RC22790 (W0305-017) May 5, 2003
Mathematics

IBM Research Report

Erratum to Stable ergodicity and Julienne Quasi-conformality, J.Eur. Math.Soc. 2, 1-52

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Theorem B of [**PS**] is correct but its proof is not complete. The following result supplies the missing ingredient, which we tacitly assumed in the proof of Proposition 10.6. We recall the context.

G is a connected Lie group, $A : G \to G$ is an automorphism, B is a closed subgroup of G with $A(B) = B$, $g \in G$ is given, and the *affine diffeomorphism*

$$f : G/B \to G/B$$

is defined as $f(xB) = gA(x)B$. It is covered by the diffeomorphism

$$\bar{f} = L_g \circ A : G \to G,$$

where $L_g : G \to G$ is left multiplication by g,

$\bar{f}$ induces an automorphism of the Lie algebra $\mathfrak{g} = T_e G$, $\mathfrak{a}(\bar{f}) = ad(g) \circ T_e A$, where $ad(g)$ is the adjoint action of g, and $\mathfrak{g}$ splits into generalized eigenspaces,

$$\mathfrak{g} = \mathfrak{g}^u \oplus \mathfrak{g}^c \oplus \mathfrak{g}^s,$$

such that the eigenvalues of $\mathfrak{a}(\bar{f})$ are respectively outside, on, or inside the unit circle. The eigenspaces and the direct sums $\mathfrak{g}^{cu} = \mathfrak{g}^u \oplus \mathfrak{g}^c$, $\mathfrak{g}^{cs} = \mathfrak{g}^c \oplus \mathfrak{g}^s$ are Lie subalgebras and hence tangent to connected subgroups G^u, G^c, G^s, G^{cu}, G^{cs}.

Theorem 1. *Let $f : G/B \to G/B$ be an affine diffeomorphism as above such that G/B is compact and supports a smooth G-invariant volume. Let H be any of the groups $G^u, G^c, G^s, G^{cu}, G^{cs}$. Then the orbits of the H-action on G/B foliate G/B. Moreover, f exponentially expands the G^u-leaves, exponentially contracts the G^s-leaves, and affects the G^c-leaves subexponentially.*

Proof. Since $gA(H)g^{-1} = H$, and since H acts by left multiplication, it follows that the H orbit partition of G is $\bar{f}$-invariant:

$$\bar{f}(Hx) = gA(Hx) = gA(H)A(x) = gA(H)g^{-1}gA(x) = H\bar{f}(x).$$

Since the orbits are right cosets, they are leaves of a foliation of G. Likewise, the orbit partition of G/B is f-invariant. Its orbits are sets of the form HxB, but it is not a priori clear that they foliate G/B. We distinguish two cases.

Case 1: The automorphism A is the identity map, i.e. $f(xB) = gxB$. Then under the assumption that G/B supports a smooth G-invariant volume, it is proved

The second author was supported in part by NSF Grant #DMS-9988809.

The third author was supported by the Leading Scientific School Grant, No 457.2003.1.

in [**S**] that for two of the subgroups H, namely G^u and G^s, the H-orbits do foliate G/B.

The orbits of a Lie group action are non-overlapping smooth manifolds. Let E^u, ..., E^{cs} and J^u, ..., J^{cs} be the tangent bundles to the H-orbit partitions of G and G/B with respect to the H-actions, $H = G^u$, ..., G^{cs}. Continuity of the tangent bundle is equivalent to the orbit partition being a foliation. Thus $E^u, \dots, E^{cs}$, J^u, and J^s are continuous. The other three bundles are continuous except for dimension discontinuity. We claim that

$$J^u + J^{cs} = T(G/M) \qquad \text{and} \qquad J^u \cap J^{cs} = 0, \tag{0.1}$$

from which it follows that J^{cs} is continuous. The first assertion is clear from the facts that $TG = E^u \oplus E^{cs}$, $T\pi(TG) = T(G/B)$, $T\pi(E^u) = J^u$, and $T\pi(E^{cs}) = J^{cs}$, where $\pi : G \to G/B$ is the natural projection.

There is a sixth $T\bar{f}$-invariant subbundle of TG, the tangent bundle of the foliation of G by left B-cosets xB, which we call F. It is the kernel of $T\pi$. Since E^u and J^u are tangent to foliations, and π takes the leaves of the E^u-foliation to those of the J^u-foliation, the rank of the restriction of $T\pi$ to E^u is constant. Hence $F \cap E^u$ is continuous.

Choose an inner product on $T_eG = \mathfrak{g}$ so that $ad(g)$ expands $\mathfrak{g}^u$, contracts $\mathfrak{g}^s$, and is neutral on $\mathfrak{g}^c$. Extend the inner product to a right invariant Riemann metric on G, and let E_1^u be the orthogonal complement of $F \cap E^u$ in E^u. Fix any Riemann metric on G/B. From the compactness of G/B it follows that there exist $a, b > 0$ such that each vector $w \in J^u$ lifts to $v_1 \in E_1^u$, $T\pi(v_1) = w$, with $a||w|| \le ||v_1|| \le b||w||$. The derivative $T\bar{f}^n : TG \to TG$ exponentially stretches the component of v_1 in E_1^u for $n > 0$, so the same is true of w – it is exponentially stretched by positive iterates of Tf.

On the other hand, any $w \in J^{cs}$ lifts to a vector in E^{cs} which is not exponentially stretched by positive iterates of $T\bar{f}$, so the same is true of w – it is not exponentially stretched by positive iterates of Tf. Thus, $J^u \cap J^{cs} = 0$, which completes the proof of (0.1), and hence of continuity of J^{cs}.

Symmetrically, J^{cu} is continuous. Then, working inside J^{cu}, the same reasoning shows that continuity of J^u leads to continuity of J^c. The H-orbits foliate G/B.

<u>Case 2:</u> The automorphism A is not the identity. Here we use a standard trick similar to the suspension of a diffeomorphism. With no loss of generality we assume that G is simply connected (replacing if needed B by its inverse image in the universal cover). Then the automorphism group $\mathrm{Aut}(G)$ is algebraic. Let $\mathrm{Aff}(G)$ be the semidirect product of G and $\mathrm{Aut}(G)$. Thus $f = L_g \circ A \in \mathrm{Aff}(G)$. Since $\mathrm{Aut}(G)$ is algebraic, the Zariski closure of the cyclic subgroup $A^{\mathbb{Z}} \subset \mathrm{Aut}(G)$ is an abelian group with finitely many connected components. In particular, there exist a one-parameter subgroup $C \subset \mathrm{Aut}(G)$ and a nonzero $k \in \mathbb{Z}$ such that $A^k \in C$. Let G_1 be the semidirect product of G and C, and B_1 the semidirect product of B and $A^{k\mathbb{Z}}$. Then G_1/B_1 fibres over the circle $C/A^{k\mathbb{Z}}$ with fibres isomorphic to G/B, and hence has a smooth G_1-invariant volume [**R**]. Clearly, G_1 is a connected Lie group and $f^k = L_h \circ A^k \in G_1$ for some $h = h(g, A, k) \in G$. Apply Case 1 to the left translation of G_1 by f^k. The resulting stable and unstable leaves are contained in the G/B-fibres while the center leaves are transverse to the fibres. Thus the H-orbits foliate G/B. $\square$

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