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ON A MATRIX INEQUALITY AND ITS APPLICATION TO THE SYNCHRONIZATION IN COUPLED CHAOTIC SYSTEMS

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Abstract. We study a matrix inequality problem which was found to be useful in deriving a sufficient condition for the synchronization in arrays of coupled chaotic systems. We consider classes of matrices for which this problem has an exact solution and solve the general case by solving sequentially a series of semidefinite programming problems.

Key words. convex programming, eigenvalue analysis, semidefinite programming, synchronization of chaos.

AMS subject classifications. 15A18, 15A45, 15A48, 34C28, 90C22

1. Introduction. Recently, synchronization in arrays of coupled chaotic systems has received considerable attention [3–5, 8]. In [7, 8], a sufficient condition for synchronization was obtained by means of a quadratic Lyapunov function. In [6] it was shown that this condition is reduced to a condition which depends on the smallest nonzero eigenvalues of the symmetric part of the coupling matrix. On the other hand, considering the case of coupled linear systems, one gets a condition which depends on the eigenvalues of the coupling matrix. Since the eigenvalues of a matrix can differ quite a bit from the eigenvalues of its symmetric part, the question is whether we can bridge the gap between these two sets of eigenvalues. We show that this question can be answered by solving an optimization problem with nonlinear semidefinite constraints. In this paper, we will solve this problem by solving a sequence of linear semidefinite programming problems.

We say a real matrix G is positive (semi-)definite if its symmetric part $\frac{1}{2}(G+G^T)$ is positive (semi-)definite, i.e. $x^T(G+G^T)x > 0$ (≥ 0) $\forall x \neq 0$. We denote this by $G \succ 0$ $(G \succeq 0)$. This is equivalent to saying that the eigenvalues of $G + G^T$ are positive (nonnegative).

2. Synchronization in coupled arrays of chaotic systems. DEFINITION 2.1. *Let W be the set of real matrices with zero row sums and all off-diagonal elements nonpositive. Let W_s be the set of irreducible symmetric matrices in W.*

DEFINITION 2.2. *A function $f(y,t)$ is P-uniformly decreasing if $(y-z)^T P(f(y,t)-f(z,t)) \leq -c\|y-z\|^2$ for some $c > 0$ and all y, z, t.*

It is easy to show that matrices in W_s have a simple eigenvalue 0 [8]. We begin with the synchronization result in [5, 8]:

THEOREM 2.3. *A coupled array of n identical chaotic systems described by the state equation:*

$$\dot{x} = (f(x_1,t), \dots f(x_n,t))^T + (G \otimes D)x \tag{2.1}$$

where $x = (x_1, \dots, x_n)^T$ synchronizes in the sense that $\|x_i - x_j\| \to 0$ as $t \to \infty$ if:

- *$f(y,t) + \alpha Dy$ is P-uniformly decreasing for some symmetric $P \succ 0$;*
- *there exists a matrix $U \in W_s$ such that $U(G - \alpha I) \otimes PD \preceq 0$.*

In Eq. (2.1), the matrix G describes the coupling topology between systems whereas the matrix D describes the coupling term between two systems. The term

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αDy is the amount of feedback needed to stabilize $\dot{y} = f(y,t)$. Theorem 2.3 motivates us to define the following quantity:

DEFINITION 2.4. *Let $\mu(G)$ be the supremum of the set of real numbers μ such that $U(G - \mu I) \succeq 0$ for some $U \in W_s$.*

Using this definition, it is easy to show that the array in Eq. (2.1) synchronizes if there exists a symmetric matrix $P \succ 0$ such that $f(y,t) + \mu(G)Dy$ is P-uniformly decreasing and $PD = D^T P \preceq 0$ [7].

This suggests that $\mu(G)$ is a measure of how well the topology of the coupled array is amenable to synchronization. The larger $\mu(G)$ is, the smaller D needs to be and the easier it is to synchronize the array.

Theorem 2.3 is obtained via Lyapunov's direct method and can be a global result. There exists another class of synchronization criteria based on the computation of Lyapunov exponents. These results are local in nature and are mathematically less rigorous. In these criteria, the least[1] nonzero eigenvalue of G is important. Under certain conditions, by Corollary 3.8 this eigenvalue is larger than the smallest nonzero eigenvalue of the symmetric part of G. Studying $\mu(G)$ allows us to find out what the gap is between the applicability of these two classes of methods[2].

3. Properties of $\mu(G)$. Since matrices in W_s are positive semidefinite, the set of real numbers such that $U(G - \mu I) \succeq 0$ for some $U \in W_s$ is an interval, i.e. if $U(G - \mu I) \succeq 0$ for some $U \in W_s$, then $U(G - \lambda I) \succeq 0$ for all $\lambda \leq \mu$.

LEMMA 3.1 ([8]). *If $A \in W_s$ and either $AD \succeq 0$ or $AD \preceq 0$, then D has constant row sums.*

Lemma 3.1 implies that $\mu(G)$ is only defined when the matrix G has constant row sums. Furthermore, since a matrix with constant row sums can be made into a matrix with zero row sums by adding a multiple of the identity matrix, for the purpose of finding $\mu(G)$ we can assume without loss of generality that G has zero row sums. In other words, adding αI to G shifts $\mu(G)$ by α. Therefore we will focus on the set of zero row sums matrix. For a matrix with zero row sums, 0 is an eigenvalue with eigenvector $e = (1, \ldots, 1)^T$.

The next theorem shows that the quantity $\mu(G)$ exists for zero row sum matrices and gives a lower bound.

THEOREM 3.2. *If G has zero row sums, then $\mu(G)$ exists, i.e. there is a real number μ and a matrix $U \in W_s$ such that $U(G - \mu I) \preceq 0$. Furthermore,*

$$\mu(G) \geq \beta(G) \geq \lambda_{\min}\left(\frac{1}{2}(G + G^T)\right)$$

where β is defined as $\beta(G) = \min_{x \perp e, \|x\|=1} x^T G x$.

Proof. Let J be the n by n matrix of all 1's and let $Q = I - \frac{1}{n}J$. It is clear that $Q \in W_s$. Let $U = Q$. Define the symmetric matrix $H = \frac{1}{2}(U(G - \mu I) + (U(G - \mu I))^T) = \frac{1}{2}(G + G^T) - \mu Q - \frac{1}{2n}(JG + G^T J)$. Since $Je = ne$ and $Ge = Qe = 0$, it follows that $He = 0$. Let $x \perp e$ with $\|x\| = 1$. This means that $Qx = x$. Then $x^T H x = \frac{1}{2}x^T(G + G^T)x - \mu - \frac{1}{2n}x^T(JG + G^T J)x$. Since $x \cdot e = 0$, this implies $Jx = 0$ and thus $x^T H x = \frac{1}{2}x^T(G + G^T)x - \mu$ which means that $H \succeq 0$ if $\mu \leq \frac{1}{2}x^T(G + G^T)x$. ☐

Note that $\beta(G) = \min_{\|Kx\|=1} x^T K^T G K x = \lambda_{\min}\left(\frac{1}{2}K^T(G + G^T)K\right)$ where K is an n by $n-1$ matrix whose columns form an orthonormal basis of $e^{\perp}$, the orthogonal

[1] In the sense of the smallest real part.

[2] See [7] for further discussion between these two classes of results.

complement of e. Furthermore, by the Courant-Fischer min-max theorem, $\beta(G)$ is less than or equal to the second smallest eigenvalue of $\frac{1}{2}(G+G^T)$.

DEFINITION 3.3. *For a matrix G with zero row sums, let $L(G)$ denote the eigenvalues of G that do not correspond to the eigenvector e.*

COROLLARY 3.4. *If G is a real matrix with zero row sums and zero column sums, then $\mu(G) \geq \lambda_2^s(G)$ where $\lambda_2^s(G)$ is the smallest eigenvalue in $L(\frac{1}{2}(G+G^T))$.*

Proof. Since e is an eigenvector of $\frac{1}{2}(G+G^T)$, we have $\lambda_2^s(G) = \beta(G)$. The result then follows from Theorem 3.2 □

COROLLARY 3.5. *If $G \in W$ and has zero column sums, then $\mu(G) \geq \beta(G) \geq 0$. If in addition $G+G^T$ is irreducible, then $\mu(G) \geq \beta(G) > 0$.*

Proof. For a symmetric matrix $G \in W$, $\lambda_2(G) \geq 0$. For a matrix $G \in W_s$, $\lambda_2(G) > 0$ [8]. The theorem then follows from the fact that $G+G^T \in W$. □

Next we show an upper bound for $\mu(G)$. Let $\mathcal{RE}(\lambda)$ denote the real part of a complex number λ.

DEFINITION 3.6. *For a real matrix G with zero row sums, define $\mu_2(G)$ as:*

$$\mu_2(G) = min_{\lambda \in L(G)} \mathcal{RE}(\lambda)$$

THEOREM 3.7. *If G is a real matrix with zero row sums, then $\mu(G) \leq \mu_2(G)$.*

Proof. This is a generalization of Theorem 3 in [7] and the proof is similar. Let $\lambda \in L(G)$ with corresponding eigenvector v. Let $U \in W_s$ be such that $U(G - \mu I) \succeq 0$ for some real number μ. The kernel of U is spanned by e. By definition of $L(G)$, v is not in the kernel of U. Since $(G-\mu I)v = (\lambda - \mu)v$, this implies that $v^*U(G-\mu I)v = (\lambda-\mu)v^*Uv$. Positive semidefiniteness of $U(G-\mu I)$ implies that $\mathcal{RE}(v^*U(G-\mu I)v) \geq 0$. Since U is symmetric positive semidefinite and v is not in the kernel of U, $v^*Uv > 0$. This implies that $\mathcal{RE}(\lambda) - \mu \geq 0$. □

The following result may be of independent interest.

COROLLARY 3.8. *If G is a real matrix with zero row sums, then*

$$\lambda_{\min}\left(\frac{1}{2}(G+G^T)\right) \leq \beta(G) \leq \mu_2(G).$$

Proof. Follows from Theorems 3.2 and 3.7. □

Theorems 3.2 and 3.7 show that $\beta(G) \leq \mu(G) \leq \mu_2(G)$. Next we present two classes of matrices for which there is a closed form expression for $\mu(G)$. In particular, for normal real matrices, the lower bound is equal to the upper bound:

THEOREM 3.9. *If G is a real normal matrix with zero row sums, then $\beta(G) = \mu(G) = \mu_2(G)$.*

Proof. First note that by normality G has zero column sums (see for example [8]). Furthermore, for a real normal matrix, the eigenvalues of $\frac{1}{2}(G+G^T)$ are just the real parts of the eigenvalues of G [2]. This implies that $\mu_2(G) = \lambda_2^s(G)$. The result then follows from Corollary 3.4 and Theorem 3.7. □

Another class of matrices for which $\mu(G) = \mu_2(G)$ is the class of triangular zero row sums matrices [7]. In the next section, we show that for matrices in W the value of $\mu(G)$ is in fact very close to $\mu_2(G)$.

4. Computing $\mu(G)$ via semidefinite programming. In this section we show how $\mu(G)$ can be computed by solving a sequence of semidefinite programming (SDP) problems.

First we note that by Theorems 3.2 and 3.7, $\mu(G)$ can be bounded in the interval $[\beta, \mu_2]$. Next we show that for a fixed μ, finding $U \in W_s$ such that $U(G - \mu I) \succeq 0$ is a feasibility SDP problem.

Clearly $U(G - \mu I) \succeq 0$ is a linear matrix inequality. A matrix U is in W_s if and only if:

1. U is symmetric;
2. all off-diagonal elements of U are nonpositive;
3. each row of U sums to zero;
4. zero eigenvalue of U has multiplicity 1, i.e. $0 \notin L(G)$.

The first three requirements can clearly be cast as matrix constraints for a SDP problem. As for the 4th requirement, it is easy to show that it is equivalent to the linear matrix inequality $K^T U K \succ 0$ where K is as defined in Section 3. To ensure that we do not get a very small U, we use the constraint $K^T U K \succeq I$ instead. It is clear that this does not affect the value of $\mu(G)$. Thus the feasiblity SDP problem we need to solve is:

$$\text{Find } U = U^T \text{ such that } U(G - \mu I) \succeq 0,\ Ue = 0,\ U_{i,j} \leq 0 \quad \forall i \neq j \text{ and } K^T U K \succeq I. \tag{4.1}$$

Since the set of values of μ such that $U(G - \mu I) \succeq 0$ for some $U \in W_s$ is an interval, we can compute $\mu(G)$ by using the bisection method to successively refine μ and then solving the corresponding SDP problem in Eq. (4.1). This is shown in Algorithm 1 where ub is initially set to $\mu_2(G)$ and lb is initially set to $\beta(G)$.

Algorithm 1 Compute $\mu(G)$

```
μ ← ub
if Problem (4.1) is infeasible then
  while |ub − lb| > ε do
    μ ← ½(ub + lb)
    if Problem (4.1) is infeasible then
      ub ← μ
    else
      lb ← μ
    end if
  end while
end if
μ(G) ← μ
```

In recent years, there have been many free and commercial programs available for solving SDP problems. The reader is referred to `http://www-user.tu-chemnitz.de/~helmberg/sdp_software.html` for a list. We have elected to use CSDP 4.7 [1] with the YALMIP 3 MATLAB interface (`http://control.ee.ethz.ch/~joloef/yalmip.msql`) to solve the SDP problem.

4.1. Zero row sums matrices. Our computer results are summarized in Table 4.1. Zero sums matrices of small order are generated and their values of $\mu(G)$ are computed. For each order n, 5000 zero row sums matrices are chosen by generating the off-diagonal elements independently from a uniform distribution in the interval $[-\frac{1}{2}, \frac{1}{2}]$. The matrices are categorized into two groups depending on whether all their eigenvalues are real or not. For each group, the mean and the standard deviation of the quantities $0 \leq i(G) = \frac{\mu(G)-\beta(G)}{\mu_2(G)-\beta(G)} \leq 1$ and $r(G) = \frac{\mu(G)}{\mu_2(G)}$ are listed.

	Only real eigenvalues				Real and complex eigenvalues			
	$i(G)$		$r(G)$		$i(G)$		$r(G)$	
Order	mean	std	mean	std	mean	std	mean	std
3	0.9862	0.0621	1.0166	1.2065	0.8672	0.1724	2.1340	34.0588
4	0.9766	0.0777	1.0227	0.9181	0.5251	0.4002	0.2362	56.8556
5	0.9715	0.0809	1.0113	0.0589	0.3191	0.3353	3.0385	133.0064
10	0.9731	0.0894	1.0060	0.0214	0.1451	0.1095	-2.8780	424.5138

TABLE 4.1
Statistics of $i(G) = \frac{\mu(G)-\beta(G)}{\mu_2(G)-\beta(G)}$ *and* $r(G) = \frac{\mu(G)}{\mu_2(G)}$ *for zero row sum matrices.*

We see quite a difference between the behavior of $i(G)$ and $r(G)$ for matrices with only real eigenvalues and for matrices with complex eigenvalues. In particular, we see that $i(G)$ is close to 1 for matrices with only real eigenvalues which implies that $\mu(G)$ is close to $\mu_2(G)$ in this case. On the other hand, for matrices with complex eigenvalues, the statistics of $i(G)$ show that $\mu(G)$ is usually significantly less than $\mu_2(G)$.

4.2. Matrices in W. Our computer results for matrices in W are summarized in Table 4.2. For each order n, 5000 zero row sums matrices are chosen by generating the off-diagonal elements independently from a uniform distribution in the interval $[-1, 0]$. The matrices are categorized into two groups depending on whether all their eigenvalues are real or not. For each group, the mean and the standard deviation of the quantities $i(G)$ and $r(G)$ are listed.

	Only real eigenvalues				Real and complex eigenvalues			
	$i(G)$		$r(G)$		$i(G)$		$r(G)$	
Order	mean	std	mean	std	mean	std	mean	std
3	0.9862	0.0612	0.9984	0.0084	0.8663	0.1698	0.9777	0.0337
4	0.9816	0.0658	0.9977	0.0095	0.9196	0.1389	0.9875	0.0246
5	0.9758	0.0770	0.9971	0.0112	0.9307	0.1292	0.9901	0.0211
10	0.9818	0.0260	0.9987	0.0020	0.9333	0.1097	0.9936	0.0123

TABLE 4.2
Statistics of $i(G) = \frac{\mu(G)-\beta(G)}{\mu_2(G)-\beta(G)}$ *and* $r(G) = \frac{\mu(G)}{\mu_2(G)}$ *for matrices in* W.

In contrast to general zero row sums matrices, the behaviors of $\mu(G)$ for matrices in W with only real eigenvalues and for matrices in W with complex eigenvalues are similar. Furthermore, $\mu(G)$ is very close to $\mu_2(G)$, especially for matrices with only real eigenvalues. It remains to be seen whether the small discrepancy between $\mu(G)$ and $\mu_2(G)$ is real or an artifact of the numerical algorithm.

5. Conclusions. We have studied a quantity $\mu(G)$ of a matrix G which characterizes the coupling topology in arrays of coupled chaotic systems. This quantity is useful in determining a synchronization criterion for an array of coupled chaotic systems. We derive upper and lower bounds for $\mu(G)$, give closed form expressions of $\mu(G)$ for some subclasses of matrices and present an algorithm for determining $\mu(G)$

using semidefinite programming. The computer results suggest that $\mu(G)$ is close to the upper bound $\mu_2(G)$ when

1. $G \in W$ or
2. G is a zero row sums matrix with only real eigenvalues.

An interesting question for further investigation is what are the (non-normal) matrices for which $\beta = \mu_2$?

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