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ABSTRACT: Queueing models with different classes of customers are considered which have the property that when the arrival process for each class of customers is Poisson then the departure process for each class of customers is Poisson. A sufficient but very general condition for a queueing system to have this property is given and illustrated with examples. Several of these examples are models which were not previously known to have a Poisson departure process. Networks of queues models in which each component queue satisfies the above condition are shown to have very simple solutions for their equilibrium state probabilities.

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I. INTRODUCTION

In 1957 J. R. Jackson [1] presented the solution for the equilibrium state probabilities for open networks of queues models with one class of customers and exponential service time distributions. These results exhibited an interesting property. A state S of each a model is given by the number of customers in each service center so that $S = (n_1, n_2, \ldots, n_N)$ where n_i is the number of customers in service center i and N is the number of service centers in the network. The arrival process to service center i from outside the network is Poisson with mean rate η_i. By a conservation of flow argument the mean total arrival rate to each service center can be found. Let λ_i be the mean total arrival rate to service center i and let $P_i(n_i)$ be the equilibrium probability that there are n_i customers in the i^{th} service center when the service center has a Poisson arrival process with mean rate λ_i. Jackson's result was that the equilibrium state probabilities for the network are given by

$$P(S = (n_1, n_2, \ldots, n_N)) = P_1(n_1)P_2(n_2)\ldots P_N(n_N)$$

Thus the states of the component service centers are independent random variables and the state probabilities for the component service centers are the same as when the arrival processes to the service centers are Poisson. This result is rather surprising as Burke points out [2] since the arrival process to a service center is not Poisson in general.

More results on networks of queues are now available [3,4,5,6,7]. Some of the more recent results allow for different classes of customers (transition probabilities and in some cases service time distributions may depend on the class of customer). As with Jackson's early results the network equilibrium state probabilities are always very simply related to the product of the equilibrium state probabilities for the component service centers when the service centers have Poisson arrival processes with appropriately chosen mean rates. In general a state S of a queueing network can be expressed as $S=S_1,S_2,\ldots,S_N$ where S_i is the state of the i^{th} service center. Let λ_{ir} be the total mean arrival rate of class r customers to the i^{th} service center (the calculation of the λ_{ir} will be detailed later). Let $\{P_i(S_i)\}$ be the equilibrium state probabilities for service center i when removed from the network and the arrival process for each class of customers is Poisson. The mean arrival rate for class r customers is λ_{ir}. In terms of the $\{P_i(S_i)\}$ we can describe the forms of the solutions found so far for queueing networks.

For closed networks the known solutions have the form

$$P(S = S_1,S_2,\ldots,S_N) = CP_1(S_1)P_2(S_2)\ldots P_N(S_N)$$

where C is a normalizing constant chosen so that the state probabilities for the network sum to one.

For open networks with homogeneous Poisson arrival processes

$$P(S = S_1,S_2,\ldots,S_N) = P_1(S_1)P_2(S_2)\ldots P_N(S_N).$$

Whether or not the equilibrium state probabilities for a network model have such a product form is related to the departure processes of the component service centers when their arrival processes are Poisson. A sufficient condition for the departure processes of the different classes of customers to be Poisson when the arrival process for each class of customers is Poisson is given in Section 2 . Queueing models which satisfy this condition will be said to have the $M \Rightarrow M$ property. It is shown in Section 3 that a sufficient condition for a network of queues model to have a product form solution as described above is that each service center has the $M \Rightarrow M$ property.

2. POISSON DEPARTURE PROCESSES

An $M/M/n$ queueing system has long been known to have a Poisson departure process [2]. One of the interesting (and sometimes confusing) aspects of this result concerns the dependency of the departure process on the state of the system. When the system is empty then the departure rate is zero while if the system is not empty then the departure rate is greater than zero. Thus at any time the future departure process depends on the current state of the system. However it can be shown that the past departure process is independent of the current state of the system [2]. In particular, it can be shown that

$$\forall \text{ states, } S \quad \lim_{\Delta t \to 0} \frac{\Pr[\text{departure in } (t-\Delta t,t)/S(t) = S]}{\Delta t} = \lambda$$

where λ is the mean arrival rate and $S(t)$ is the state of the system at time t.

The departure rate with time reversed is therefore independent of the state of the system and thus is independent of time. This together with the observation that the rate of multiple departures is zero shows that the departure process is Poisson.

We now explore this property (i.e. the independence of the departure rate with time reversed and the state of the system) more formally and for more general models. Consider a queueing model with different classes of customers. Let $\mathcal{S}$ be the set of states of the model. The transition rate from state $S_i \in \mathcal{S}$ to state $S_i' \in \mathcal{S}$ will be denoted by $R(S_i \rightarrow S_i')$. Thus

$$R(S_i \rightarrow S_i') = \lim_{\Delta t \rightarrow 0} \frac{\Pr[S(t+\Delta t) = S_i' / S(t) = S_i]}{\Delta t}$$

The inverse transition rate from S_i' to S_i will be denoted by $Q(S_i' \rightarrow S_i)$. Thus

$$Q(S_i' \rightarrow S_i) = \lim_{\Delta t \rightarrow 0} \frac{\Pr[S(t-\Delta t) = S_i / S(t) = S_i]}{\Delta t}$$

It is easy to show that

$$Q(S_i' \rightarrow S_i) = \frac{P(S_i)\ R(S_i\ S_i')}{P(S_i')}$$

where $P(S_i)$ is the equilibrium probability of the system being in state S_i.

Let $|S_i|_r$ denote the number of class r customers in the system in state S_i. If $S(t)=S_i$ then a departure of a class r customer in $(t-\Delta t, t)$ corresponds to a transition (with time reversed) from S_i at time t to a state S_i' at time $t-\Delta t$ where $|S_i'|_r=|S_i|_r+1$. The departure rate for class r customers in inverse time is therefore the sum of state transition rates from the current state to a state with one more class r customer.

Let

$$B_{+r}(S_i) = \{S_{i+r}/S_{i+r} \in \mathcal{S},\ |S_{i+r}|_r = |S_i|_r + 1,\ R(S_{i+r} \to S_i) > 0\}$$

The condition for the departure process of class r customers to be Poisson becomes

$$\forall S_i \in \mathcal{S} \quad \sum_{S_{i+r} \in B_{+r}(S_i)} \lim_{\Delta t \to 0} \frac{\Pr[S(t-\Delta t)=S_{i+r}/S(t)=S_i]}{\Delta t} = \lambda_r$$

where λ_r is the mean arrival rate of class r customers.

This is clearly

$$\forall S_i \in \mathcal{S}, \quad \sum_{S_{i+r} \in B_{+r}(S_i)} Q(S_i \to S_{i+r}) = \lambda_r$$

or

$$\forall S_i \in \mathcal{S}, \quad \sum_{S_{i+r} \in B_{+r}(S_i)} \frac{P(S_{i+r})\ R(S_{i+r} \to S_i)}{P(S_i)} = \lambda_r$$

In establishing this condition for Poisson departure processes no assumption has been made about the arrival process. However in the remainder of the paper we will always deal with queueing models with Poisson arrival processes. A queueing model which satisfies the above condition when the arrival process for each class of customers is Poisson will be said to have the M⇒M (Markov implies Markov) property.

Note:

It has been implicitly assumed that all departures are associated with a change of state. Another type of departure is possible in some systems. An example is an M/M/n queueing system with finite waiting room. If customers which arrive when the waiting room is full (and therefore do not join the system) are considered departures then these departures are not associated with a state change but the departure process is Poisson. To account for this type of departure we can generalize the requirement for a Poisson departure process. Let $\alpha_r(S_i)$ be the probability that a customer of class r which arrives when the system is in state S_i will instantaneously depart. Then a sufficient condition for the departure process for class r customers to be Poisson when the arrival process is Poisson with rate λ_r is

$$\forall S_i \in \mathcal{S} \quad \sum_{S_{i+r} \in B_{+r}(S_i)} \frac{P(S_{i+r})\ R(S_{i+r} \to S_i)}{P(S_i)} + \alpha_r(S_i)\lambda_r = \lambda_r$$

In the case of the M/M/n queueing system with finite waiting room and one class of customers $\alpha(S_i)=0$ if the waiting room is not full in state S_i and $\alpha(S_i)=1$ if the waiting room is full in state S_i.

For convenience we will assume in the remainder of the paper that departures are always associated with state transitions. However, all of the results of this paper can be shown to apply in this more general case. We note in passing that departures of the type just described corresponding to Jackson's "service deletion" [3] in queueing networks.

2.1 Queueing Models with the M⇒M Property

2.1.1 Reich [8] has shown that any reversible queuing system with a Poisson arrival process has a Poisson departure process. For a reversible system

$$Q(S_i \to S_i') = R(S_i \to S_i') \quad \forall S_i, S_i' \in \mathcal{S}$$

Our condition for a Poisson departure process is

$$\forall S_i \quad \sum_{S_{i+r} \in B_{+r}(S_i)} Q(S_i \to S_{i+r}) = \lambda_r \ .$$

For a reversible system this is equivalent to

$$\forall S_i \quad \sum_{S_{i+r} \in B_{+r}(S_i)} R(S_i \to S_{i+r}) = \lambda_r$$

This condition is satisfied since the left-hand side is just the sum of the transition rates from S_i to a state with one more class r customer. This is clearly equal to the mean arrival rate of class r customers.

It follows that the $M \Rightarrow M$ property is at least as general as the reversibility requirement. The following three examples are non-reversable systems which satisfy the $M \Rightarrow M$ condition and therefore show that the $M \Rightarrow M$ condition is a less restrictive sufficient condition for the departure process to be Poisson.

2.1.2 The $M/G/\infty$ queueing model with different classes of customers has the $M \Rightarrow M$ property. The arrival process for each class of customer is Poisson and the mean arrival rate for class r customers is denoted by λ_r. Let R be the number of different classes of customers. Classes of customers are assumed to be labeled by the integers 1 through R. The service time distribution for class r customers will be represented using the method of stages [9] as illustrated in Figure 1.

The states of this model are given by the number n_{rk}, of class r customers in stage k of service for $1 \leq r \leq R$, $1 \leq k \leq m_r$. We use the notation $S=\{n_{rk}\}$.

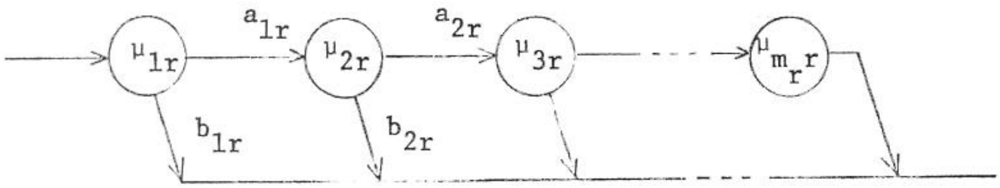
The equilibrium state probabilities for this model are

$$P(S=\{n_{rk}\}) = \exp\left(-\sum_{r=1}^{R} \frac{\lambda_r}{\mu_r}\right) \prod_{r=1}^{R} \prod_{k=1}^{m_r} \left[\frac{\lambda_r A_{rk}}{\mu_{rk}}\right]^{n_{rk}} \frac{1}{n_{rk}\,!}$$

where

$\frac{1}{\mu_r}$ is the mean service time of class r customers

$$A_{rk} = \prod_{\ell=1}^{k} a_{\ell k}$$



where $\frac{1}{\mu_{ir}}$ is the mean service time in stage i.

a_{ir} is the probability that a customer proceeds to the next stage when stage i is completed.

b_{ir} is the probability that a customer departs when stage i is completed $(b_{ir} = 1 - a_{ir})$.

m_r is the number of stages for class r customers.

Figure 1.

It is a tedious but routine task to check that the above solution satisfies the balance equations for the model and also the $M \Rightarrow M$ condition.

It is easy to see that an $M/G/\infty$ system is not in general reversible. A necessary condition for a process to be reversible is that for any two states S_i and S_i' if $R(S_i \rightarrow S_i') \neq 0$ then $R(S_i' \rightarrow S_i) \neq 0$. In an $M/E_2/\infty$ system (a special case) there is a non-zero transition rate from the state with one customer in his second stage of service to the idle state while the transition rate from the idle state to the state with one customer in his second stage of service is clearly zero. Thus the system is non-reversible.

2.1.3 The $M/G/1$ queueing model with LCFS-preemptive scheduling has the $M \Rightarrow M$ property. We will use the same notation for arrival rates, service time distribution parameters, etc. as for example 2.

The states of such a system are given by an ordered list of the customers in the queue in LCFS order and the stage of service they have achieved. We will use the notation $S=((r_1,k_1)(r_2,k_2)\ldots,(r_n,k_n))$ where (r_i,k_i) indicates that the i^{th} customer in LCFS order is a class r_i customer and is in stage k_i of service. Actually, the equilibrium probabilities for two states which are identical except for a permutation of the LCFS ordering are identical. Let n_{rk} be the number of customer of class r which are in the k^{th} stage of service.

$$P\left(S=\left((r_1,k_1),(r_2,k_2),\ldots,(r_n,k_n)\right)\right)$$

$$=\left[1-\sum_{r=1}^{R}\lambda_r\frac{1}{\mu_r}\right]\prod_{\ell=1}^{n}\frac{\lambda_{r_\ell}}{\mu_{r_\ell k_\ell}}$$

$$=\left[1-\sum_{r=1}^{R}\lambda_r\frac{1}{\mu_r}\right]\prod_{r=1}^{R}\prod_{k=1}^{n_r}\left(\frac{\lambda_r}{\mu_{rk}}\right)^{n_{rk}}$$

Again it is a routine task to check that these solutions satisfy the balance equations for the model and that the model has the $M \Rightarrow M$ property.

This example and the next are models which have not previously been shown to have Poisson departure processes. A proof that the $M/G/\infty$ system has a Poisson departure process can be found in [10].

2.1.4 The last example is the $M/G/1$ queueing model with the processor-sharing scheduling discipline. With the processor-sharing discipline each customer in the system shares equally the service capacity of the server. Thus if there are n customers in the system at a given time then each is served at the rate of $\frac{1}{n}$ seconds of work/second. This service discipline is an approximation to the round-robin service discipline which is common in computer system scheduling.

As in the case of the $M/G/\infty$ model the state of the system is given by the number of customers in each class in each stage of service, i.e., $S=\{n_{rk}\}$. The equilibrium state probabilities are given by

$$P(S = \{n_{rk}\}) = \left(1 - \sum_{r=1}^{R} \frac{\lambda_r}{\mu_r}\right) n! \prod_{r=1}^{R} \prod_{k=1}^{n_r} \left(\frac{\lambda_r A_{rk}}{\mu_{rk}}\right)^{n_{rk}} \frac{1}{n_{rk}!}$$

where n is the total number of customers in the system in state S, i.e., $n = \sum_{r=1}^{R} \sum_{k=1}^{n_r} n_{rk}$.

This solution can be validated from the balance equations for the model and the model can be shown to have the $M \Rightarrow M$ property.

2.1.5 State Dependent Service Rates

In this section, we consider various types of state dependent service rates.

Consider a queueing model with the $M \Rightarrow M$ property and state transition rates denoted by $[R(S_i \to S_i')]$. Now, consider a new model which is the same as the original model except that the state transition rates which do not correspond to arrivals are state dependent. The arrival rates are not state dependent. We denote the new state transition rates by $[R'(S_i \to S_i')]$.

We consider two types of state dependency.

(a) $R'(S_i \to S_i') = R(S_i \to S_i')\ X\ (|S_i|)$

where the transition $S_i \to S_i'$ does not correspond to an arrival and $X(|S_i|)$ is an arbitrary but strictly positive function of the number of customers in the system in state S_i.

(b) $R''(S_i \to S_i') = R(S_i \to S_i')\ X_r\ (|S_i|_r)$

where the transition $S_i \to S_i'$ does not correspond to an arrival and $X_r\ (|S_i|_r)$ is an arbitrary function of the number of class r customers in the system in state S_i. Here we assume that any change of state (not by an arrival) can be associated with a customer either leaving the system or moving to the next stage of service. r is the class of the customer associated with the transition $S_i \to S_i'$.

For the type of state dependent service rates described in (a) it can be shown that if the equilibrium state probabilities $[P'(S_i)]$ exist for the new model, they are given by

$$P'(S_i) = C' \frac{P(S_i)}{\prod_{j=1}^{|S_i|-1} X(j)}$$

where C' is a normalization constant.

Moreover, the new model has the $M \Rightarrow M$ property.

For the type of state dependent service rates described in (b) it can be shown that the equilibrium state probabilities $[P''(S_i)]$ exist for the new model, they are given by

$$P''(S_i) = C'' \frac{P(S_i)}{\prod_{r=1}^{R} \prod_{j=1}^{|S_i|_r - 1} X_r(j)}$$

where C'' is a normalization constant.

Moreover the new model has the $M \Rightarrow M$ property.

3. NETWORKS OF SERVICE CENTERS WITH THE $M \Rightarrow M$ PROPERTY

3.1 Introduction

The network models treated in this paper contain an arbitrary but finite number of service centers. A service center is described by the usual characteristics of a queueing model, i.e., number of servers, queueing discipline, etc. There is an arbitrary but finite number of different classes of customers. Customers travel through the network according to transition probabilities. We permit customers to change their class membership. Thus a customer of class r which completes service at service center i will next require service at service center j and enter class s with probability $p_{i,r;j,s}$. Customers in different classes may have different service time distributions at the various service centers.

Two types of queueing networks are generally distinguished: (1) closed networks in which customers neither enter or leave the network and (2) open networks in which there are external arrivals to the network and departures from the network. For closed networks

$$\forall i,r \quad \sum_{j,s} p_{i,r;j,s} = 1$$

while for open networks $1 - \sum_{j,s} p_{i,r;j,s}$ is the probability that a class r customer leaves the network after completing service at service center i.

We assume that service time distributions are represented using the method of stages so that the state space of the model is discrete. Any service time distribution with a rational Laplace Transform has a stages representation (Cox [9]).

3.2 Mean Arrival Rates in Closed Networks

Let λ_{ir} denote the mean arrival rate of class r customers to service center i in a queueing network with transition matrix $P=[p_{i,r;j,s}]$. The $\{\lambda_{ir}\}$ must satisfy the following simultaneous linear equations:

$$\lambda_{ir} = \sum_{j,s} \lambda_{js} \, p_{j,s;ir} \qquad \forall i,r$$

The λ_{ir} are not uniquely determined by these equations. The transition matrix $P=[p_{i,r;js}]$ may be thought of as defining a Markov chain consisting of several ergodic subchains in general. The states of the Markov chain are the pairs (i,r). The λ_{ir} corresponding to a given ergodic subchain are determined within a multiplicative constant. Thus these equations define the relative mean arrival rates for the service centers of the network. This will be enough. The multiplicative constants are assumed to be chosen so that no service center is saturated.

3.3 Mean Arrival Rates in Open Networks

An open network has arrivals to the network and departures from the network. The exogenous arrival process of class r customers to service center i is assumed to be a homogeneous Poisson process with mean rate η_{ir}. The total mean arrival rate of class r customers to service center i is denoted by λ_{ir}. The $\{\lambda_{ir}\}$ are the solution to the following simultaneous linear equations.

$$\lambda_{ir} = \eta_{ir} + \sum_{j,s} \lambda_{js}\, p_{j,s;i,r} \qquad \forall i,r$$

These equations will be assumed to have a unique solution.

3.4 Product Solutions for Networks of Queues

The following theorem is the main result of Section 3.

Theorem: Let N be a network of queues model as defined earlier where there are R classes of customers and N service centers. Let λ_{ir} be the mean arrival rate of class r customers to service center i (for closed networks the multiplicative constants are arbitrary except that no service station should be saturated). Let $\{P_i(S_i)\}$ be the equilibrium state probabilities for service center i when the arrival process for class r customers is Poisson with mean rate λ_{ir}, $1 \leq r \leq R$.

(a) If N is a closed network then the equilibrium state probabilities for the network are given by

$$P(S=S_i, S_2, \ldots, S_N) = C\, P_1(S_1)\, P_2(S_2) \ldots P_N(S_N)$$

where C is a normalizing constant chosen so that the network state probabilities sum to one.

(b) If N is an open network then the equilibrium state probabilities for the network are given by

$$P(S=S_i, S_2, \ldots, S_N) = P_1(S_1)\, P_2(S_2) \ldots P_N(S_N)$$

Proof: Only the proof of part (a) is given since the proof of part (b) follows along the same lines.

The first part of the proof shows that the equilibrium state probabilities of service center with the $M \Rightarrow M$ property satisfy a simplified balance equation. This result is used in the second part of the proof which is to show that the proposed solution satisfies the balance equations for the network.

Balance Equations for a Service Center with the $M \Rightarrow M$ Property

Given that a service center has the $M \Rightarrow M$ property the balance equations can be shown to reduce to a simplified form. Since we are making no assumptions about the service center other than the $M \Rightarrow M$ property it will be necessary to introduce some general notation in order to express the balance equations. For the readers convenience we repeat the definition of $B_{+r}(S_i)$.

Let $S_i \in \mathcal{S}_i$.

$$B_{+r}(S_i) = \{S_{i+r} / S_{i+r} \in \mathcal{S}_i,\ |S_{i+r}|_r = |S_i|_r + 1,\ R(S_{i+r} \to S_i) > 0\}$$

i.e., the set of nearest neighbor states of S_i which contain one more class r customer and such that the transition rate <u>into</u> S_i is nonzero.

$$B_{-r}(S_i) = \{S_{i-r} / S_{i-r} \in \mathcal{S}_i,\ |S_{i-r}|_r = |S_i|_r - 1,\ R(S_{i-r} \to S_i) = \lambda_{ir}\}$$

i.e., the set of nearest neighbor states of S_i with one less class r customer and such that an arrival of a class r customer causes a transition <u>into</u> S_i.

$$B_0(S_i) = \{S_{i-0} / S_{i-0} \in \mathcal{S}_i,\ \forall r |S_{i-0}|_r = |S_i|_r,\ R(S_{i-0} \to S_i) \neq 0\}$$

i.e., the set of nearest neighbor states of S_i with the same number of each class of customer and there is a non-zero transition rate into S_i. Thus these are the states from which it is possible to move into S_i (with one transition) without an arrival or departure.

$$A_{-r}(S_i) = \{S'_{i-r} / S'_{i-r} \in \mathcal{S}_i,\ |S'_{i-r}|_r = |S_i|_r - 1,\ R(S_i \to S'_{i-r}) \neq 0\}$$

i.e., the set of nearest neighbor states of S_i which can be reached from S_i in one transition by a departure of a class r customer.

$$A_0(S_i) = \{S_{i+0} / S_{i+0} \in \mathcal{S}_i,\ \forall r |S_{i+0}|_r = |S_i|_r,\ R(S_i \to S_{i+0}) \neq 0\}$$

i.e., the set of nearest neighbor states of S_i with the same number of each class of customer and which can be reached from S_i in one transition.

Now we can express the balance equation for the i^{th} service center with poisson arrivals.

$$\sum_{S_{i+0} \in A_0(S_i)} P(S_i)\ R(S_i \to S_{i+0}) + \sum_r \sum_{S'_{i-r} \in A_{-r}(S_i)} P(S_i)\ R(S_i \to S'_{i-r})$$
$$+ \sum_r \lambda_{ir}\ P(S_i) = \sum_r \sum_{S_{i-r} \in B_{-r}(S_i)} P(S_{i-r})\ \lambda_{ir}$$
$$+ \sum_r \sum_{S_{i+r} \in B_{+r}(S_i)} P(S_{i+r})\ R(S_{i+r} \to S_i)$$
$$+ \sum_{S_{i-0} \in B_0(S_i)} P(S_{i-0})\ R(S_{i-0} \to S_i)$$

The three terms on the left hand side of the the balance equation correspond respectively to flow out of S_i due to (1) change of state without an arrival or departure (2) change of state due to a departure and (3) change of state due to an arrival. Similarly the three terms on the right hand side of the balance equation correspond respectively to (1) transition into S_i due to an arrival (2) transition into S_i due to a change of state without an arrival or departure. However, from the $M \Rightarrow M$ property we have

$$\sum_{S_{i+r} \in B_{+r}(S_i)} Q(S_i \to S_{i+4}) = \lambda_{ir}$$

$$\text{or} \quad \sum_{S_{i+r} \in B_{+r}(S_i)} \frac{P(S_{i+r})\, R(S_{i+r} \to S_i)}{P(S_i)} = \lambda_{ir}$$

$$\text{or} \quad \sum_{S_{i+r} \in B_{+r}(S_i)} P(S_{i+r})\, R(S_{i+r} \to S_i) = \lambda_{ir}\, P(S_i)$$

$$\text{or} \quad \sum_r \sum_{S_{i+r} \in B_{+r}(S_i)} P(S_{i+r})\, R(S_{i+r} \to S_i) = \sum_r \lambda_{ir}\, P(S_i)$$

Thus these terms always cancel in the balance equations so that each $P(S_i)$ satisfies the following equation.

$$\sum_{S_{i+0} \in A_0(S_i)} P(S_i)\, R(S_i \to S_{i+0}) + \sum_r \sum_{S'_{i-r} \in A_{-r}(S_i)} P(S_i)\, R(S_i \to S'_{i-r})$$
$$= \sum_r \sum_{S_{i-r} \in B_{-r}(S_i)} P(S_{i-r})\, \lambda_{ir} + \sum_{S_{i-0} \in B_0(S_i)} P(S_{i-0})\, R(S_{i-0} \to S_i)$$

<u>Verifying the Solution $P(S) = C\, P_1(S_1)\, P_2(S_2) \ldots P_N(S_N)$</u>

In writing the general form of a balance equation for a network we shall use the previously defined notation, i.e., $B_{-r}(S_i), B_{+r}(S_i)$, etc., with the understanding that the terms corresponding to states which are not possible in the network are to be deleted.

The general form of a balance equation in the network is given by

$$\sum_{i=1}^{N} \left\{ \sum_{S_{i+0}\, A_0(S_i)} P(S=S_1, S_2, \ldots, S_N)\, R(S_i \to S_{i+0}) + \sum_{r} \sum_{S'_{i-r} \in A_{-r}(S_i)} P(S=S_1, S_2, \ldots, S_N)\, R(S_i \to S'_{i-r}) \right\}$$

$$= \sum_{i=1}^{N} \left\{ \sum_{r} \sum_{S_{i-r} \in B_{-r}(S_i)} \left[\sum_{j=1}^{N} \sum_{q} \left(\sum_{S_{j+q} \in B_{+q}(S_j)} P(S=S_1, \ldots, S_{j+q}, \ldots, S_{i-r}, \ldots, S_N)\, R(S_{j+q} \to S_j) \right) P_{j,q;i,r} \right] + \sum_{S_{i-0} \in B_0(S_i)} P(S=S_1, \ldots, S_{i-0}, \ldots, S_N)\, R(S_{i-0} \to S_i) \right\}$$

The left hand side of this equation is the sum of flow out terms for each service center. There are two terms for each service center. The first corresponds to flow out due to a change in state of a service center without a departure. The second term corresponds to flow out due to a departure from the service center. The right hand side of the equation is the sum of flow in terms for each service center. The first term accounts for all possible ways of moving into the state by a customer departing from some service center and joining the i^{th} service center. The second term corresponds to transitions into the state by changes in state of a service center without departures. We now show that the proposed solution satisfies these equations.

Consider the expression

$$\sum_{S_{j+q} \in B_{+q}(S_j)} P(S=S_1,\ldots,S_{j+q},\ldots,S_{i-r},\ldots,S_N) \quad R(S_{j+q} \to S_j)$$

which appears in the right hand side of the balance equation. Substituting the proposed solution this can be written

$$C\ P_1(S_1)\ldots P_{j-1}(S_{j-1})P_{j+1}(S_{j+1})\ldots P_i(S_{i-r})\ldots P_N(S_N) \sum_{S_{j+q} \in B_{+q}(S_j)}$$

$$P_j(S_{j+q}) \ R(S_{j+q} \to S_j)$$

But from the $M \Rightarrow M$ property

$$\sum_{S_{j+q} \in B_{+q}(S_j)} P_j(S_{j+q}) \ R(S_{j+q} \to S_j) = P_j(S_j) \ \lambda_{jq}$$

Thus

$$\sum_{S_{j+q} \in B_{+q}(S_j)} P(S=S_1,\ldots,S_{j+q},\ldots,S_{i-r},\ldots,S_N) \ R(S_{j+q} \to S_j)$$

$$= C\ P_1(S_1)\ldots P_j(S_j)\ldots P_i(S_{i-r})\ldots P_N(S_N) \ \lambda_{jq}$$

Thus

$$\sum_j \sum_q \left(\sum_{S_{j+q} \in B_{+q}(S_j)} P(S=S_1,\ldots,S_{j+q},\ldots,S_{i-r},\ldots,S_N)\ R(S_{j+q} \to S_j) \right) P_{j,q;i,r}$$

$$= \sum_j \sum_q C\ P_1(S_1)\ldots P_j(S_j)\ldots P_i(S_{i-r})\ldots P_N(S_N)\ \lambda_{jq}\ \ P_{jq;ir}$$

$$= C\ P_1(S_1)\ldots P_j(S_j)\ldots P_i(S_{i-r})\ldots P_N(S_N) \sum_j \sum_q \lambda_{jq}\ P_{j,q;i,r}$$

$$= C\ P_1(S_1)\ldots P_j(S_j)\ldots P_i(S_{i-r})\ldots P_N(S_N)\ \lambda_{ir}$$

Thus the balance equation for the network becomes

$$\sum_{i=1}^{N} \left\{ \sum_{S_{i+0} \in A_0(S_i)} P(S=S_1,\ldots,S_N)\ R(S_i \to S_{i+0}) + \sum_r \sum_{S'_{i-r} \in A_{-r}(S_i)} P(S=S_1,\ldots,S_N)\ R(S_i \to S'_{i-r}) \right\}$$

$$= \sum_{i=1}^{N} \left\{ \sum_r \sum_{S_{i-r} \in B_{-r}(S_i)} P(S=S_1,\ldots,S_{i-r},\ldots,S_N)\ \lambda_{ir} + \sum_{S_{i-0} \in B_0(S_i)} P(S=S_1,\ldots,S_{i-0},\ldots,S_N)\ R(S_{i-0} \to S_i) \right\}$$

Both sides of this equation involve a sum over the service centers. We now show that these sums match term for term. Thus for fixed i:

$$\sum_{S_{i+0} \in A_0(S_i)} P(S=S_1,S_2,\ldots,S_N)\ R(S_i \to S_{i+0}) + \sum_r \sum_{S'_{i-r} \in A_r(S_i)} P(S=S_1,\ldots,S_N)\ R(S_i \to S'_{i-r})$$

$$= \sum_{r} \sum_{S_{i-r} \in B_{-r}(S_i)} P(S=S_1,\ldots,S_{i-r},\ldots,S_N)\,\lambda_{ir}$$
$$+ \sum_{S_{i-0} \in B_0(S_i)} P(S=S_1,\ldots,S_{i-0},\ldots,S_N)\; R(S_{i-0} \to S_i)$$

But substituting $P(S=S_1,\ldots,S_N)=C\,P_1(S_1)\ldots P_N(S_N)$ and dividing by $C\,P_1(S_1)\ldots P_{i-1}(S_{i-1})P_{i+1}(S_{i+1})\ldots P_N(S_N)$ this equation becomes:

$$\sum_{S_{i=0} \in A_0(S_i)} P_i(S_i)\; R(S_i \to S_{i+0}) + \sum_{r} \sum_{S'_{i-r} \in A_{-r}(S_i)} P(S_i)\; R(S_i \to S'_{i-r})$$
$$= \sum_{r} \sum_{S_{i-r} \in B_{-r}(S_i)} P_i(S_{i-r})\,\lambda_{ir} + \sum_{S_{i-0} \in B_0(S_i)} P_i(S_{i-0})\; R(S_{i-0} \to S_i)$$

But this is the equation we showed that the $P_i(S_i)$ satisfy if the i^{th} service station has the M⇒M property. Therefore we know this equation is satisfied which is the final step in showing that the network balance equation is satisfied by the proposed solution.

Q.E.D.

3.5 Network Hierarchies

We can generalize the definition of the $M \Rightarrow M$ property. We assume that the queueing model has certain input classes of customers and certain output classes of customers. We say that the queueing model has the $M \Rightarrow M$ property if the departure process for each output class of customer is Poisson when the arrival process for each input class of customer is Poisson. In an open network of queues model in which each service center has the $M \Rightarrow M$ property it is easy to show that the entire network has the $M \Rightarrow M$ property. In the network model, input and output classes of customers are now identified by pairs (i,r) where i identifies a service center and r a customer class. Considering a subnetwork as a service center, it is clear that we can treat a network of subnetworks. Service rates within a subnetwork may depend in various ways on the "state" of the subnetwork. We can also model service deletions for subnetworks. Thus, the number of customers in a subnetwork can be limited to some maximum. If a new arrival to this subnetwork would cause this maximum to be exceded then the customer procedes instantaneously to the next subnetwork as though it had just completed service in the subnetwork.

4. CONCLUSION

A sufficient condition for the departure process of a queueing system to be Poisson has been given along with examples of queueing systems which were not previously known to have this property. It was shown that a network of queueing systems which have Poisson departures when the arrival processes are Poisson has a product form solution. While both of these results are only sufficiency conditions, they appear to be very general. In particular, the author is not aware of any general results for networks of queues models which do not satisfy the $M \Rightarrow M$ property. The necessity of this condition is an open question.

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