RC22866 (W0308-012) August 5, 2003
Mathematics

IBM Research Report

Diffusion Approximation for Random Walk Reflected Away from Boundary

Yingdong Lu
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

Diffusion Approximations for Random Walks Reflected Away from Boundary

Yingdong Lu
IBM T.J. Watson Research Center
Yorktown Heights, NY 10598
Tel. (914) 945-3738, e-mail: yingdong@us.ibm.com

July 2003

Abstract

We investigate the asymptotic behavior of reflected random walks. Diffusion approximations will be our major technical methodology. We demonstrate that under mild conditions, the reflected random walks can be approximated by perturbed Brownian motions; the limit process will be further studied through applications in a queueing system.

1 Introduction

Diffusion approximations have been successfully applied in performance analysis and control of complex system. Whitt [6] provides a good survey on the methodology and applications on the subject. Most of studies are restricted on the cases of unit increment, classic queueing context, especially at the case of the reflecting. In Kushner [4], queues and reflection with batch processes were examined, diffusion approximations results, especially the reflecting angels in some special cases were obtained.

In this paper, we intend to establish some rigorous foundation for this type of problem. The problem we focus on will the the diffusion approximations of reflected random walks with general distributions. Like in the queueing case, the "reflecting" happens when the process in "forced" to stay nonnegative. However, in our case, the process will stay at the position of the previous step, instead of the boundary, as in some studies. This phenomenon of "reflecting in the interior" poses difficulties in expressing the reflecting process as functional of the cumulative process of the pure random walk, as did in the classic reflecting results. To overcome this barrier, we introduce a sequence of processes that can be expressed as the functional of cumulative process of the pure random walks or itself. We show that this sequence will eventually overside with the reflected random walk, The diffusion approximation, therefore, can be obtained with the aid of this sequence of processes. The limiting process turns out to be a perturbed reflecting Brownian motion, this is in line with the fact that we apply extra "force" to push the process back to the nonnegative domain in the case of reflected random walk.

This approximation can be easily applied to the cases of queueing system with batch arrivals and finite buffer, which in turn can be used to model a large class of applications, such as manufacturing systems as considered in Kushner [4]. The applications of this results can be found in Lu [5].

2 Diffusion approximation of reflected random walks

We will study the following "reflected" random walk,

$$S_n = \sum_{i=1}^{n} X_i \mathbf{1}\{S_{i-1} + X_i \geq 0\}, \qquad S_0 = 0. \tag{1}$$

where X_is are i.i.d random variables with density $F(dx)$, and denote μ as its mean and σ^2 the variance. Apparently, S_n will stay nonnegative.

S_n evolves like a regular random walk while in nonnegative half line, however, whenever the increment leads it into negative, it will abandon the step, i.e. stay unchanged. Apparently, $S_n \geq 0$ holds for all n. The difference between this process and conventional reflected random walk is that it is not reflected exactly at boundary, therefore, the so called "regulation" process is more than just the running maximum of the negative part.

In the following, we will introduce another representation of the dynamics of S_n. In this representation, we will introduce a sequence of stochastic processes, and S_n will be the limit of this sequence. This representation will enable us to introduce fluid and diffusion scaling much easier, hence, leads to the result of diffusion approximations.

The dynamic can be characterized equivalently by the following,

$$\begin{aligned}
Z_n^1 &= \sum_{i=1}^{n} X_i, \qquad \text{for}, n = 1, 2, \cdots, \\
\tau^1 &= \inf\{n : Z_n^1 < 0\} \\
Z_n^2 &= Z_n^1 + \mathbf{1}\left\{\sup_{k\leq i-1}\{-Z_k^1\} < \sup_{k\leq i}\{-Z_k^1\}\right\}\left\{\left[Z_{i-1}^1 - \sup_{k\leq i-1}\{-Z_k^1\}\right] + \left[\sup_{k\leq i}\{-Z_k^1\} - \sup_{k\leq i-1}\{-Z_k^1\}\right]\right\} \\
\tau^2 &= \inf\{n : Z_n^1 < 0\} \\
&\cdots \\
Z_n^\ell &= Z_n^{\ell-1} + \mathbf{1}\left\{\sup_{k\leq i-1}\{-Z_n^{\ell-1}\} < \sup_{k\leq i}\{-Z_k^{\ell-1}\}\right\} \\
&\quad \left\{\left[Z_{i-1}^{\ell-1} - \sup_{k\leq i-1}\{-Z_k^{\ell-1}\}\right] + \left[\sup_{k\leq i}\{-Z_k^{\ell-1}\} - \sup_{k\leq i-1}\{-Z_k^{\ell-1}\}\right]\right\} \\
\tau^\ell &= \inf\{n : Z_n^{\ell-1} < 0\} \\
&\cdots
\end{aligned}$$

Remark: The intuition behind this representation is that every time random walk is reflected, the regulation process contains two parts, one, represented by the running maximum of the negative part of the random walk, put the process back to level zero; another one, the difference between last step and level zero, put the process back to the position of the previous step.

We can conclude that

$$Z_n^\ell = S_n, \text{ for } \ell \text{ large enough},$$

More specifically, let $\eta_n := \inf\{\ell \geq 0, \tau^\ell \leq n\}$, then, $S_n = Z_n^{\eta_n}$. Therefore, we have,

$$\begin{aligned}
Z_n^{\eta_n} &= Z_n^{\eta_n-1} + \mathbf{1}\left\{\sup_{k\leq i-1}\{-Z_n^{\eta_n-1}\} < \sup_{k\leq i}\{-Z_k^{\eta_n-1}\}\right\} \\
&\quad \left\{\left[Z_{i-1}^{\eta_n-1} - \sup_{k\leq i-1}\{-Z_k^{\eta_n-1}\}\right] + \left[\sup_{k\leq i}\{-Z_k^{\eta_n-1}\} - \sup_{k\leq i-1}\{-Z_k^{\eta_n-1}\}\right]\right\}
\end{aligned}$$

$$Z_n^{\eta_n} = Z_n^1 + \sum_{\ell=2}^{\eta_n} \sum_{k=1}^{\tau_\ell - 1} \left\{ \sup_{k \le i-1} \{-Z_n^{\ell-1}\} < \sup_{k \le i} \{-Z_k^{\ell-1}\} \right\}$$
$$\left\{ \left[Z_{i-1}^{\ell-1} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\} \right] + \left[\sup_{k \le i} \{-Z_k^{\ell-1}\} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\} \right] \right\}$$

Another important piece of the puzzle is the asymptotic equivalency of the following two quantity,

$$\sup_{k \le i} \{-Z_k^{\ell-1}\} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\}$$

and

$$Z_{i-1}^{\ell-1} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\}$$

when $\ell \to \infty$. More precisely,

$$\sup_{k \le i} \{-Z_k^{\ell-1}\} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\} \to S_0, a.s.$$

$$Z_{i-1}^{\ell-1} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\} \to S_0, a.s.$$

where S_0 follow the distribution. The reason is that $\sup_{k \le i} \{-Z_k^{\ell-1}\} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\}$ can be treat as overshoot of the random walk for a level that goes to infinite, and $Z_{i-1}^{\ell-1} - \sup_{k \le i-1} \{-Z_k^{\ell-1}\}$ can be treated as undershoot. They will both approach S_1, see details in [1].

2.1 Diffusion approximations

The diffusion approximation will be discussed in two parts. First, using tightness argument, we will establish the weak convergence of the process under consideration. Then, equations developed in last section will help us to identify the limiting process.

Define $\hat{S}^n(t) = (1/\sqrt{n}) S_{\lfloor nt \rfloor}$.The goal is to obtain the limit of $\hat{S}^n(t)$ as $n \to \infty$. Adapting the argument in [4], we can show that the random sequence $Z^\epsilon(t)$ is tight. Then, from the dynamic we developed in the previous section, we can obtain the dynamic equation that the limit process will satisfy. More specifically, as we illustated in the last section, the "force" of extra push will have the same point wise limit as the running maximum of the negative part of the random walk. Then the following result Theorem 5.5 from Billingsley [2] can guarantee that they will not be discerned through the weak convengence of the probability measures.

Lemma 1 *h_n and h to be maps, and $h_n \to h, a.s.$, P_n and P are probability measures, $P_n \Rightarrow P$, then, $P_n h_n^{-1} \Rightarrow P h^{-1}$.*

In summary, we have,

Theorem 2 *The scaled process $\hat{S}^n(t)$ is tight, and its weak limit satisfies $Z(t)$ the following equation,*

$$Z(t) = W(t) + 2 \inf_{s \le t} [Z(s)] \tag{2}$$

Where $W(t)$ is a Brownian motion.

In Davis [3], it is shown that (2) uniquely identifies a diffusion process, and it follows the law of Bessel process with dimension 2.

References

[1] Asmussen, S. *Applied Probability and Queues*, 2nd Edition, Springer, 2000.

[2] Billingsley, P. *Convergence of Probability measures,* John Wiley & Sons, 1968.

[3] Davis, B. Weak Limits of Perturbed Random Walks and the Equation $Y_t = B_t + \alpha \sup\{Y_s \colon s \leq t\} + \beta \inf\{Y_s \colon s \leq t\}$. *Ann. Probab.* 24 (1996), no. 4, 2007–2023.

[4] Kushner,H. Control and Optimal Control of Assemble to Order Manufacturing Systems under Heavy Traffic. *Stochastics Stochastics Rep.* 66 (1999), no. 3-4, 233–272.

[5] Lu, Y. Performance Analysis and Control of Assemble-to-Order Systems with Batch Arrivals under Heavy Traffic, *Preprint*, IBM T.J. Watson Research Center.

[6] Whitt, W *Convergence of Stochastic Processes,* 2002