RC22787 (W0304-215) April 25, 2003
Mathematics

IBM Research Report

A New Approach to Data Storage Using Localized Structures

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A NEW APPROACH TO DATA STORAGE USING LOCALIZED STRUCTURES.

P. COULLET, C. RIERA, AND C. TRESSER

Abstract. In this paper we describe how to use the bifurcation structure of static localized solutions in one dimension to store information on a medium in such a way that no extrinsic grid is needed to locate the information. We demonstrate that these principles, deduced from the mathematics adapted to describe one dimensional media, also allow to store information on two-dimensional media.

In this paper we describe how to use the bifurcation structure of static localized solutions in one dimension, as they exist near the robust existence of stable fronts between homogeneous solutions and periodic patterns, to store information on a medium. Furthermore, information storage and retrieval can be done, we argue, in such a way that no extrinsic grid is needed to locate the information. We then demonstrate that these principles, deduced from the mathematics adapted to describe one dimensional media, also allow to store information on two-dimensional media. The mathematical theory of reversible system and elements of stability theory, allow to isolate a parameter range for some partial differential equations in one dimension, where the localized structures are decomposable in the sense that packets of bumps or holes can essentially be created at will (when and where needed). This parameter range is where information storage and retrieval would be most efficient. The practical value of these finding is enhanced by the fact that a similar phenomenology can be observed in two-dimensional simulations.

In a recent paper [1] (see also [2]), we provided mathematical foundation, in the context of one-dimensional media, to the intuitive idea that static localized structures can be interpreted as bits of structured solutions living on an homogeneous substrate. We also described how the domain of existence (in parameter space) of such static localized structures relates to the domain of existence of the kind of stable static fronts between homogeneous and structured solutions which had been described in 1986 by Pomeau [3]. The study of static stable localized structures has attracted a great deal of attention, at least since the time one expected to use magnetic bubbles as storage elements (see, *e.g.*, [4]). While some of the hopes one had put on that technology have not yet been fulfilled as far as fast access memory in concerned, garnets provide quite successful permanent storage elements based

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C.T. is partially supported by NSF.

on bubbles. As is well known, similar localized structures arise in liquid crystals [5] and in chemistry [6] [7] [8]. There has been a new surge of interest in the context of optics where such structures have again been envisioned as high performance storage elements: see [9] and [10] for more recent advance. It is indeed the possible application of such phenomena which motivates the present paper. It is usual to use the presence or absence of bubbles (meant here as a generic atom of localized structure) as bits of information: on a sample of the medium meant to carry the information, for instance a piece of garnet, bubbles are placed or not at predetermined positions to represent some string of ones and zeros. In the case of classical application of garnets, the information is then pushed to some reader by a field. One can also imagine moving around the reader and pick the bits of information where they have been frozen as parts of a solution to some field equation. Then, one needs to superimpose some pattern to have a reasonable way of knowing where the bits are read, and the extrinsic pattern has to be strongly and very precisely attached to the sample for the positions of the bits to be reliably recovered. We will show that an extension of the analysis provided in [1] allows to define a new strategy: instead of localizing bubbles on an homogeneous substrate, we propose to localize small pieces of homogeneous solution on a substrate carrying a periodic pattern of bubbles. The same periodic pattern can then be used to locate the bits, which thus allows to avoid the need of the extrinsic grid, and the often hard problem of registration (marking some origin using which positions can be defined and recognized at later uses).

The mathematical analysis which supports our approach is in some sense dual from the one we proposed in [1], so we next briefly remind the reader of what was done there. Recall first that, in the case when a stable homogeneous solution coexists with a stable periodic pattern, Pomeau described the mechanism of robust existence of stable stationary fronts between the two states [3]: there is a region in parameter space whose boundaries correspond to unpinning transitions, where such fronts exist. In [1], we reported on the fact that the region $\mathcal{P}$ is shadowed by a region of existence of stable localized structures in one dimension, as well as in two dimensions when the periodic pattern has compact elementary cells such as hexagons. In fact, the one dimensional case was justified following arguments from the theory of reversible dynamical systems, and we demonstrated that similar phenomenology existed in two dimension, with the existence of a parameter region like $\mathcal{P}$ similarly shadowed by the domain of existence of localized structures. As in [1], we consider here systems described either by variational or by non-variational equations such as:

$$\begin{aligned} \partial_t u &= -\tfrac{\partial V}{\partial u} - v + D_u \nabla^2 u\,, \\ \partial_t v &= -\gamma v + cu + D_v \nabla^2 v\,, \end{aligned} \tag{0.1}$$

or:

$$\partial_t u = -\frac{\partial V}{\partial u} - \nu \nabla^2 u - \nabla^4 u\,, \tag{0.2}$$

where $V = -\mu u^2/2 + u^4/4 - \eta u$.

Equation 0.1 describes a chemical reaction while equation 0.2 is a generalization of the Swift-Hohenberg model [11], which appears in a variety of contexts. We will also use the same overall strategy: a fully mathematically founded study in one dimension, and verification that the phenomenology so described and understood can also be observed numerically in the most important 2-dimensional case (higher dimension is both harder to observe,and less evidently useful for application to information fast storage and retrieval).

In one dimension, the second member of equations 0.1 or 0.2, which describes the stationary solutions, can be rewritten as a 4-dimensional vector field (we assume the dimension to be $2n \geq 4$ in the rest of the discussion) that inherits from the space isotropy the reversibility property [13] which is the essential ingredient in the analysis to follow.

An orbit of the vector fields needs not correspond to a stable solution of the PDE. For critical points and periodic solutions, stability for the PDE generically implies there is no purely imaginary eigenvalue nor purely imaginary Floquet multiplier ik_0, $k_0 \neq 0$, as otherwise, the corresponding zero growth rate at k_0 would generically induce positive and negative growth rates at nearby wavelengths (see also [2]). Natural coordinates for the phase space are (u, u_x, v, v_x) for equation 0.1 and $(u, u_x, u_{xx}, u_{xxx})$ for equation 0.2.

The only ODE's we will have to consider here are reversible differential equations as they are the one that appear as second member of PDE's of the form

"time derivative=ODE in space."

Thus the reversibility of the ODE follows from the $x \mapsto -x$ invariance. This is very different from the motivations of Devaney. Another definition of reversibility has been proposed and studied by Arnol'd and others (see [14]), but we will not use that point of view here. We recall some basic terms and facts about reversible systems as defined in [13]. The phase portrait of a reversible system in dimension $2n$ is invariant under the reflection symmetry R about a n-dimensional plane Π. The time reversal $(t \mapsto -t)$ being denoted by T, the system Φ is *reversible* if and only if $T\Phi(Rx) = \Phi(x)$ where Φ stands for for either the flow defined by a differential equation, or a map. In a reversible system, orbits invariant under R are called *reversible.* A critical point in Π is a *reversible critical point.* Any orbit that cuts twice the hyperplane Π is a *reversible periodic orbit.* The spectrum of eigenvalues at a *finite orbit*, periodic orbit or critical point, is symmetrical with respect to both the real and imaginary axis. Recall that a homoclinic is an orbit bi-asymptotic to some finite orbit; when the finite orbit is reversible and the homoclinic orbit cuts Π, we have a *reversible homoclinic orbit*: it was already shown in Devaney that reversible homoclinic orbits are robust, *i.e.,* are preserved up to small modifications under small enough parameters changes. Combining the previous stability argument with the reversibility property implies that all stable and unstable manifolds of reversible critical points and periodic orbits have dimension n.

In the above cases where the fields components are scalars, Π corresponds to the odd derivatives set equal to zero, so that all the critical points, which correspond to the stationary homogeneous solutions of the PDE are reversible. A (stationary) localized solution of the PDE corresponds to a homoclinic curve bi-asymptotic to a critical point of the vector field: we have already recalled that if the homoclinic curve is reversible, it is robust. Reversible periodic orbits arise in one parameter families [13], with the period of the orbit generically varying along the family. Because of the symmetry of the spectrum, and by the very fact that any periodic orbit carries necessarily a zero eigenvalue along the orbit, a reversible periodic orbit carries necessarily a second zero eigenvalue, indeed necessitated by the fact that they come in families. We consider only families that are as hyperbolic as possible, *i.e.,* with only two zero eigenvalues and $n-1$ eigenvalues on each side of the imaginary axis.

A heteroclinic orbit corresponding to a Pomeau front joins a reversible critical point to an as hyperbolic as possible reversible periodic orbit. It selects one the periods in a family of reversible periodic orbits when a parameter λ varies (for instance, η in the examples given above), so that the one parameter family of reversible periodic orbits generates the robustness of the stationary front under a generic transversality assumption. This can be captured on a Poincaré map on a $(2n-1)$-dimensional section S to the family of periodic orbits containing Π, as illustrated in figure (1) in the case when $n = 2$. In the $2n-1$-dimensional hyperplane S,

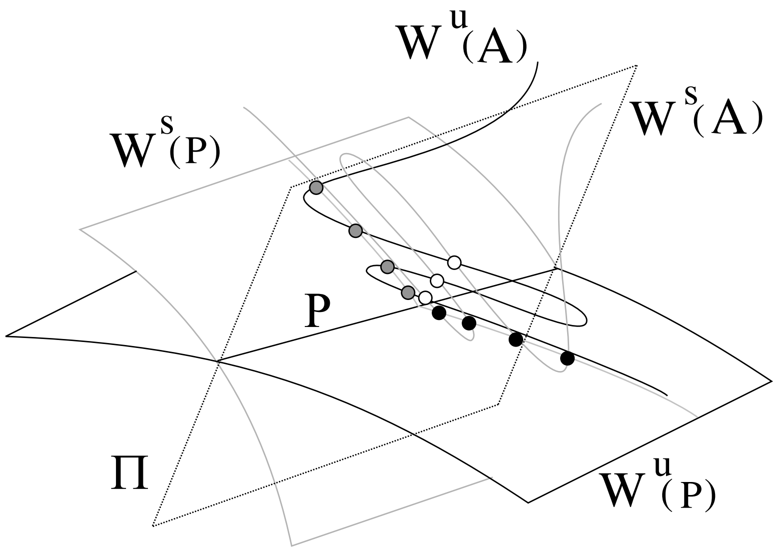
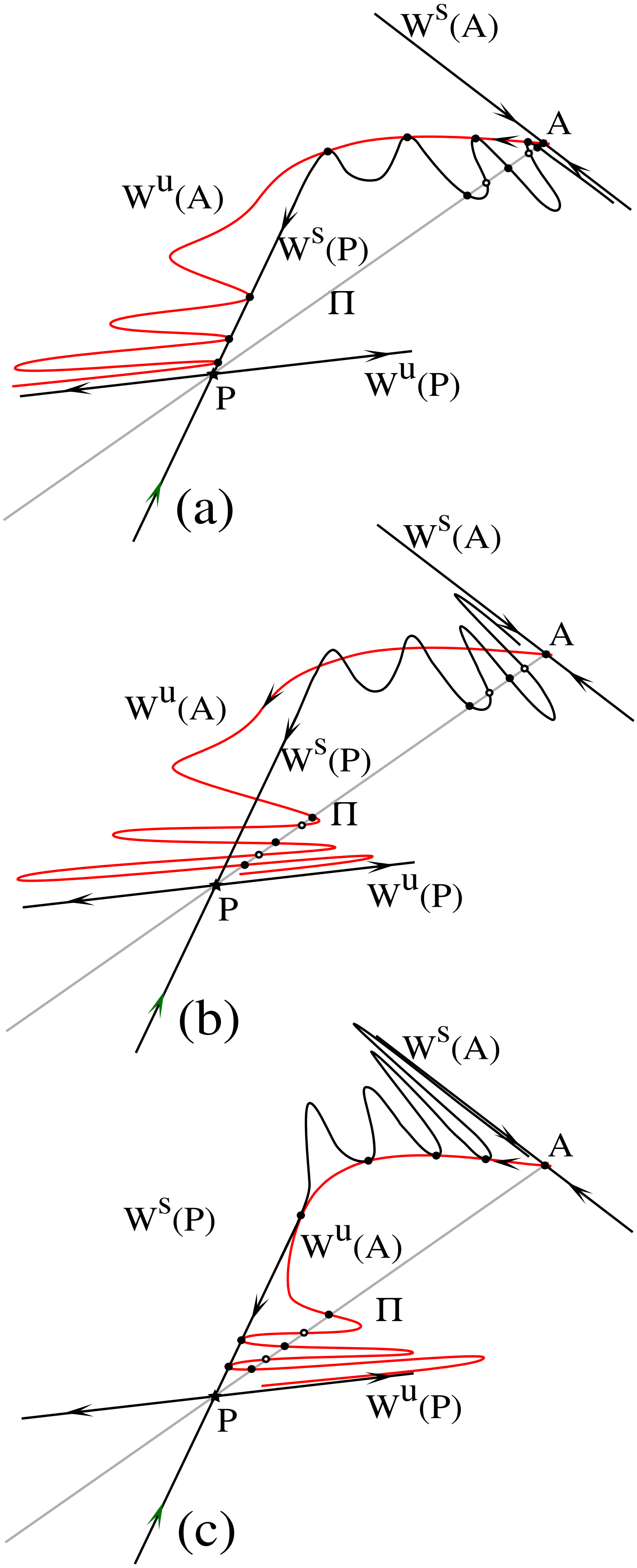


FIGURE 1. Phase portrait for a Poincaré section near a family of reversible fixed points when there exists a Pomeau front.

(1) The family of reversible periodic orbits appears as a line P of fixed points P_λ , so that the respective collections of the stable and unstable manifolds form two n-dimensional surfaces, $W^s(P)$ and $W^u(P)$, that are symmetrical with respect to Π.
(2) The invariant manifolds of the critical point A appear as $(n-1)$-dimensional surfaces $W^s(A)$ and $W^u(A)$, so that $W^u(A)$ intersects $W^s(P)$ transversally.

When λ varies, the selected fixed point P_λ changes, and the relative positions of $W^u(A)$ and $W^s(P)$, and of $W^u(A)$ and Π, change as illustrated in figure (2), where the boundaries of $\mathcal{P}$ correspond to Figures 2-a and 2-c. As illustrated in Figure



W^s(A)
A
W^u(A)
W^s(P)
Π
W^u(P)
P
(a)
W^s(A)
A
W^u(A)
W^s(P)
Π
W^u(P)
P
(b)
W^s(A)
A
W^s(P)
W^u(A)
Π
W^u(P)
P
(c)

2-b, between the boundaries of $\mathcal{P}$, there are transversal intersections of $W^u(A)$ with $W^s(P)$ (for early relationship between pinning and transversality, see, *e.g.*, [17]). This implies transversal intersections of $W^u(A)$ and Π, which by reversibility are also transversal intersections of $W^s(A)$ and Π, thus correspond to homoclinic orbits bi-asymptotic to A and hereby to localized solutions of the PDE. These solutions arise in pairs, being created and destroyed by saddle-node bifurcations; using the continuity of the spectrum and the stability of the structured solutions, this indicates that the localized solutions of the PDE arise in stable-unstable pairs. The transversality of the intersections of $W^s(P)$ with Π that can be seen in Figure 2-c implies that such homoclinic intersections subsist beyond one boundary of $\mathcal{P}$, until they successively disappear by saddle-node bifurcations.

The present paper addresses potential applications of localized structures to storage, on the basis of:

(1) the analyze in [1] that we have briefly summarized above,
(2) and another analysis, that we present below, and which can be thought of as dual to that presented in [1].

In brief, in [1] we showed that the heteroclinic orbit between the homogeneous and structured solutions (the Pomeau fronts) induces homoclinic orbits bi-asymptotic to the homogeneous solution which precisely are the localized bubble solutions. This analysis was made near the intersection points of a family of structured solutions with an appropriate hypersurface of section, using a sort of Poincaré map that does not take care of the direction of crossing, and contains the critical point A. By the duality we mentioned, a similar analysis can be done near A itself, and in the same way we can deduce the existence of homoclinic orbits bi-asymptotic to the structured (periodic) solutions.

Notice that there is another duality, of trivial nature, where fronts that separate an homogeneous solution on the left and a structured solution on the right are replaced by fronts that separate a structured solution on the left and an homogeneous solution on the right. While the former correspond, as illustrated in Figure 2 to an orbit of intersections of $W^u(A)$ with $W^s(P)$, the later would correspond to an orbit of intersections of $W^u(P)$ with $W^s(A)$, as one would get on a figure obtained from Figure 2 by exchanging the labels A and P.

The duality we invoke here is of deeper nature, but can be seen in the same Figure 2 we used before to recall the analysis in [1]: it corresponds to the fact the heteroclinic intersection also induces crossing of the invariant manifolds of A with Π, hence, again as a result of reversibility, reversible homoclinic orbits that are now bi-asymptotic to A, and correspond to localized flats on a periodic substrate. The duality exchanges the roles of creations and destructions of the Pomeau fronts, and one sees for instance from Figure 2-a that there are reversible homoclinic orbits to A before a boundary of $\mathcal{P}$, in the same way we have inferred that there were reversible homoclinic to P after the other boundary boundary of $\mathcal{P}$ from Figure 2-c: see also Figure 3 which assemble phenomena from both types of localized structures. The details of the phenomenology are however different for these two type of localized structures. In terms of dynamical systems, this corresponds to the difference between taking a section to a periodic orbit and taking a section that contains a critical point. Back to the PDE, the main point is that a periodic

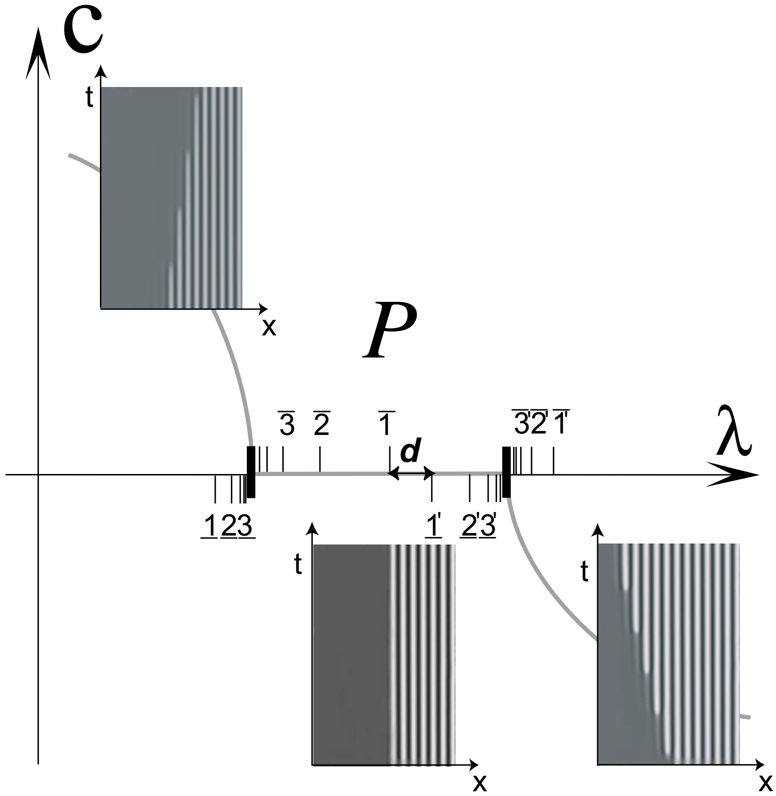


Figure 3. The bifurcation structure near the pinning region. The horizontal axis is the parameter λ. The vertical axis is the mean velocity of the front. Overbars correspond to bubbles, underbars to flats. d is the region where the structure is fully decomposable: this is where one should be for data storage applications.

orbit has a characteristic length associated to it: the period of the orbit, while the homogeneous solution has none. The consequence is that any number of consecutive bubbles can appear, with proper adjustment of the parameter. To the contrary, approaching and leaving the neighborhood of A also carries a characteristic length. This does not prevent the existence of arbitrary blocks to appear in the midst of a structured solution, but that is not enough for the application we seek: we want the homoclinic orbits bi-asymptotic to the structured solution to be obtainable by very local perturbation, which would not need the far away bubbles to be displaced, and in fact, *we want to be able to switch local flats on and off at will by very local perturbations*. This then needs to adjust further some other parameters so that not only the homoclinic orbits exit, but also, the characteristic length near A are adapted to the characteristic length of the structure, and in fact ideally equal to it.

To obtain better control of the positions of the bubbles that, by being switched on and off would carry information (hereby allowing easier read and write), as well as to improve the stability of the memory elements, we suggest to define blocks of the periodic structure and only use central sub-blocks of these blocks for carrying the bits of information. For instance, on one dimensional substrates, if one is greedy in density, one could use only the even numbered bubbles to be switched on or off. One could instead, for better stability and readability, define blocks of length seven , and mark the cental sub-blocks of three bubbles in those length-seven blocks: then the characteristic length near A would ideally be either equal to or triple of the length of the period of the structured solution. Clearly, a wide variety of choices can be made, some of which carry some obvious bit by bit error correction capabilities which could then be complemented with usual Error Correcting Codes (see for instance [20]).

One dimensional media can be wrapped around and packed on a two-dimensional support such as a disk, so that the dynamical system analysis presented here can be rather directly used in some implementations. It might be preferable, although yet mathematically unfounded, to use the strategy described here in genuine two dimensional media. We have illustrated that in Figure 4 using equation 0.2 and its two-dimensional analogue. Both in dimensions 1 and 2, we shown how to use super-blocks of minimal size to protect information in their centers, and possibly to help locate points on the medium: larger blocks could be used with the usual tradeoffs between stability and storage density.

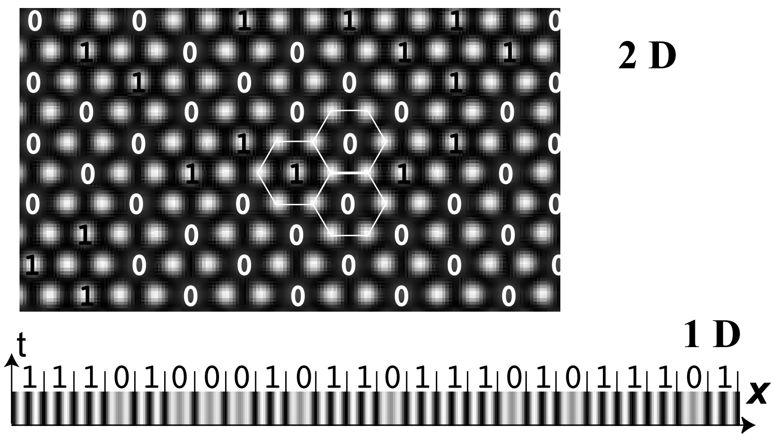


FIGURE 4. Examples of coding in one and two dimension of space.

Acknowledgments. This work is partially supported by the European Community grants EP 28235 - PIANOS and FMRX-CT96-0010, and by NSF Grant DMS-0073069. Useful discussions with P. Chaudhari, M. Martens, and J. Tredicce are gratefully acknowledged.

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