optimal production schedule(s). Many feasible production vectors might have the properties. However, the following theorem shows a somewhat surprising result that there is only one feasible production schedule satisfying the local optimal structure.

Uniqueness Theorem If there are two feasible production schedules B and C satisfying the necessary optimal conditions in the Local Property Theorem, then B=C.

Proof Let b_i and c_i be the production quantity of the i[th] period for production schedule B and C respectively, i=1, 2, …, n.

If B≠C, let m (≤n) be the first period such that $b_1=c_1$, $b_2=c_2$, …, $b_{m-1}=c_{m-1}$, but $b_m>c_m$ (without loss of generality). Then m must be a positive-inventory period in the production schedule B (i.e., I_m^B>0), because otherwise $I_m^B = 0 \Leftrightarrow \sum_{j=1}^{m} d_j = \sum_{j=1}^{m} b_j > \sum_{j=1}^{m} c_j$ implies C is not feasible since demands in the first m periods can't be met by outputs from C. Let u (>m) be the next zero-inventory period in the production schedule B, i.e., I_u^B=0, I_i^B>0 for m≤i<u. We claim that $b_i \geq c_i$ for all m≤i≤u.

Since each period i in m≤i<u is a positive-inventory period, by property ii) in the Local Property Theorem, $b_{i+1} \geq b_i$ for all m≤i≤u and there are two cases:

Case 1: If b_{m+1}=X (capacity limit), then b_i=X for m<i≤u since $b_{i+1} \geq b_i$ for i, m≤i≤u. And $b_i \geq c_i$ for i (m≤i≤u) follows from the feasibility of C (because of the capacity constraints of $c_i \leq X$ for i, 1≤i≤n).

Case 2: If b_{m+1}<X, then $F'(b_{m+1}) = F'(b_m) + h_m$ by (10). Moreover, $F'(c_{m+1}) \leq F'(c_m) + h_m$ by (9). By convexity assumption (6), $b_m > c_m$ implies $F'(b_m) \geq F'(c_m)$, and therefore $F'(b_{m+1}) \geq F'(c_{m+1})$ which implies $b_{m+1} \geq c_{m+1}$. By induction i, we can show that $b_i \geq c_i$ for i, m≤i≤u in this case as well.

Therefore $b_i=c_i$ for 1≤i<m and $b_i \geq c_i$ for m≤i≤u, which implies $\sum_{j=1}^{u} b_j \geq \sum_{j=1}^{u} c_j$. Thus c_i = b_i for every i (1≤i≤u) so that C can be feasible (i.e. all demands in the first u periods are met by outputs from C), because I_u^B=0 (i.e., $\sum_{j=1}^{u} d_j = \sum_{j=1}^{u} b_j$).

By excluding the first u planning periods, the remaining problem is a problem of (n-u) periods. By induction on n, the unique theorem is established. ■

Since the uniqueness theorem only requires the (local) necessary conditions of the optimal production schedules, the uniqueness of the (Global) optimal solution is immediately obtained as follows.

Corollary 1 There is one and only one optimal solution to the production planning problem defined in (3)-(5).

Proof: The uniqueness claim directly follows from the unique theorem since optimal solutions must both be feasible and satisfy the necessary optimal conditions in the theorem. The existence of the optimal solution is obvious because the total cost function C(Z) is convex and it is defined over a convex domain of $\{ 0 \le z_j \le X \mid j = 1, \cdots, n \}$. ■

Moreover, the unique theorem can also dramatically simplify the complex and long proof of optimality of solution algorithms proposed in the existing literature. For example, the proof of optimality for Modigliani-Hohn algorithm (1955) becomes trivially a direct corollary of the theorem, i.e:

Corollary 2 The production schedule obtained by the Modigliani-Hohn algorithm (1955) is optimal.

Proof: Let $X=\infty$ and the corollary is true because the solution derived from the Modigliani-Hohn algorithm satisfies all the local optimal properties. ■

4. A Solution Algorithm

In addition to establish the uniqueness of the global optimal solution and to simplify the proof of optimality of historical algorithms, the uniqueness theorem can also help construct alternative solution algorithms for solving the production planning problem. This section presents such a new solution algorithm based on the uniqueness theorem.

The new algorithm is essentially "divide-and-conquer", breaking an n-period problem into a set of problems with shorter planning horizons and then merging them together to form the optimal solution for the entire problem.

The "divide" aspect of the algorithm is based on the local structure between two zero-inventory periods by the local property theorem. Let (i-1) and j be two consecutive zero-inventory periods in the optimal solution, then the production levels (z_i, z_{i+1}, ..., z_j) must satisfy:

1) $z_i < z_{i+1} < \ldots < z_j$, and $F'(z_{m+1}) = F'(z_m) + h_m$ for $i \le m < j$ by (10). Or,

2) $z_i < z_{i+1} < \ldots < z_t = X = \ldots = z_j$, and $F'(z_{m+1}) = F'(z_m) + h_m$ for $i \le m < t$ by (10) and $F'(z_t) \le F'(z_{t-1}) + h_{t-1}$ by (9) (Note that $F'(z_{m+1}) \le F'(z_m) + h_m$ is obviously true for $t \le m < j$ since $z_t = X = \ldots = z_j$ and the holding cost is positive).

Borrowing terminology from Carlson (1976), we define a single path solution to be a feasible solution (z_i, z_{i+1}, ..., z_j) that satisfies conditions in 1); and a path jump solution to be a feasible solution (z_i, z_{i+1}, ..., z_j) that satisfies the conditions in 2). (Note that feasibility here means all demands from period i to period j are satisfied without backorders). A production subvector (z_i, z_{i+1}, ..., z_j) is a good subvector if it satisfies one of the following:

a) $z_m = d_m$ (i.e., z_m exactly equals the demands in the m^th^ period), m=i, ..., j.

b) (z_i, z_{i+1}, ..., z_j) is a single path solution

c) (z_i, z_{i+1}, ..., z_j) is a path jump solution

Two good subvectors V=(z_i, ..., z_j) and W=(z_{j+1}, ..., z_t) are well matched if condition (9) is met, i.e., $F'(z_{j+1}) \le F'(z_j) + h_j$. If two good subvectors V and W are not well matched, we can merge V and W to form a good subvector that is longer (in terms of number of planning periods in the vector) than W by the following merging procedure.

Procedure of Merging Two Good Subvector Given two good subvectors $V=(z_i, \ldots, z_j)$ and $W=(z_{j+1}, \ldots, z_t)$ that are not well matched (i.e., $F'(z_{j+1}) > F'(z_j)+h_j$), which implies $z_{j+1}>z_j$, and therefore $z_j<X$ (capacity limit) because $z_{j+1}\leq X$ by feasibility constraints. There are two possible cases:

Case 1: When W is a single path solution. A new good subvector can be formed from V and W by reducing productions in periods (j+1) through t in W and simultaneously increasing productions in periods i through j in V. Specifically, let $z_m^{'} = z_m + \theta_m$ for $i\leq m\leq j$ and $z_m^{'} = z_m - \theta_m$ for $j+1\leq m\leq t$, where $\{\theta_m \mid i\leq m\leq t\}$ are positive numbers such that:

$$F'(z_{j+1} - \theta_{j+1}) = F'(z_j + \theta_j) + h_j \tag{11}$$

to ensure the goodness of period j (i.e. $F'(z_{j+1}^{'}) = F'(z_j^{'}) + h_j$). To ensure the goodness for periods j+1 to t and periods i to j, we must also have:

$$F'(z_{m+1} - \theta_{m+1}) = F'(z_m - \theta_m) + h_m \quad \text{for } j+1\leq m<t \tag{12}$$

$$F'(z_{m+1} + \theta_{m+1}) = F'(z_m + \theta_m) + h_m \quad \text{for } i\leq m<j \tag{13}$$

Finally, total production must remain the same as before:

$$\sum_{m=i}^{t} z_m = \sum_{m=i}^{t} z_m^{'} \text{, or equivalently } \sum_{m=i}^{m=j} \theta_m - \sum_{m=j+1}^{m=t} \theta_m = 0 \tag{14}$$

Since the cost function F(z) are strictly convex, its first derivative function $F'(z)$ is monotonically increasing, and the equation $F'(z) = \alpha$ has one and only one solution for any given value of α. Therefore, the (t-i+1) equations (with t-i+1 number of variables) in systems (11)-(14) yield one and only one solution $(\theta_i, \theta_{i+1}, \ldots, \theta_t)$. The merging procedure thus generates a unique good subvector with (t-i+1) number of planning periods included, from two good subvectors V (with j-i+1 periods) and W (with t-j periods).

Case 2: When W is a path jump solution. Let $W = (z_{j+1}, .., z_p, X, \ldots, X)$ where $z_p<X$. Then $(z_{j+1}, \ldots, z_p)$ is a single path solution, and a new good subvector can be generated by merging V and $(z_{j+1}, \ldots, z_p)$ using the procedure in Case 1. Denote the newly generated good subvector as V and W =(X, …, X), repeat the procedure in Case 1 until (V, W) becomes a good subvector.

The grand scheme of the new algorithm is simple. It starts with a feasible solution, which is composed of a set of good subvectors $(V_1, V_2, \ldots, V_s)$. But the subvectors may not well matched, the algorithm then repeatedly applies the above merging procedure until a single good production vector is obtained for the n-period planning problem. The algorithm is formally started as follows.

The Solution Algorithm Let $(V_1, V_2, \ldots, V_s)$ be the starting set of good subvectors, k=s-1, $W=V_s$ and $V=V_k$.

Step 1: If W and V are well matched, go to step 3; otherwise go to step 2.

Step 2: Apply the above merging procedure to subvectors V and W. Let W be the resulting good subvector, k:=k-1 and $V:=V_{k-1}$ and go to step 4.

Step 3: Let W := (V, W); k:=k-1 and $V:=V_{k-1}$ and go to step 4.

Step 4: If k=0, terminate; otherwise go to step 1.

The remaining problem is that at least one set of good subvectors is required to start the algorithm with. Recall that the demand vector is $(d_1, \ldots, d_n)$. Due to production capacity constraints, the demand vector may not be feasible. Fortunately, a set of good subvectors can be derived from the demand vector by the following initialization procedure.

The Initialization (for a set of good subvectors) Let k:=n, $Z=(z_1, z_2, \ldots, z_n) = (d_1, d_2, \ldots, d_n)$ (the original demand vector)

Step 1: If $z_k>X$ (capacity), let $z_{k-1} := z_{k-1} + (z_k-X)$ and k:=k-1

Step 2: Repeat step 1 until k=1.

The Initialization procedure either yields a feasible solution that can serve as a starting point for the algorithm, or ends up with $z_1>X$. The scenario of $z_1>X$ is equivalent to $\sum_{i=1}^{m} d_i > mX$ for some $m \le n$, which implies that the original problem itself is infeasible since there is no way to satisfy the demands in the first m periods without backorders.

It is straightforward to check that the resulting production vector from the algorithm meets all requirements in the local property theorem. By the uniqueness theorem, it follows that the algorithm yields the optimal solution.

Moreover, the complexity to solve (t-i+1) equations in (11)-(14) for (t-i+1) number of variables $(\theta_i, \theta_{i+1}, \ldots, \theta_t)$ is $O(t-i+1)^2$. Since each iteration in the algorithm increases the number of planning periods in W by at least one, the number of merges required for the algorithm is not greater than the number of production period n. Therefore, the proposed is a polynomial algorithm, whose complexity is $O(n^3)$.

5. Quadratic Production Cost Functions

This section thoroughly treats a special case when the production cost functions are quadratic $F(z) = az^2 + bz + c$, where a is positive (to ensure convexity) and b is positive (to ensure meaningful positive marginal cost). Then $F'(z) = 2az + b$ and condition (9) becomes:

$$z_{j+1} - z_j \le h_j/(2a) \tag{15}$$

While condition (10) becomes:

$$z_{j+1} - z_j = h_j/(2a) \tag{16}$$

Moreover, solving systems (11)-(14) for $(\theta_i, \theta_{i+1}, \ldots, \theta_t)$ is simplified dramatically since (12) and (13) become:

$$\theta_{j+1} = \theta_{j+2} = \cdots = \theta_t = \alpha \tag{17}$$

$$\theta_i = \theta_{i+1} = \cdots = \theta_j = \beta \tag{18}$$

Which is equivalent to $z'_m = z_m + \beta$ for all m ($i \le m \le j$) with β independent of m; and $z'_k = z_k - \alpha$ for all k ($j+1 \le k \le t$) with α independent of k. The two variables α and β completely determine $(\theta_i, \theta_{i+1}, \ldots, \theta_t)$, and α and β satisfy:

$$\begin{cases} \alpha + \beta = (z_{j+1} - z_j - h_j)/(2a) & (19) \\ (j-i+1)\beta - (t-j)\alpha = 0 & (20) \end{cases}$$

Where equation (19) is from (11) and (20) from (14).

The simplified systems (15)-(20) afford us to study how the optimal production schedule changes with respect to the parameter a, inventory holding costs $\{h_1, h_2, \ldots, h_n\}$ and capacity limit X. For simplicity, we assume $h_1=h_2=\ldots=h_n=h$ in the rest of the paper.

To explore the impact of holding cost h on the optimal production schedule, we consider two holding cost $h'' < h'$. Let the optimal solution for h' be $Z' = (Z'_1, \cdots Z'_u)$, where each $Z_i^{'}$ is a good subvector and all consecutive subvectors are well matched. Notice that the solution $Z^{'}$ is also feasible for the problem with holding cost $h^{''}$. So the optimal solution of the problem with holding cost $h^{''}$ can be obtained by applying the algorithm in Section 4, using the solution $Z^{'}$ as starting point (instead of the output of the initialization procedure).

It is worthwhile to study how the algorithm changes $Z^{'}$ to obtain the optimal solution $Z^{''}$ for $h^{''}$. Without loss of generality, we focus on just one good subvector $Z_{ij}^{'} = (z_i^{'}, z_{i+1}^{'}, \cdots, z_j^{'})$ in $Z^{'}$.

Case 1: When $Z_{ij}^{'}$ is a single path solution, then $z_{k+1}^{'} = z_k^{'} + h^{'}/(2a)$ for all k by (16), i<k<j. Therefore $z_k^{'} = z_i^{'} + (k-i)h^{'}/(2a)$ for all i≤k≤j, which means that $Z_{ij}^{'}$ is completely determined by the starting production level $z_i^{'}$ and the increment $h^{'}/(2a)$. With respect to $h^{''}$, the increment should be $h^{''}/(2a)$. It is obvious that $Z_{ij}^{'}$ is not a single path solution with respect to $h^{''}$. But a single path solution for $h^{''}$, which has the same structure as that for $h^{'}$, can be easily obtained as follows.

To ensure $\Sigma_{k=i}^{j} z_k^{'} = \Sigma_{k=i}^{j} z_k^{''}$, $z_k^{'} = z_i^{'} + (k-i)h^{'}/(2a)$, $z_k^{'''} = z_i^{''} + (k-i)h^{''}/(2a)$ for every k that i≤k≤j, the starting production level $z_i^{''}$ for the problem with respect to $h^{''}$ must satisfy $\Sigma_{k=i}^{j}[z_i^{'} + (k-i)h^{'}/(2a) = \Sigma_{k=i}^{j}[z_i^{''} + (k-i)h^{''}/(2a)]$. Thus:

$$\begin{cases} z_i^{''} = z_i^{'} + \dfrac{(h^{'} - h^{''})}{2a}\dfrac{(j-i)}{2} \\ z_k^{''} = z_i^{''} + (k-i)\dfrac{h^{''}}{2a} = z_i^{'} + \dfrac{h^{'}}{2a}\dfrac{(j-i)}{2} + \dfrac{h^{''}}{2a}\dfrac{(2k-i-j)}{2}, \; i \le k \le j \end{cases} \tag{21}$$

Case 2: When $z_{ij}^{'} = (z_i^{'} < \cdots < z_{t-1}^{'} < z_t^{'} = \ldots = z_j^{'} = X)$ is a path jump solution for some t, then $z_{k+1}^{'} = z_k^{'} + h^{'}/(2a)$ for all k (i<k<t) by (16) and $X = z_t^{'} \le z_{t-1}^{'} + h^{'}/(2a) = z_i^{'} + (t-i)h^{'}/(2a)$ by (15). We can first apply the procedure in Case 1 to $Z_{it}^{'} = (z_i^{'} < \cdots < z_{t-1}^{'} < z_t^{'} = X)$ for a single path solution for $h^{''}$ (from periods i to t), and the new t[th] period production is reduced from X to $z_t^{''} = z_i^{'} + \dfrac{(h^{'} + h^{''})}{2a}\dfrac{(t-i)}{2}$. Note that $z_t^{''} + \dfrac{h^{''}}{2a} = z_i^{'} + \dfrac{h^{'}}{2a}\dfrac{(t-i)}{2} + \dfrac{h^{''}}{2a}\dfrac{(t+2-i)}{2}$, though $X = z_t^{'} \le z_{t-1}^{'} + \dfrac{h^{'}}{2a} = z_i^{'} + \dfrac{h^{'}}{2a}(t-i)$, condition (15) for the problem (with $h^{''}$) may not true since $h^{''} < h^{'}$. If $X = z_{t+1}^{''} \le z_t^{''} + h^{''}/(2a)$, let $z_{t+1}^{''} = \cdots = z_j^{''} = X$; otherwise ($X > z_t^{''} + h^{''}/(2a)$), replace t by (t+1) and Z_{it} by $Z_{i(t+1)}$, repeat the process until either t=j or $X \le z_t^{''} + h^{''}/(2a)$, and the resulting $Z_{ij}^{''} = (z_i^{''}, \ldots, z_j^{''})$ is a path jump solution for the problem with $h^{''}$.

We must point out that the demand vector (d_1, ..., d_n) is out of picture in the above procedure. Therefore it remains to check the feasibility of the resulting $Z^{''}$. Fortunately, feasibility directly follows from four observations:

1. By (21), it follows that, as long as $h^{''} < h^{'}$, $\sum_{k=i}^{l} z_k^{''} \geq \sum_{k=i}^{l} z_k^{'}$ for all $l, i \leq l \leq j$ (i.e., total production in earlier periods should become larger[1]).
2. Since both the procedure in Case 2 above and the "Procedure of Merging Two Good Subvectors" reduce production level(s) in later period(s) while increase the production level(s) in earlier period(s), $\Sigma_{k=1}^{l} z_k^{''} \geq \Sigma_{k=1}^{l} z_k^{'}$ is true for all $l, 1 \leq l \leq n$ as well.
3. The feasibility of $z^{'} = (z_1^{'},...,z_n^{'})$ says that all the demands are met by $z^{'}$ without backorders. Thus $\Sigma_{k=1}^{l} z_k^{''} \geq \Sigma_{k=1}^{l} z_k^{'}$ for all $l, 1 \leq l \leq n$, implies that $z^{''}$ can do so as well, i.e., $z^{''}$ satisfies the inventory constraints in (4).
4. The capacity constraints in (5) are met by the way $z^{''}$ is created.

Since the procedure from $z^{'}$ to $z^{''}$ maintains all properties in the local property theorem, $z^{''}$ is thus the optimal solution to the problem with $h^{''}$ by the uniqueness theorem. Notice that the observation 1 is essential to the feasibility of $Z^{''}$, and it relies on the condition that $h^{''} < h^{'}$. If the procedure is reversed (i.e., from $Z^{''}$ to $Z^{'}$), the feasibility of $Z^{'}$ cannot be guaranteed from the feasibility of $Z^{''}$. Thus we have the following:

Proposition 1 If $h^{''} < h^{'}$, the optimal solution associated with $h^{''}$ can be constructed from the optimal solution associated with $h^{'}$, without using the demand vector at all. But the reverse is not true. In other words, the optimal production vector Z_h (for the problem with holding cost h) is "Morkov" with respect to "time" -h.

The initialization procedure in section 4 in essence assumes h=∞. Proposition 1 lends credence to the initialization procedure and the algorithm in Section 4. Similarly, we studied the impact of the cost parameter a on the optimal solution, and we had:

Proposition 2 If $a^{''} < a^{'}$, the optimal solution associated with $a^{'}$ can be constructed from the optimal solution associated with $a^{''}$, without using the demand vector at all. But the reverse is not true. In other words, the optimal production vector Z_a (for the problem with cost parameter a) is "Morkov" with respect to "time" a.

Asymptotically, when a goes to ∞ and h goes to 0, the following proposition holds.

Proposition 3 As h goes to 0 (or a goes to ∞), the optimal solution associated with parameter h (or a) will become piecewise constant, i.e., (C_1,C_1,...,C_1; C_2,C_2, ..., C_2; ...; C_t,...,C_t), where $C_1 \geq C_2 \geq ... \geq C_t$. And $C_1 = C_2 = ... = C_t = C$ if and only if production schedule (C, C, ...,C) is feasible (where C is the average demand of n periods).

Proof When h goes to 0 (or ∞), (15) and (16) become $z_j \geq z_{j+1}$ and $z_j = z_{j+1}$ respectively. The proposition follows from the way the optimal solution obtained. ■

To demonstrate how the algorithm works, we end this section with numerical examples.

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[1] This shouldn't come across as any surprise. When the unit inventory holding cost decreases (i.e., $h^{''} < h^{'}$), it is intuitively natural to increase production outputs in earlier periods to take advantage of the cheaper inventory holding costs

Example: Considering a four-period problem with demands $(d_1, d_2, d_3, d_4) = (1, 8, 7, 8)$, we assume $F(z) = z^2/2$, identical unit inventory holding cost of $h_1=h_2=h_3=h_4=h$ (where the parameter h to be determined), and capacity X(>0) to be determined as well.

Case 1: $h \geq 7$ and $X \geq 8$. The initial demand vector (1, 8, 7, 8) is optimal since it is both feasible and satisfies the local optimal properties.

Case 2: $5/3 \leq h < 7$ and $X \geq 8$. The initialization procedure yields the demand vector (1, 8, 7, 8) for the algorithm to start with. However, period 1 and period 2 are not well matched. By the merging procedure, optimal production levels z_1 and z_2 satisfy:

$$\begin{cases} z_1 + z_2 = 1 + 8 \\ z_2 - z_1 = h \end{cases} \quad (22)$$

Solving (22) for z_1 and z_2 yields $z_1=4.5-h/2$ and $z_2=4.5+h/2$. Since $h \geq 5/3$, $z_3-z_2 = 7 - (4.5+h/2) \leq h$ and thus z_2 and z_3 are well matched, and z_3 and z_4 are well matched as well. Therefore (4.5-h/2, 4.5+h/2, 7, 8) is the optimal solution.

Case 3: $4/3 \leq h < 5/3$ and $X \geq 8$. Using proposition 1, we use the optimal solution (4.5-h/2, 4.5+h/2, 7, 8) in the Case 2 as starting point. Because $h<5/3$, z_2 and z_3 are not well matched (i.e., $z_3-z_2 = 7-(4.5+h/2) > h$). Merging (z_1, z_2) with (z_3), we obtain a new set of good subvectors (16/3-h, 16/3, 16/3+h, 8). Moreover, $z_4 - z_3 = 8 - (16/3 + h) \leq h$ since $4/3 \leq h$. All subvectors are thus well matched, and (16/3-h, 16/3, 16/3+h, 8) is the minimum cost solution.

Case 4: $0 < h < 4/3$ and $X \geq 8$. Similar to the Case 3, the optimal solution is (6-1.5h, 6-0.5h, 6+0.5h, 6+1.5h).

Case 5: $X < 8$, say $X=7$. The initialization procedure generates (3, 7, 7, 7) for the algorithm to start with. Similar to the above discussion from Case 1 to Case 4, we have:

1. The optimal solution is (3, 7, 7, 7) when $h \geq 4$.
2. The optimal solution is (5 - 0.5h, 5 + 0.5h, 7, 7) when $4/3 \leq h < 4$.
3. The optimal solution is (17/3 - h, 17/3, 17/3 + h, 7) when $2/3 \leq h < 4/3$.
4. The optimal solution is (6 - 1.5h, 6 - 0.5h, 6 + 0.5h, 6 + 1.5h) when $0 < h < 2/3$.
5. The optimal solution is (6, 6, 6, 6) when h=0.

6. Conclusion

This paper revisits the deterministic capacitated production planning problem, where the production cost function is convex and no backlog is allowed. We first explore the relationship of production levels between two consecutive periods, characterizing the local structure of the optimal solution(s). We then show that there is a unique feasible solution satisfying the local optimal structure, thus establish the uniqueness of the global optimal solution. The uniqueness theorem not only significantly simplifies the proof of optimality of solution algorithms proposed in the literature, but also helps constructing alternative solution approaches. We also present one new solution algorithm, and apply the algorithm to solve problems with quadratic production functions.

Finally, Duan (2002) has extended the results in this paper to environments where production costs (or capacities) across planning periods are not identical, and discount rate is applied. Furthermore, similar results have been established when backorders (or lost sales) are allowed and the production cost functions are concave.

This paper and its extensions all consider problems with deterministic demands. It remains an open question whether the uniqueness theorem (or its likes) exists when demands are stochastic.

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