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On the Relationship between Pinning Control Effectiveness and Graph Topology in Complex Networks of Dynamical Systems

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Abstract

This paper concerns pinning control in complex networks of dynamical systems, where an external forcing signal is applied to the network in order to align the state of all the systems to the forcing signal. By considering the control signal as the state of a virtual dynamical system, this problem can be studied as a synchronization problem. The main focus of this paper is to study how the effectiveness of pinning control depends on the underlying graph. In particular, we look at the relationship between pinning control effectiveness and the complex network asymptotically as the number of vertices in the network increases. We show that for vertex balanced graphs, if the number of systems receiving pinning control does not grow as fast as the total number of systems, then the strength of the control needed to effect pinning control will be unbounded as the number of vertices grows. Furthermore, in order to achieve pinning control in systems coupled via locally connected graphs, as the number of systems grows, both the pinning control and the coupling among *all* systems need to increase. Finally, we give evidence to show that applying pinning control to minimize the distances between all systems to the pinned systems can lead to a more effective pinning control.

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The object of study here is a set of coupled dynamical systems where the underlying coupling topology is a complex network. In particular, we look at pinning control of this network, where a subset of systems are forced in order to exert influence on all systems. Intuitively, pinning control on a system will have less influence on systems which are far away from the control. We show that this is indeed the case and show how in a large locally connected network it is hard to achieve control of the entire network by applying control to a few systems. We also look at locations where pinning control should be applied and how it relates to the network topology.

I. INTRODUCTION

Recently, there has been much activity in studying synchronization in complex networks of nonlinear dynamical systems [1–3] where criteria are derived that ensure all dynamical systems are synchronized to the same behavior. An interesting question in this study is how these criteria are related to the topology of the network. A related research area is the problem of pinning control in such complex networks [4–6], where a subset of dynamical system is forced to bring the entire network to follow a specific trajectory. In particular, it was shown that by forcing the behavior of a few systems pinning control can be achieved. In [6] it was shown that such control is possible if the underlying topology contains a spanning directed tree. In [7] it was studied how the topology of the network influences the effectiveness of pinning control. In this paper we continue this investigation and present new results.

II. COMPLEX NETWORK OF DYNAMICAL SYSTEMS

We consider a network of n coupled dynamical systems whose state equations are written in the following form:

$$\frac{dx_i}{dt} = f(x_i, t) - \alpha \sum_j G_{ij} D(t) x_j \tag{1}$$

where x_i is the state vector of the i-th system and $\alpha > 0$ is a scalar coupling coefficient. The total number of systems is denoted by n (i.e. $1 \leq i \leq n$). The matrix $D(t)$ describes the linear coupling between two systems.

We assume that the matrix G is a matrix with nonpositive off-diagonal elements and has zero row sums, i.e, G is the Laplacian matrix of its directed graph. We say the system in Eq. (1) synchronizes (globally) if $\|x_i - x_j\| \to 0$ as $t \to \infty$. Conditions for global and local synchronization have been obtained using a variety of techniques [8–12]. In many cases, the synchronization conditions depend on the nonzero eigenvalues of αG.

III. PINNING CONTROL IN NETWORKS OF DYNAMICAL SYSTEMS

In pinning control, forcing terms are applied to a subset of systems in Eq. (1) in order to drive the entire network to follow a prescribed trajectory. In particular, we consider linear pinning control of the form:

$$\frac{dx_i}{dt} = f(x_i, t) - \alpha \left(\sum_j G_{ij} D(t) x_j + c_i D(t)(x_i - u(t)) \right) \tag{2}$$

where $u(t)$ is the desired target trajectory and $c_i > 0$ if control is applied to the i-th system and $c_i = 0$ otherwise. We define P as the set of systems where pinning control is applied, i.e., $i \in P \Leftrightarrow c_i > 0$. We call P the set of *pinned* systems.

Assume that $u(t)$ is a trajectory of the individual dynamical system in the network, i.e.

$$\frac{du(t)}{dt} = f(u(t), t) \tag{3}$$

Then Eq. (3) is a virtual system and by setting $x_{n+1}(t) = u(t)$, we obtain a network of $n+1$ dynamical systems with state equations

$$\frac{dx_i}{dt} = f(x_i, t) - \alpha \sum_j \tilde{G}_{ij} D(t) x_j \tag{4}$$

where $\tilde{G}$ is related to G as

$$\tilde{G} = \begin{pmatrix} G_{11} + c_1 & G_{12} & & \dots & G_{1n} & -c_1 \\ G_{21} & G_{22} + c2 & G_{23} & \dots & G_{2n} & -c_2 \\ \vdots & & & & & \vdots \\ & & & \dots & G_{nn} + c_n & -c_n \\ 0 & & & & & 0 \end{pmatrix}$$

Thus we have reduced the pinning control problem to a synchronization problem. Pinning control is achieved in Eq. (2), i.e. every system's state vector x_i follows the trajectory $u(t)$ if the extended system in Eq. (4) synchronizes. We next look at how properties of G is useful to derive a criterion for achieving pinning control in Eq. (2).

IV. PINNING CONTROL AND GRAPH TOPOLOGY

Definition 1 *Let B be an irreducible square matrix B with nonpositive off-diagonal elements. The quantities $\beta(B)$ and $\gamma(B)$ are defined as follows. Decompose B uniquely as $B = L + U$, where L is a zero row sum matrix and U is a diagonal matrix. Let w be the unique positive vector such that $w^T L = 0$ and $\max_v w_v = 1$. The vector w exists by Perron-Frobenius theory [13]. Let $W = diag(w)$. Then $\gamma(B) = \min_{x \neq 0, x \perp \mathbf{1}} \frac{x^T W B x}{x^T \left(W - \frac{ww^T}{\sum_v w_v}\right) x}$ and $\beta(B) = \min_{x \neq 0} \frac{x^T W B x}{x^T W x}$. We define $\gamma(0) = +\infty$.*

The matrix $\tilde{G}$ can be written as

$$\begin{pmatrix} G + C & -c \\ 0 & 0 \end{pmatrix}$$

where C is a diagonal matrix with c_i on the diagonal and c is the vector of c_i's. Consider $\tilde{G}$ written in Frobenius normal form [14], i.e.

$$\tilde{G} = A \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1q} \\ & B_2 & \cdots & B_{2q} \\ & & \ddots & \vdots \\ & & & B_q \end{pmatrix} A^T \tag{5}$$

where A is a permutation matrix and B_i are square irreducible matrices. The Frobenius normal form is not unique, but we pick A such that $B_q = 0$ is a scalar corresponding to the virtual system.

In [7] it was shown that an important relationship that determines pinning control is

$$\alpha \beta_{\min} \geq 1 \tag{6}$$

where $\beta_{\min} \stackrel{\text{def}}{=} \min_{i<q} \beta(B_i)$.

In the rest of this paper, we will use $\alpha\beta_{\min}$ to denote the *effectiveness* of the pinning control.

Consider the two types of parameters in the coupled network in Eq. (2): α and c_i. The parameter α describes the strength of the coupling between *all* the systems, whereas the parameters c_i (in conjunction with α) describe the strength of the pinning control. This is schematically shown in Fig. 1.

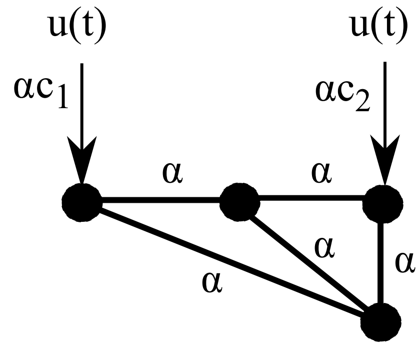

FIG. 1: The parameters α and c_i in Eq. (2). α describes the coupling strength between all systems. c_i describes the strength of the pinning control.

Thus the network is harder to achieve pinning control if a larger c_i or α are needed. Next we study how the topology of the network affects the requirements for these 2 types of parameters. If $\beta_{\min}$ is small, then according to Eq. 6 a large α is needed to achieve pinning control.

First let us assume that the graph of G is undirected, i.e. G is symmetric. This means that $\beta_{\min} = \beta(G + C) = \lambda_1(G + C)$. In [7] it was shown that for the same parameters c_i, $\beta_{\min}$ is large if $\lambda_2(G)$ (also known as the algebraic connectivity of the graph of G) is large.

Let m be the number of systems that have control signals applied to. It is the cardinality of the set P, i.e., the number of coefficients c_i that are nonzero. In [7] the case of only a single system being applied pinning control (i.e. $m = 1$) is studied.

In [15] it was shown that $\beta_{\min} \leq \frac{1}{n}\sum_i c_i$. This shows that:

If c_i, α are bounded and m grows slower than n, then pinning control is not achievable as $n \to \infty$.

This is illustrated in Fig. 2 where we have computed $\beta_{\min}$ for fully connected graphs where $c_i = 1$, $\alpha = 1$ and $m = \lceil\sqrt{n}\rceil$.

This means that if the number of systems where pinning control is applied is small compared with the total number of systems, then the applied control (expressed as αc_i) need to be large. However, this is not sufficient if the network is locally connected. In particular, we show that for locally connected networks, if m grows slower than n, then pinning control is not possible for a bounded α, regardless of how large the parameters c_i are.

Definition 2 ([16, 17]) *A locally connected network is defined as a network where the*

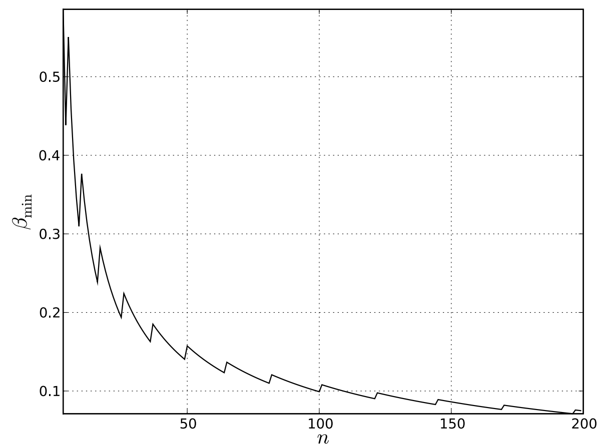

FIG. 2: The value of $\beta_{\min}$ as the number of vertices is varied for a fully connected graph. The number of systems with pinning control applied is $\lceil \sqrt{n} \rceil$.

nodes are located on a integer lattice $\mathbb{Z}^d$ and are connected by an edge only if they are at most a distance r apart. The parameters d and r are assumed to be fixed.

It is clear that a subgraph of a locally connected network is also locally connected. An example of a locally connected network for $d = 2$, $r = 1$ is shown in Fig. 3.

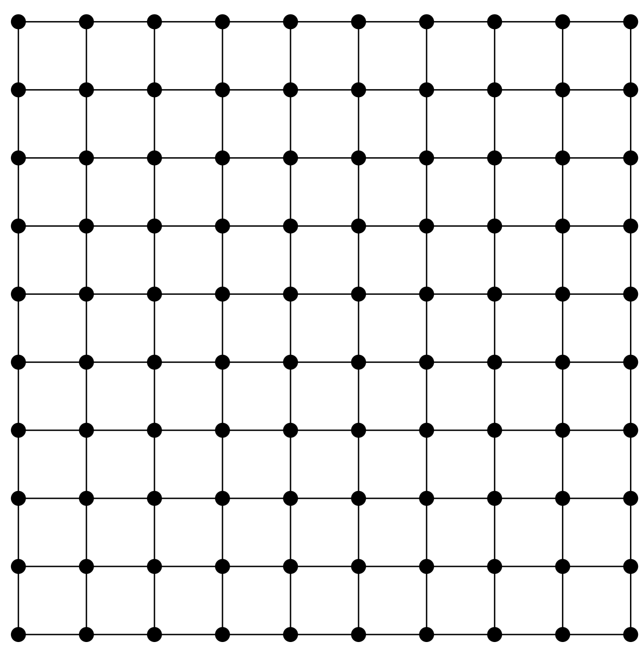

FIG. 3: A locally connected graph.

Let us assume that α is fixed and that the underlying graph is a locally connected graph.

First consider the case where the underlying graph is of the following form: the vertices are arranged in a circle and are connected by an edge if and only if they are less than or equal to d vertices apart. Let us denote this graph as $\mathcal{G}_d$ with Laplacian matrix G_d. For $d = 1$, this is the cycle graph. The eigenvalues of G_d are given by:

$$\lambda_i = 2\left(d - \sum_{l=1}^{d} \cos\left(\frac{2\pi il}{n}\right)\right), \quad i = 0, \ldots, n-1$$

It is not hard to show that for $m \in o(n)$, the smallest $m+1$ eigenvalues of G all converges to 0 as $n \to \infty$. From Weyl's eigenvalue interlacing theorem [18], $\lambda_1(G + C) \leq \lambda_{m+1}(G) + \lambda_{n-m}(C)$. Since only m of the c_i's are nonzero, $\lambda_{n-m}(C) = 0$. This implies that $\beta_{min} \to 0$ as $n \to \infty$.

Next, consider a general locally connected graph with parameters r and d and Laplacian matrix G. It is easy to see that it is a subgraph of a locally connected graph that can be decomposed as the strong product of r graphs of the form $\mathcal{G}_d$. Since the eigenvalues of this graph can be derived from sums and products of eigenvalues of multiple $\mathcal{G}_d$ [19], it is also true that for $m \in o(n)$, the smallest $m+1$ eigenvalues of $G \to 0$ as $n \to \infty$. The same argument as above shows that $\beta_{\min} \to 0$ as $n \to \infty$ in this case as well.

Thus we have shown the following:

For a fixed parameter α, and a locally connected network of n dynamical systems with pinning control applied to m systems, pinning control is not possible as $n \to \infty$ if m grows slower than n.

This is illustrated in Fig. 4, where we show how $\beta_{\min}$ changes as $n \to \infty$. For each n, the graph is a cycle graph of n vertices. We choose $c_i = 100n$, $\alpha = 1$ and $m = \lceil \sqrt{n} \rceil$. We see that $\beta_{\min} \to 0$ as $n \to \infty$.

The above analysis is also valid if the graph is not undirected, but vertex-balanced, i.e. the indegree of each vertex is equal to its outdegree. In this case, the analysis is applied to the symmetric zero row sums matrix $\frac{1}{2}(G + G^T)$.

V. LOCALIZATION OF PINNING CONTROL SITES

Let us now consider the problem of which vertices to put pinning control, i.e. determining the set P. If the goal is to find $c_i \geq 0$ such that $\sum_i c_i \leq \gamma$ and $\beta_{\min}$ is maximized, then the solution is clear for a vertex-balanced graph. Simply set $c_i = \frac{\gamma}{n}$ for all i. The reason that

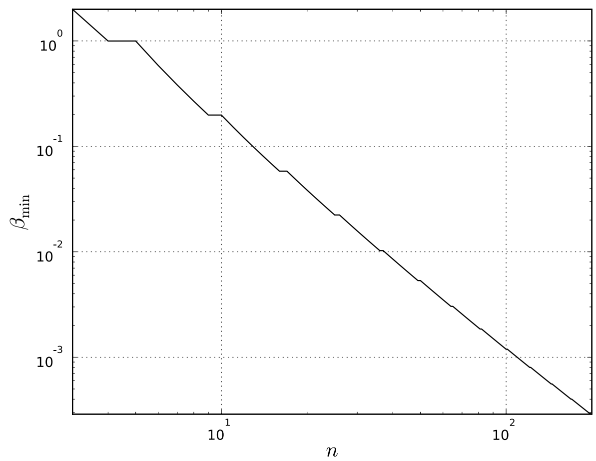

FIG. 4: The value of $\beta_{\min}$ as the number of vertices is varied for a cycle graph.

this is optimal is as follows. First note that $C = \frac{\gamma}{n} I$ and thus $\beta_{\min} = \lambda_1(\frac{1}{2}(G + G^T) + C) = \lambda_1(\frac{1}{2}(G + G^T)) + \frac{\gamma}{n} = \frac{\gamma}{n}$ where we have used the fact that $G + G^T$ is a singular matrix. Next it was shown in [15] that $\beta_{\min} \leq \frac{\gamma}{n}$ for all choices of c_i such that $\sum_i c_i \leq \gamma$.

But the situation is different if we can only apply pinning control to a small number of vertices. Consider the 2D dimensional grid graph of n vertices with Laplacian matrix G_n (Fig. 3). Suppose that all the nonzero pinning control strengths are equal, i.e. if $c_i > 0$, then $c_i = c$. Where should the pinning control be applied to maximize $\beta_{\min}$? We performed the following simple experiments to study this question. First m vertices are randomly chosen where pinning control is applied. Then for each vertex with pinning control, its pinning control is moved to another location that increases $\beta_{\min}$. This operation is reiterated until no such move will increase $\beta_{\min}$. We show the resulting configuration of pinning control in Fig. 5. We use $n = 100$, $m = 20$, $\alpha = 1$ and $c = 100$. The locations where pinning control is applied are shown larger and in gray. We see that pinning control is applied to vertices whose locations are spread out in the graph.

We repeated the same experiment, but now to minimize $\beta_{\min}$. The result is shown in Fig. 6. We see now that the pinning control is applied to vertices whose locations are close to each other. The difference in β_{min} between these two configuration is more than 20-fold. This experiment suggests that it is more beneficial to apply pinning control at locations which are spread out. What is meant by spread out vertices is that the (graph-theoretical) distance between pinning vertices in the graph should not be small. For the grid graph

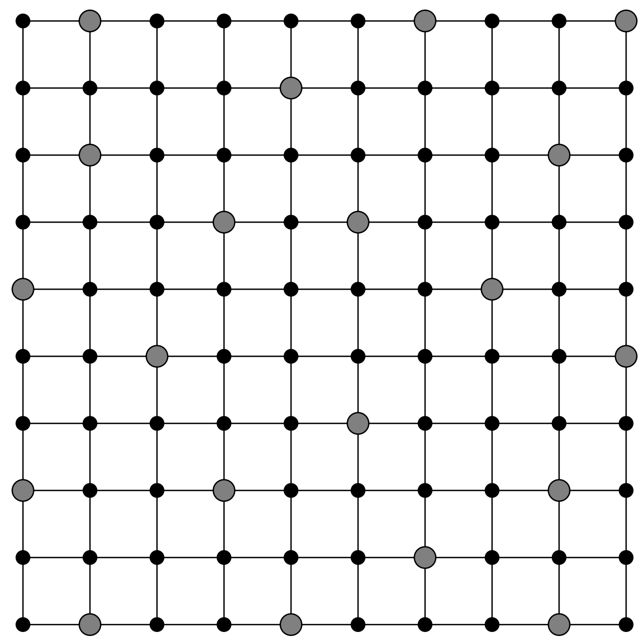

FIG. 5: Pinning control where the value of $\beta_{\min}$ is large. The locations where pinning control is applied are shown larger and in gray. $\beta_{\min} = 0.7687$.

in Fig. 3, the graph-theoretical distance is equivalent to the l_1 distance on the plane. We conjecture that $\beta_{\min}$ is maximized for the pinning control configuration that minimizes the distances from all vertices to the controlled vertices. More precisely, let V be the set of vertices of the graph and $P \subset V$ be the subset of vertices where pinning is applied.

Conjecture 1 *Under the constraint that $|P| = m$, $\beta_{\min}$ is maximized for a set P such that*

$$D_P = \max_{v \in V} \min_{p \in P} d(v, p)$$

is minimized.

Here $d(v, p)$ is the distance between v and p, i.e. the length of the shortest path between v and p. Note that the conjecture talks about *a* set P since there are in general many sets P which minimize D_P, and the conjecture states that one of them will maximize $\beta_{\min}$.

The quantity D_P can be used to derive a lower bound on $\beta_{\min}$. In [15] it was shown that

$$\beta_{\min} \geq \frac{c}{2} \left(2 \left(r + \frac{1}{2}(2r)^{-D_P} \right) \right)^{-D_P} > 0$$

where r is the maximal outdegree among unpinned vertices of the graph and $c = \min_{i \in P} c_i$.

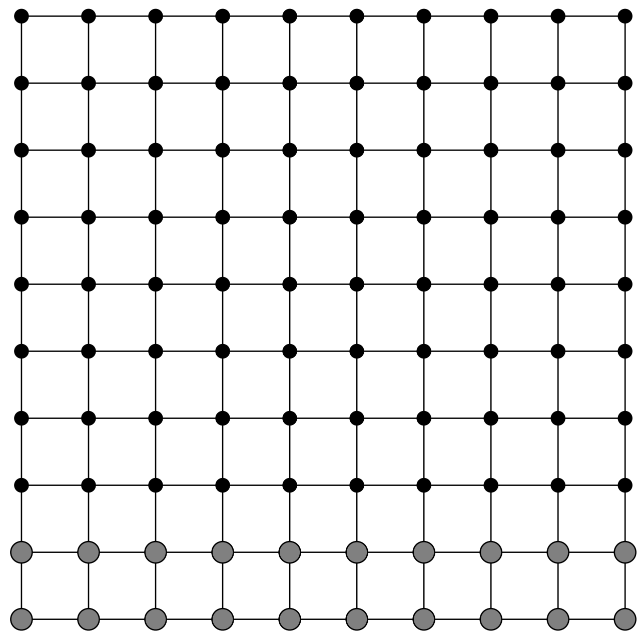

FIG. 6: Pinning control where the value of $\beta_{\min}$ is small. The locations where pinning control is applied are shown larger and in gray. $\beta_{\min} = 0.034$.

To support Conjecture 1, we performed the following experiment. 100000 random sets of 20 pinning locations are chosen on the grid graph (Fig. 3) and $\beta_{\min}$ and D_P are computed. The results are shown in Fig. 7. The configurations in Figs. 5 and 6 with $D_P = 2$ and $D_P = 8$ resp. are also plotted in Fig. 7. It is clear that $\beta_{\min}$ tend to be larger for smaller D_P.

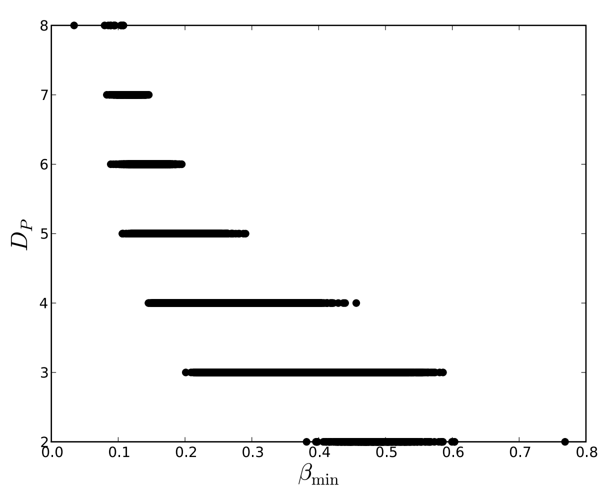

FIG. 7: $\beta_{\min}$ versus D_P for 100000 random sets of pinning control locations on the grid graph.

This relationship is more evident when we repeated the experiment with a path graph and parameters $n = 100$, $m = 5$, $c = 100$ (Fig. 8).

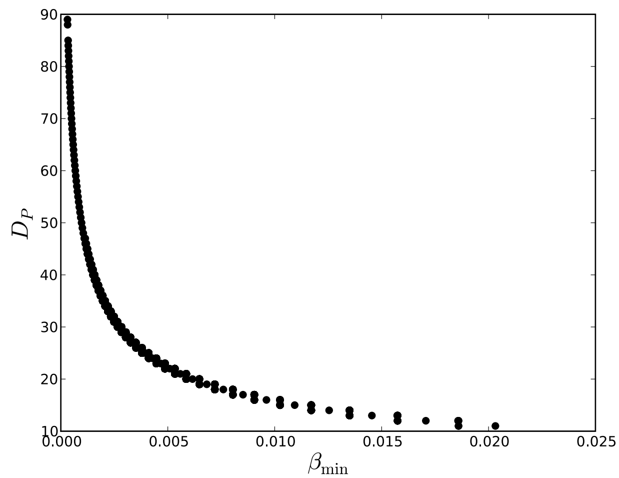

FIG. 8: $\beta_{\min}$ versus D_P for 50000 random sets of pinning control locations on the path graph with 100 vertices and $m = 5$.

VI. CONCLUSIONS

We study the locations where effective pinning control should be applied in a complex network of coupled dynamical systems. We show that for the case where the number of controlled systems is small compared with the total number of systems, pinning control strength needs to be increased in order to maintain pinning control for large n. If the graph is locally connected, then the coupling among all systems also need to increase to maintain pinning control. Finally, we show that applying pinning control to vertices that minimizes the distances to all vertices in the graph could lead to more effective pinning control.

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