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An Algorithm for Finding Invariant Algebraic Curves of a Given Degree for Polynomial Planar Systems

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AN ALGORITHM FOR FINDING INVARIANT ALGEBRAIC CURVES OF A GIVEN DEGREE FOR POLYNOMIAL PLANAR SYSTEMS

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Abstract. Given a system of two autonomous ordinary differential equations whose right–hand sides are polynomials it is very hard to tell if any nonsingular trajectories of the system are contained in algebraic curves. We present an effective method of deciding, if a given system has an invariant algebraic curve of a given degree. The method also allows the construction of examples of polynomial systems with invariant algebraic curves of a given degree. We present the first known example of degree 6 algebraic saddle–loop for polynomial system of degree 2 which has been found using the described method. We also present some new examples of invariant algebraic curves of degrees 4 and 5 with an interesting geometry.

1. Preliminary definitions and introduction.

Since Darboux [7] has found in 1878 connections between algebraic geometry and the existence of first integrals of polynomial systems, algebraic invariant curves are a central object in the theory of integrability of polynomial systems in $\mathbb{R}^2$. Today, after more than a century of investigations the theory of invariant algebraic curves is still full of open questions. One of the reason for that is the fact that the examples of polynomial systems with invariant algebraic curves are extremely hard to find. Even with the help of computers it is far from obvious how to look for such examples. In the present paper we propose an approach, which turned out to be very effective for low–degree polynomial systems. Before we proceed with the introduction we give some definitions.

A *polynomial system* of a degree k in $\mathbb{R}^2$ is a system of two autonomous differential equations

$$\begin{aligned} \dot{x} &= p(x,y), \\ \dot{y} &= q(x,y), \end{aligned} \tag{1.1}$$

where p, q are coprime polynomials of a given degree k

$$p(x,y) = \sum_{i,j=0}^{k} p_{i,j} x^i y^j, \quad q(x,y) = \sum_{i,j=0}^{k} q_{i,j} x^i y^j \ .$$

We say that the algebraic curve is an *invariant algebraic curve of degree* n if it is contained in the union of trajectories of (1.1) and it is given by zeroes of a polynomial φ of a degree n.

$$\varphi(x,y) = \sum_{i,j=0}^{n} = \varphi_{i,j} x^i y^j.$$

From basic properties of polynomials follows the fundamental fact that the algebraic curve $\varphi(x,y) = 0$ is an invariant algebraic curve of system (1.1) if and only if there exists a polynomial $\kappa = \kappa(x,y)$ satisfying

$$p\frac{\partial \varphi}{\partial x} + q\frac{\partial \varphi}{\partial y} - \kappa\varphi = 0. \tag{1.2}$$

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The polynomial κ is called a *cofactor* of the curve $\varphi = 0$. Of course, the degree of the cofactor can be at most $k-1$, so

$$\kappa(x,y) = \sum_{i,j=0}^{k-1} k_{i,j} x^i y^j. \tag{1.3}$$

An invariant algebraic curve $\varphi = 0$ is called *irreducible* if the polynomial φ is irreducible. In the rest of the paper all the invariant algebraic curves are assumed to be irreducible unless stated otherwise.

A trajectory γ of system (1.1) is a *limit cycle* if it is nonconstant periodic and there are no other periodic trajectories in some neighborhood of γ. The orbit γ is an *algebraic limit cycle* of system (1.1) if it is a limit cycle of and if it is contained in some irreducible algebraic invariant curve $\varphi = 0$ of system (1.1). Algebraic saddle–loop is defined analogously.

A polynomial system which has enough invariant algebraic curves must be integrable. For example, the classical result of Darboux is

Theorem 1.1. *(Darboux) If a polynomial system of degree k has more than $\frac{k(k+1)}{2}$ irreducible invariant algebraic curves φ_i, then it has a first integral in the form of Darboux*

$$\prod_{i=1}^{\frac{k(k+1}{2}} \varphi_i^{\alpha_i},$$

where α_i are some constants. When the number of invariant algebraic curves is equal to $\frac{k(k+1)}{2}$, the system has an integrating factor in the form of Darboux.

Nevertheless, the above conditions are in general too strong. This motivates the following problem:

Problem 1. What are the connections between the possible degrees and number of invariant algebraic curves of a polynomial system of degree k and the existence and type of its first integral.

Understanding the significance of invariant algebraic curves Poincaré [13] has formulated slightly different question: Estimate the greatest possible degree $n = n(k)$ of an invariant algebraic curve for polynomial system of degree k. In this formulation the question has a trivial answer, the system

$$\begin{aligned} \dot{x} &= nx, \\ \dot{y} &= y \end{aligned} \tag{1.4}$$

has the invariant algebraic curve $x - y^n = 0$, therefore even $n(1)$ is unbounded. Nevertheless, the system (1.4) has a rational first integral xy^{-n}, so each of it's trajectories is contained in some algebraic curve. Therefore the Problem 1 is often referred to as the "Poincaré's problem". Another approach is to look for "nontrivial" examples of invariant algebraic curves of high degrees, like algebraic limit cycles or algebraic saddle–loops.

One of the main problems in the development of the theory of invariant algebraic curves is the fact, that there are not many examples known. Even for systems of degree 2 the structure of invariant algebraic curves turned out to be much more complex then it has been expected. For example it has been conjectured that

Conjecture (Lins–Neto)**.** *There exists a number $N(2)$ such that if a quadratic system has an invariant algebraic curve of a degree $n > N(2)$, then the system has a rational first integral.*

It has been proved to be false by Christopher and Llibre [5], who have found a class of quadratic systems which can have an invariant algebraic curve of any degree, and not have a rational first integral. Their example has a rational integrating factor. Later Chavarriga and Grau [3] have found a family of quadratic systems which can have an invariant algebraic curve of any degree and without a rational integrating factor. It has an integrating factor in the form of Darboux and it is still an open question if the following conjecture is true

Conjecture (Weakened Lins–Neto). *There exists a number $N(2)$ such that if a quadratic system has an invariant algebraic curve of a degree $n‘ > N(2)$, then the system has an integrating factor in the form of Darboux.*

The problem classification of algebraic limit cycles for quadratic system is also open, for almost 30 years there have been only 3 examples known, one of degree 2 [14] and two of degree 4, [15], [11]. It was also known ([8], [9], [10]) that there are no quadratic systems with algebraic limit cycles of degree 3. Then in the year 2000 two more families of quadratic systems with algebraic limit cycles of degree 4 have been found, see [2]. It has also been proved in [4] that there are no other families of quadratic systems with algebraic limit cycles of degree 4. The question if there exist quadratic systems with algebraic limit cycles of degree greater then 4 remained open till recently two new examples, one of degree 5 and one of degree 6 have been found [6]. Also in [6] is presented the first example of algebraic saddle–loop of degree 5. In subsection 4.3 we give the first example of algebraic saddle–loop of degree 6.

Another simple and interesting class of polynomial systems for which one may ask a question about existence of algebraic limit cycles are the Liénard systems $\dot{x} = y$, $\dot{y} = -F_k(x)y - G_m(x)$ (F_k, G_m– polynomials of degrees n, m respectively). In this case the question has been answered by Żołądek [16] for all the values of k, m except for $k = 1$, $m = 3$, for which the question still remains open.

These, and many more similar questions motivate the need for an efficient algorithm which would enable to find efficiently examples of families of polynomial systems with invariant algebraic curves. Till now most of the attempts were based on looking for the algebraic curves in some special form (usually hyperelliptic) for the sake of simplifying the calculations. However successful this simple approach was in many cases, it is far form being general and fails completely when one tries to look for the invariant curves of a high degree. This is the reason while there have been practically no known examples of invariant algebraic curves of degrees higher then 4. For quadratic systems even the invariant algebraic curves of degree 4 are not well investigated. As one of the examples of the application of the presented algorithm we give in Section 4 two examples of invariant algebraic curves of degree 4 with an interesting geometry, which to our knowledge has not been known before.

With the method described in the present paper we have been able to successfully investigate some families of quadratic systems with invariant algebraic curves of degrees as high as 14.

2. The problem of invariant algebraic curves from the point of view of linear algebra.

The method we present is based on the observation, that the problem of existence and finding a solution to the equation (1.2) is a purely linear problem. To be more precise, we look for a polynomial $\varphi(n)$ of a degree less or equal to n. Such polynomials form a linear space $\mathcal{V}_n$ of dimension $\frac{(n+1)(n+2)}{2}$. Given a polynomial system (1.1) of degree k and a polynomial $\kappa(x, y)$ of degree $k - 1$ we define an operator $\Xi : \mathcal{V}_n \to \mathcal{V}_{n+k-1}$ as

$$\Xi[\varphi] = p\frac{\partial\varphi}{\partial x} + q\frac{\partial\varphi}{\partial y} - \kappa\varphi.$$

Obviously Ξ is a linear operator. An obvious consequence of the definition is:

Proposition 2.1. *Polynomial system (1.1) has an invariant algebraic curve φ of degree less or equal to n with cofactor κ if and only if the operator Ξ has nontrivial kernel.*

To investigate the kernel of Ξ we shall use the language of matrices. We introduce the following basis $\mathcal{B}$ in $\mathcal{V}_n$:

$$x^i y^j = e_{\mu(i,j)},$$

where $\mu(i, j) = \frac{(i+j)(i+j+1)}{2} + i$. This comes from linear ordering of the homogenous monomials in the following way: $x^i y^j > x^k y^l$ if and only if $i + j > k + l$ or $i + j = k + l$ and $i > k$.

Remark 2.2. Note that the function μ is a bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, so it has an inverse function. Therefore it makes sense both to say $\mu = \mu(i, j)$ and $i = i(\mu)$, $j = j(\mu)$.

Every polynomial $\varphi \in \mathcal{V}_n$ has a unique representation as a vector in the basis $\mathcal{B}$–its coordinates are simply the coefficients of the polynomial φ. Now the operator Ξ is represented in the basis $\mathcal{B}$ by a $\frac{(n+k)(n+k+1)}{2} \times \frac{(n+1)(n+2)}{2}$ matrix $A = (a_{IJ})$. The terms a_{IJ} satisfy

$$a_{IJ} = i(J)p_{i(I)-i(J)+1,j(I)-j(J)} + j(J)q_{i(I)-i(J),j(I)-j(J)+1} - k_{i(I)-i(J),j(I)-j(J)}, \tag{2.1}$$

where $i(I)$, $j(I)$, $i(J)$, $j(J)$ are the unique numbers satisfying $\mu(i(I), j(I)) = I$, and $\mu(i(J), j(J)) = J$ (see Remark 2.2). We apply the convention that we put p_{ij}, q_{ij} k_{ij} equal to 0 if (i, j) is out of range of the definition, i.e. one of them is negative, or their sum is greater then the degree of the polynomial who's coefficient they are.

Matrix A has the following block–multi–diagonal form

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
B_n^{n+k-1} & & & & & & & & & & & & & \\ \hline
B_n^{n+k-2} & B_{n-1}^{n+k-2} & & & & & & & & & & & & \\ \hline
\vdots & \vdots & \ddots & & & & & & & & & & & \\ \hline
B_n^n & B_{n-1}^n & \cdots & B_{n-k+1}^n & & & & & & & & & & \\ \hline
B_n^{n-1} & B_{n-1}^{n-1} & \cdots & B_{n-k+1}^{n-1} & B_{n-k}^{n-1} & & & & & & & & & \\ \hline
 & B_{n-1}^{n-2} & \cdots & B_{n-2}^{n-2} & B_{n-3}^{n-2} & B_{n-4}^{n-2} & & & & & & & & \\ \hline
 & & \ddots & \vdots & \vdots & \vdots & \ddots & & & & & & & \\ \hline
\multicolumn{14}{|c|}{\vdots} \\ \hline
 & & & & & & \ddots & \vdots & \vdots & \vdots & \ddots & & & \\ \hline
 & & & & & & & B_{k+2}^{k+1} & B_{k+1}^{k+1} & B_k^{k+1} & \cdots & B_2^{k+1} & & \\ \hline
 & & & & & & & & B_{k+1}^k & B_k^k & \cdots & B_2^k & B_1^k & \\ \hline
 & & & & & & & & & B_k^{k-1} & \cdots & B_2^{k-1} & B_1^{k-1} & B_0^{k-1} \\ \hline
 & & & & & & & & & & \ddots & \vdots & \vdots & \vdots \\ \hline
 & & & & & & & & & & & B_2^1 & B_1^1 & B_0^1 \\ \hline
 & & & & & & & & & & & & B_1^0 & B_0^0 \\ \hline
\end{array}$$

where each of the blocks B_i^j is a $(i + 1) \times (j + 1)$ matrix.

Let M_0 denote the set of all the minors of maximum dimension (determinants of $\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2}$ submatrices) of the matrix A. M_0 is a set of polynomials in the variables p_{ij}, q_{ij} and k_{ij}. The number of polynomials in the set M_0 is equal to

$$\#M_0 = \begin{pmatrix} \frac{(n+k)(n+k+1)}{2} \\ \frac{(n+1)(n+2)}{2} \end{pmatrix}$$

and each of its elements depends in general on $\frac{(n+1)(3n+4)}{2}$ variables. From fundamental facts of linear algebra there follows

Theorem 2.3. *Polynomial system (1.1) has an invariant algebraic curve φ of degree less or equal to n with cofactor κ if and only if all the polynomials in M_0 vanish simultaneously.*

Theorem 2.3 suggests the following algorithm. If we want to find a polynomial system of a given degree k with an invariant algebraic curve of degree less or equal to n, we calculate the corresponding matrix A for the system 1.1, and the corresponding set M_0. Next we try to solve the equation $M_0 = 0$. (In the language of algebraic geometry this means that we look for a simple description of the algebraic set $V(M_0)$). The methods of solving of systems of polynomial equations are very well developed, there is a whole theory of Gröbner Basis and multi–variable resultants. Nevertheless, one can immediately see, that if we try to use this straightforward approach, we end up with an enormous number of equations in many variables.

Fortunately, when we look for the examples of polynomial systems with invariant algebraic curves, we usually consider certain families, depending only on a few parameters. Therefore the number of variables is usually not a problem.

The key to reducing the number of equations is a standard linear–algebra approach. First we note, that if there is a row i in the matrix A containing only a single nonzero constant term $a_{i,j}$, A, then each of the vectors in the kernel of A must have 0 at the j–th coordinate. Therefore we can remove the column j from the matrix A, limiting our considerations to a certain subspace of the space $\mathcal{V}_n$. Moreover, after the removal there can appear more rows with only one nonzero constant term in them, so sometimes the size of the matrix A can be reduced significantly in that way. We also can remove all the rows containing only zeroes. We obtain the *reduced matrix* B.

Once we have found the matrix B we apply the Gauss–Jordan elimination. When the polynomial system is expressed in a normal form, one may expect the matrix B to have a lot of terms which are constants–not dependent on the parameters of the system and the coefficients of the cofactor. Therefore we get the following algorithm:

3. The algorithm

Given a family of polynomial systems

$$\begin{aligned} \dot{x} &= \sum_{i,j=0}^{k} p_{i,j} x^i y^j \\ \dot{y} &= \sum_{i,j=0}^{k} q_{i,j} x^i y^j \end{aligned} \tag{3.1}$$

whose coefficients $p_{i,j}$, $q_{i,j}$ depend on some parameters $p_1, \ldots, p_s$ and an integer n we want to find those values of the parameters for which the system has an invariant algebraic curve of degree n.

The procedure

(1) We use changes of variables to transform simultaneously the system (3.1) and the potential cofactor $\kappa(x,y) = \sum_{i,j=0}^{k-1} k_{i,j} x^i y^j$ to the simplest form. Usually we strive to make as many as possible of the coefficients $p_{i,j}$, $q_{i,j}$, $k_{i,j}$ zeroes or constants and the other ones we treat as the parameters of the family. We shall call the family obtained in this way the *simplified family.*

(2) We find the matrix A for the simplified family.

(3) We generate a vector $\tilde{W} \in K[x,y]^{\frac{(n+1)(n+2)}{2}}$, whose i–th coordinate is a monomial $e_{\mu(i)}$, i.e. $\tilde{W} = (x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n, x^{n-1}, x^{n-2}y, \ldots, x, y, 1)$. We create an extended matrix $\tilde{A}$ obtained by adding the vector $\tilde{W}$ as the last row to the matrix A. This is done only for the reason to make the transformation of the obtained vector–solution into a corresponding polynomial more convenient.

(4) We perform the preliminary simplification of the extended matrix $\tilde{A}$: if there is any row i in the matrix $\tilde{A}$ containing only a single nonzero constant term $a_{i,j}$, we remove the j–th column from the matrix $\tilde{A}$. We keep repeating this process till there are no more rows with only one nonzero constant terms. Then from the obtained matrix we remove all the rows with only zeroes in them. We denote the *extended reduced matrix* matrix we have obtained by $\tilde{B}$.

(5) We denote the last row of the matrix $\tilde{B}$ by W. We remove it. The matrix we obtain is the reduced matrix B for the simplified family.

(6) We apply the process of Gauss–Jordan elimination to the matrix B using only nonzero constant terms. Namely, starting from the leftmost column we pick a nonzero constant term and use

row reduction to make all the other terms in that column equal to zero. Then we proceed to the next column. If there is a column with all the terms in it depending on the parameters, we skip it in the process. We denote the obtained matrix by C

(7) We apply the process described in the step 4 to the matrix C. In other words this means that we remove all the columns with precisely only constant term in them, and then we remove all the rows with only zeroes in them. The matrix we obtain we denote by D.

(8) We calculate the set M_1 of minors of maximum dimension of the matrix D. From the standard facts from linear algebra it follows that M_0 vanishes if and only if M_1 vanishes.

(9) We try to solve the system of equations $M_1 = 0$. We find a set of solutions $\{S_1, S_2, \dots, S_d\}$.

(10) For each S_i we substitute it to the matrix B, obtaining a matrix $B_i = B|_{S_i}$. Next, we solve the linear system of equations $B_i \cdot X = 0$. Of course each of the matrices B_i is a degenerate matrix, so for each i we have a nonempty set of l_i solutions $\{X_i^l\}_{l=1}^{l_i}$, $l_i \geq 1$. Note that in most of the cases B_i is a family of matrices–after the substitution of the solution S_i, B usually still depends on some parameters, and so does each of the corresponding vectors X_i^l. Therefore we shall refer to each of X_i^l as to a family of solutions, although in some cases it can be a constant family.

(11) For each l the family of polynomials $\varphi_i^l(x, y) = W \cdot X_i^l$ defines a family of invariant algebraic curves for the subfamily of the simplified family (3.1) defined by the conditions S_i. Note that S_i usually contains some equations that must be satisfied by the coefficients of the cofactor, as well as the coefficients of the system.

Remark 3.1. One may notice that steps 3–5 of our algorithm seem unnecessary. Indeed, one could apply Gauss–Jordan elimination immediately to the matrix A. Nevertheless, the form of the vector W and the simplified matrix B contain some information about the structure of invariant algebraic curve we are trying to find. This is particularly helpful, when we try to determine if the family of systems we are investigating is a good candidate. Sometimes it can suggest how to change the family. Another advantage is, that performing this preliminary reduction makes the elimination process run faster.

Remark 3.2. In most of the cases, the system of linear equations $B_i \cdot X = 0$ in step of our algorithm has only one solution X_i^1. In case $l_i > 1$ the polynomial system corresponding to S_i has a rational first integral. Indeed, invariant algebraic curves φ_i^1 and φ_i^2 have the same cofactor κ, so

$$\left(\frac{\varphi_i^1}{\varphi_i^2}\right)^{\cdot} = \frac{\kappa\varphi_i^1\varphi_i^2 - \varphi_i^1\kappa\varphi_i^2}{(\varphi_i^2)^2} = 0$$

4. Examples

4.1. **Degree 4 invariant algebraic curves for a certain family of quadratic systems.** We look for invariant algebraic curves of degree 4 within the family of quadratic systems

$$\begin{aligned}\dot{x} &= x + y + xy,\\ \dot{y} &= Kx + Ly + \alpha x^2 + \beta xy + 2y^2\end{aligned}$$

with cofactor $4y$. This family depends on 4 parameters $\{K, L, \alpha, \beta\}$. We perform steps 1–3 of our algorithm. The extended matrix $\tilde{A}$ for the system is

$$\left[\begin{array}{ccccccccccccccc}
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta & 2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2\beta & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 3\beta & 4\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 4\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & K & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 3+L & 2K & 0 & 0 & -1 & \beta & 2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 2+2L & 3K & 0 & 0 & 0 & 2\beta & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1+3L & 4K & 0 & 0 & 1 & 3\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4L & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & K & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 2+L & 2K & 0 & -2 & \beta & 2\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1+2L & 3K & 0 & -1 & 2\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & K & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1+L & 2K & -3 & \beta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2L & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & K & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & L & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
x^4 & x^3y & x^2y^2 & xy^3 & y^4 & x^3 & x^2y & xy^2 & y^3 & x^2 & xy & y^2 & x & y & 1
\end{array}\right]$$

The reduced matrix

$$B = \begin{bmatrix}
4 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -1 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & K & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 3 & 2+L & -2 & \beta & 2\alpha & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & -1 & 2\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & K & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 2 & 1+L & 2K & -3 & \beta & 0 \\
0 & 0 & 0 & 0 & 1 & 2L & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & K & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & L & -4
\end{bmatrix}$$

and the monomial vector

$$W = (x^4, x^3, x^2y, x^2, xy, y^2, x, y, 1).$$

We proceed to the step 7 of our algorithm and we get

$$D = \begin{bmatrix}
2\alpha - 2\beta + 3\alpha\beta - 3\beta^2 - 2L + 3\alpha L - 6\beta L - 2KL - L^2 & 2 - 2\alpha + 5\beta + 2K + L \\
3\alpha\beta - 3\beta^2 - \beta K + \alpha L - 3\beta L - KL & -\alpha + 3\beta + K \\
2\alpha - 2\beta + 3\alpha\beta - 3\beta^2 + 2K - 4L + 3\alpha L - 6\beta L - 3L^2 & 4 - 3\alpha + 6\beta + 3K + 3L
\end{bmatrix}$$

Now we are ready to calculate M_1. It consists of three terms, which after multiplication by a constant are equal to

$$\begin{aligned}
&(\alpha - \beta - K)\left(-2\alpha + 3\alpha\beta - 6\beta^2 - 2\beta K - \alpha L\right), \\
&(\alpha - \beta - K)\left(-4 + 2\alpha - 8\beta + 3\alpha\beta - 3\beta^2 - 4K - 8L + 3\alpha L - 12\beta L - 6KL - 3L^2\right), \\
&(\alpha - \beta - K)\left(-2\alpha - 6\beta + 6\alpha\beta - 9\beta^2 - 2K - 3\beta K - 9\beta L - 3KL\right).
\end{aligned}$$

The set of equations $M_1 = 0$ can be solved explicitly, and we have the following solutions

$$\begin{aligned}
S_1 &= \{\alpha = \beta + K\} \\
S_2 &= \{K = -1 \wedge L = -1 \wedge \alpha = -2 \wedge \beta = -1\} \\
S_3 &= \{K = -1 \wedge L = -1 \wedge \beta = \frac{1}{3}\} \\
S_4 &= \{\beta = -1 \wedge K = 2\alpha + 3 \wedge K \neq -1 \wedge L = -1\} \\
S_5 &= \{\left(3\beta + 2\sqrt{-2-3L} = 2 + 3L \vee 2 + 2\sqrt{-2-3L} + 3L = 3\beta\right) \wedge \\
& \quad L \neq -1 \wedge 2\alpha + \frac{(1+\beta)(1+K)}{1+L} = 5 + 3\beta + 3K + 3L\}
\end{aligned}$$

The kernel of $B_1 = B|_{S_1}$ is generated by the vector $X_1^1 = ((\beta+K)^2, 4K(\beta+K), -4(\beta+K), 2K(\beta+3K)-2(\beta+K)L, -8K, 4, 4K(K-L), -4K+4L, (K-L)^2)^T$. Therefore to S_1 corresponds an invariant algebraic curve $W \cdot X_1^1 = (L - \beta x^2 - K(1+x)^2 + 2y)^2$, which is reducible.

Similarly, to S_2 corresponds the reducible invariant algebraic curve $(x + x^2 + y)^2 = 0$

To S_3 corresponds the invariant curve $18x^2 + 4x^3 - 12\alpha x^3 - 3\alpha x^4 + 36xy + 12x^2y + 18y^2 = 0$, which for $\alpha < 2/3$ has a form of a cuspidal loop containing all three singular points of the system.

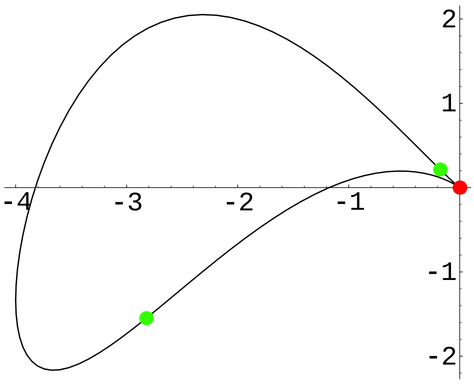

Figure 1: The curve corresponding to $K = -1$, $L = -1$, $\alpha = -\frac{4}{3}$, $\beta = \frac{1}{3}$.

To S_4 corresponds the invariant curve $\left(x^2 + 2y\right)(x(4+3x) + 2y) - Kx^2(2+x)^2 = 0$.

The solution S_5 corresponds in fact to several families of algebraic invariant curves. Here we present only one example, belonging to a 2–parameter family

$$\alpha = \frac{K\left(2 - \sqrt{-2-3L} + 3L\right) + (2+3L)\left(4 + \sqrt{-2-3L} + 3L\right)}{3(1+L)}, \qquad \beta = \frac{2}{3}\left(1 + \sqrt{-2-3L}\right) + L$$

with an invariant algebraic curve

$$\varphi_{0,0} + \varphi_{1,0}x + \varphi_{2,0}x^2 + \varphi_{3,0}x^3 + \varphi_{4,0}x^4 + \varphi_{0,1}y + \varphi_{1,1}xy + \varphi_{2,1}x^2y + \varphi_{0,2}y^2 = 0$$

$$\begin{aligned}
\varphi_{0,0} &= 27\,(K-L)\,(1+L)^3\\
\varphi_{1,0} &= 108K(1+L)^3\\
\varphi_{2,0} &= 18\,(1+L)\left(\left(4+\sqrt{-2-3L}+3L\right)\left(2+5L+3L^2\right)+K\left(8-\sqrt{-2-3L}+18L+9L^2\right)\right)\\
\varphi_{3,0} &= 4\,(2+3L)\left(6\left(4+\sqrt{-2-3L}\right)+17\left(4+\sqrt{-2-3L}\right)L+\left(60+9\sqrt{-2-3L}\right)L^2+\right.\\
&\qquad \left. 18L^3+K\left(10-2\sqrt{-2-3L}+21L+9L^2\right)\right)\\
\varphi_{4,0} &= (2+3L)\,K\left(10-2\sqrt{-2-3L}+21L+9L^2\right)+\\
&\qquad (2+3L)^2\left(14+8\sqrt{-2-3L}+3\left(7+2\sqrt{-2-3L}\right)L+9L^2\right)\\
\varphi_{0,1} &= -108(1+L)^3\\
\varphi_{1,1} &= -36\,(1+L)\left(6+\left(13+\sqrt{-2-3L}\right)L+6L^2\right)\\
\varphi_{2,1} &= -12\left(4+\sqrt{-2-3L}+3L\right)\left(2+5L+3L^2\right)\\
\varphi_{0,2} &= 18\left(1+\sqrt{-2-3L}\right)(1+L)
\end{aligned}$$

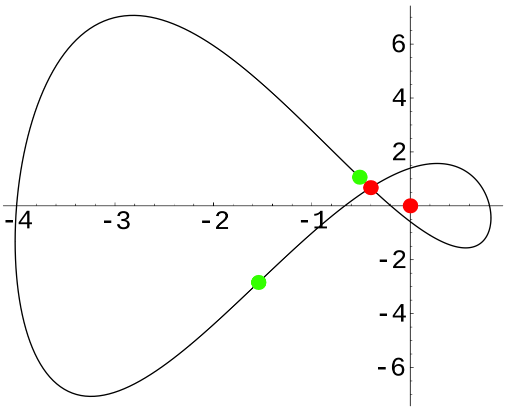

Figure 2: The curve corresponding to $K=-\frac{17}{3}$, $L=-\frac{8}{5}$, $\alpha=-\frac{1}{135}(1358+43\sqrt{70})$, $\beta=\frac{2}{15}(\sqrt{70}-7)$.

4.2. **Degree 5 invariant algebraic curves for quadratic systems.** We present two examples of degree 5 invariant algebraic curves for quadratic systems. They are isolated examples, not belonging to any families of quadratic systems with invariant algebraic curve of degree 5.

The system

$$\begin{aligned}
\dot{x} &= x+y+xy,\\
\dot{y} &= \tfrac{375(8836\sqrt{21}-1828897)}{722131963}x+\tfrac{46875(748\sqrt{21}-2331)}{2888527852}x^2+\tfrac{5(170\sqrt{21}-41951)}{219961}y+\tfrac{375(9\sqrt{21}+182)}{439922}xy+\tfrac{5}{4}y^2
\end{aligned}$$

has the invariant algebraic curve

$$\begin{aligned}
&-3.1973\cdot 10^{57}+2.06748\cdot 10^{60}x-3.7594\cdot 10^{62}x^2+1.32337\cdot 10^{64}x^3-1.46055\cdot 10^{64}x^4+2.21\cdot 10^{62}x^5+\\
&2.22619\cdot 10^{60}y-8.09555\cdot 10^{62}xy+4.27331\cdot 10^{64}x^2y-6.20874\cdot 10^{64}x^3y-4.36964\cdot 10^{62}y^2+\\
&4.63717\cdot 10^{64}xy^2-1.09432\cdot 10^{65}x^2y^2+1.69051\cdot 10^{64}y^3-9.20718\cdot 10^{64}xy^3-3.02394\cdot 10^{64}y^4=0
\end{aligned}$$

with cofactor $5y$. We present the coefficients in numerical form because the exact formula is over 2 pages long.

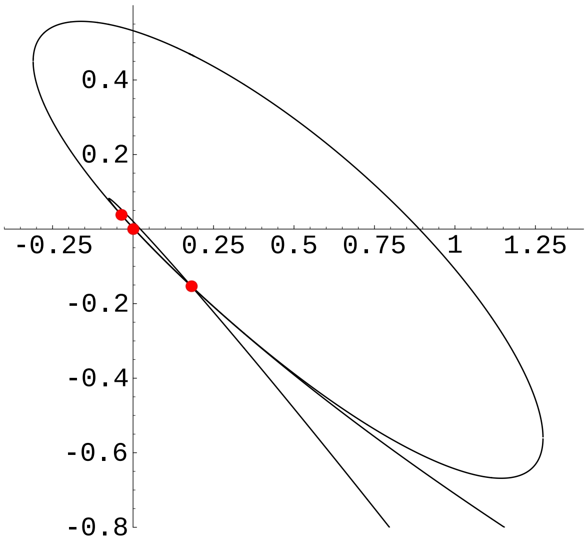

Figure 3: Invariant algebraic curve of degree 5.

The system

$$\dot{x} = x + y + xy,$$
$$\dot{y} = 189x + \frac{405}{4}x^2 - 11y - \frac{27}{2}xy + \frac{5}{4}y^2$$

has the invariant algebraic curve

$$25600000 + 120960000x + 224272800x^2 + 203163552x^3 + 89367381x^4 + 15116544x^5 - 640000y - 2030400xy - 2137104x^2y - 746496x^3y + 16800y^2 + 35136xy^2 + 18306x^2y^2 - 208y^3 - 216xy^3 + y^4 = 0$$

with cofactor $5y$.

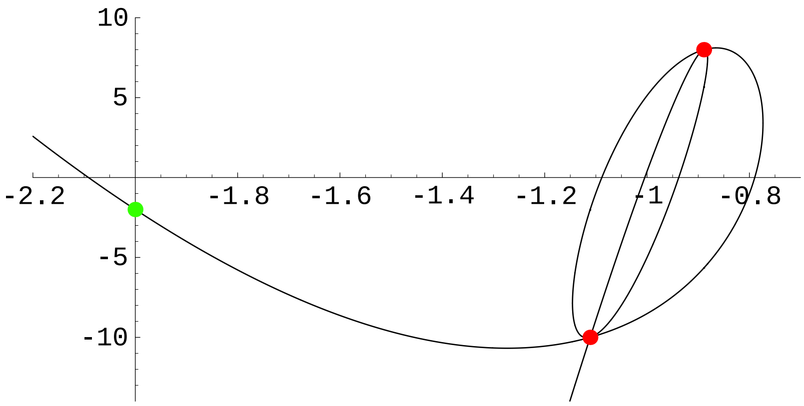

Figure 4: Another invariant algebraic curve of degre 5.

4.3. **Degree 6 invariant algebraic curve containing a saddle-loop for a certain family of quadratic systems.** Application of our algorithm to the family of systems

$$\begin{aligned}\dot{x} &= 1 + x + xy,\\ \dot{y} &= (K-\alpha) + Kx + Ly + \alpha x^2 + \beta xy + 2y^2\end{aligned}$$

with cofactor $6y$ and $n = 6$ leads to the discovery of degree 6 algebraic saddle–loop. As far as we know this is the first known example of an algebraic saddle–loop of degree greater than 5 for quadratic systems.

Theorem 4.1. *The system*

$$\begin{aligned}\dot{x} &= 1 + x + xy,\\ \dot{y} &= \tfrac{-22-47L-21L^2}{10} - \tfrac{34+87L+60L^2+9L^3}{10}x + Ly - \tfrac{(3+L)(2+3L)^2}{10}x^2 + \tfrac{(3+L)(2+3L)}{10}xy + 2y^2\end{aligned}$$

has an invariant algebraic curve defined by

$$\begin{aligned}&\varphi_{0,0} + \varphi_{1,0}x + \varphi_{2,0}x^2 + \varphi_{3,0}x^3 + \varphi_{4,0}x^4 + \varphi_{5,0}x^5 + \varphi_{6,0}x^6 + \varphi_{0,1}y + \varphi_{1,1}xy +\\ &\varphi_{2,1}x^2y + \varphi_{3,1}x^3y + \varphi_{4,1}x^4y + \varphi_{0,2}y^2 + \varphi_{1,2}xy^2 + \varphi_{2,2}x^2y^2 + \varphi_{0,3}y^3 = 0\end{aligned}$$

where

$$\begin{aligned}
\varphi_{0,0} &= -200\left(192 + 1104L + 2184L^2 + 1732L^3 + 463L^4\right)\\
\varphi_{1,0} &= -2400\left(88 + 474L + 937L^2 + 834L^3 + 325L^4 + 42L^5\right)\\
\varphi_{2,0} &= -60(2+3L)^2\left(1296 + 3160L + 2506L^2 + 701L^3 + 47L^4\right)\\
\varphi_{3,0} &= 20(2+3L)^3\left(-884 - 1496L - 615L^2 + 4L^3 + 19L^4\right)\\
\varphi_{4,0} &= 12(2+3L)^4\left(-138 - 109L + 33L^2 + 24L^3 + 2L^4\right)\\
\varphi_{5,0} &= 6(L-2)(3+L)^3(2+3L)^5\\
\varphi_{6,0} &= (L-2)(3+L)^3(2+3L)^6\\
\varphi_{0,1} &= -8000\left(12 + 29L + 33L^2 + 6L^3\right)\\
\varphi_{1,1} &= -1200\left(192 + 584L + 544L^2 + 152L^3 + 3L^4\right)\\
\varphi_{2,1} &= -600(2+3L)^2\left(28 + 20L - 11L^2 - 3L^3\right)\\
\varphi_{3,1} &= -120(L-2)^2\,(3+L)\,(2+3L)^3\\
\varphi_{4,1} &= 60(L-2)(3+L)^2(2+3L)^4\\
\varphi_{0,2} &= 120000L\,(1+L)\\
\varphi_{1,2} &= 24000\left(6 + 17L + 14L^2 + 3L^3\right)\\
\varphi_{2,2} &= -600\left(4 + 4L - 3L^2\right)^2\\
\varphi_{0,3} &= 80000
\end{aligned}$$

with cofactor $6y$. For $1 < L < 2$ this curve contains a saddle–loop.

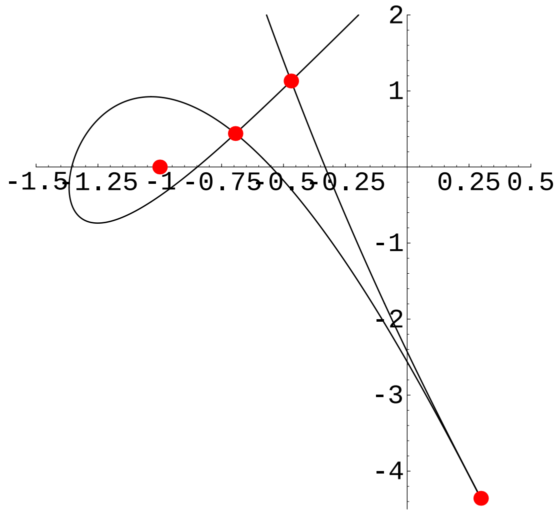

Figure 5: Degree 6 algebraic saddle loop for $L = \frac{11}{7}$

Remark 4.2. Most of the examples presented in the paper belong to a very special class of quadratic systems. There are certain conditions that must be satisfied for a quadratic system to have an invariant algebraic curve of a high degree. They have been studied in [12] and all the quadratic systems admitting high–degree limit cycles have been classified. In particular the family

$$\begin{aligned}\dot{x} &= x + y + xy,\\ \dot{y} &= Kx + Ly + \alpha x^2 + \beta xy + \gamma y^2\end{aligned}$$

with cofactor ny (denoted by S_n^n in [12]) is a very promising class of systems. Many other examples of quadratic systems with invariant algebraic curves have been found using the described algorithm, but they usually do not have such interesting geometry. Similar conditions to some of the ones presented in [12] has been found for polynomial systems (not necessarily quadratic) in the paper [1].

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