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Research Report

MANDELBROT RANDOM BEADSETS AND BIRTH PROCESSES WITH INTERACTION

Jacques Peyrière

IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598

IBM Research Division
San Jose · Yorktown · Zurich

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ABSTRACT A class of birth processes arising from a work of B. Mandelbrot is investigated. It involves a stationary Markov chain in which the states are words made up of letters from a certain prescribed alphabet. In the transition from the word X_n, the n-th term of the chain, to the word X_{n+1}, each letter of the former is replaced by a random subword. The subwords replacing neighboring letters may be statistically dependent. Let $|X_n|$ be the length of X_n. Under suitable hypotheses there exists a $\lambda > 1$ such that, with probability 1, $\lambda^{-n} |X_n|$ has a finite, nonzero limit. In addition, given any word w, and denoting by $\sigma_n(w)$ the number of times that w appears as a subword of X_n, the ratio $\sigma_n(w)/|X_n|$ tends to a constant with probability 1. Among other applications, Mandelbrot uses these birth processes to construct a variety of fractal random curves, and he makes conjectures concerning their fractal Hausdorff dimension. Proofs of generalized forms of some of these conjectures are given.

1. *INTRODUCTION.*

Some domains in Euclidean spaces can be covered iteratively by nested tiles similar to the domains themselves. Classical examples are plane triangles and hypercubes in $\mathbb{R}^d$. Other examples, less familiar but more sophisticated, are domains having as boundary a fractal curve (Mandelbrot [1,2]). A beadset of order k is defined in Mandelbrot [4] as a collection of tiles of the k-th generation that nest with the following properties: (a) the beads are ordered linearly, and such that two consecutive beads share part of their boundaries, and (b) the beadset of order (k+1) is a subset of the beadset of order k. In the cases of interest, this subset is chosen at random, and the area of the k-th beadset tends to zero as $k \to \infty$, hence the intersection of the sequence of beadsets is a random curve.

Diverse shapes occurring in nature (polymer molecules, rivers, ...) may be modeled by such random curves. Beadsets are also an alternative to self avoiding random walks. A variety of such processes have been described by Mandelbrot [3,4]. He shows that each involves a generalized birth process imbedded geometrically, and such that individuals interact with their neighbors. He also expresses a number of conjectures concerning the beadset limits' fractal properties, notably their Hausdorff Besicovitch dimension.

The purposes of the present paper are (a) to offer a mathematical formulation covering many instances of this new birth process; (b) to prove Mandelbrot's conjectures concerning the Hausdorff Besicovitch dimension, and to extend them.

I owe much to stimulating discussions with B. Mandelbrot, for which I wish to thank him.

2. *NOTATIONS, DEFINITIONS AND RESULTS.*

When there is no risk of confusion, finite probability spaces will be referred to through their underlying sets.

2.1. *Couplings.*

A coupling between two finite probability spaces, (Ω_1,p_1) and (Ω_2,p_2), is a probability, c, on $\Omega_1 \times \Omega_2$ whose marginal distributions are p_1 and p_2. The coupling $p_1 \times p_2$ is said to be the null one.

If $(\Omega_j,p_j)_{1\le j\le n}$ is a sequence of finite probability spaces $(n\ge 2)$ and if for each $j=1,2,\ldots,n-1$ a coupling, c_j, between Ω_j and Ω_{j+1} is given, a probability p on $\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ is defined by

$$p(\{\omega_j\}) = \begin{cases} [\prod_{1\le j<n} c_j(\omega_j,\omega_{j+1})]\,[\prod_{1<j<n} p_j(\omega_j)]^{-1} \text{ if } \prod_{1<j<n} p_j(\omega_j) > 0 \\ 0 \text{ otherwise} \end{cases}$$

This new probability space is denoted $\Omega_1 *_{c_1} \Omega_2 *_{c_2} \cdots *_{c_{n-1}} \Omega_n$.

The projection π_j of $\Omega_1 *_{c_1} \cdots *_{c_{n-1}} \Omega_n$ on Ω_j has p_j as distribution. More generally, if k and ℓ are integers such that $1\le k<\ell\le n$, then the distribution of the projection of $\Omega_1 *_{c_1} \cdots *_{c_{n-1}} \Omega_n$ on $\Omega_k \times \ldots \times \Omega_\ell$ is the probability on $\Omega_k *_{c_k} \cdots *_{c_{\ell-1}} \Omega_\ell$. On the other hand, if π_k and π_ℓ are independent, so are π_i and π_j for $1\le i\le k$ and $\ell\le j\le n$.

A way to define a coupling between (Ω_1,p_1) and (Ω_2,p_2) is to consider a couple (d_1,d_2) of mappings from Ω_1 and Ω_2 to the same set such that the images of p_1 and p_2 by d_1 and d_2 respectively are identical. This defines a coupling, c:

$$c(\omega_1,\omega_2) = \begin{cases} 0 \quad \text{if } d_1(\omega_1) \ne d_2(\omega_2) \text{ or } p_1(d_1^{-1}(d_1(\omega_1))) = 0 \\ p_1(\omega_1)p_2(\omega_2)/p_1(d_1^{-1}(d_1(\omega_1))) \text{ otherwise.} \end{cases}$$

For instance, consider a sequence $(\Omega_j,p_j)_{1\le j\le n}$ of probability spaces and for each $j=1,2,\dots,n-1$ a coupling, c_j between Ω_j and Ω_{j+1}. Let k and ℓ be two integers such that $1\le k\le \ell\le n$ and set

$$(U_1,q_1) = \Omega_1 *_{c_1} \cdots *_{c_{\ell-1}} \Omega_\ell \qquad (= \Omega_1 \text{ if } \ell=1),$$

and $$(U_2,q_2) = \Omega_k *_{c_k} \cdots *_{c_{n-1}} \Omega_n \qquad (= \Omega_n \text{ if } k=n).$$

Let d_1 and d_2 be the projections of U_1 and U_2 on $\prod_{k\le j\le \ell} \Omega_j$. The couple (d_1,d_2) defines a coupling, c, between U_1 and U_2. The probability spaces $U_1 *_c U_2$ and $\Omega_1 *_{c_1} \cdots *_{c_{n-1}} \Omega_n$ are isomorphic.

2.2. *Beadsets.*

The following are given:

(i) a finite set $\underline{L}$ which shall be called alphabet,

(ii) for each $a\in\underline{L}$, a finite probability space (Ω_a,p_a),

(iii) for each couple, (a,b), of letters in $\underline{L}$, a subset $\mathcal{C}_{a,b}$ of the set of couplings between Ω_a and Ω_b.

A beadset is a couple $X = (\{a_j\}_{1\le j\le \nu}, \{c_j\}_{1\le j<\nu})$ where $\{a_j\}_{1\le j\le \nu}$ is a word, also denoted w(X), built with the alphabet $\underline{L}$ and each c_j is a coupling in $\underline{\mathcal{C}}_{a_j,a_{j+1}}$. One sets $|X| = \nu$ and defines a vector $L(X) = \{L_a(X)\}_{a\in\underline{L}}$ so

$$L_a(X) = \text{card}\,\{j;\ 1\le j\le \nu,\ a_j=a\}$$

The set of beadsets will be denoted **B**.

Let X and X′ be two beadsets. If w(X) ends with a and w(X′) begins with b, and if c is a coupling in $\underline{\mathcal{C}}_{a,b}$, one gets a new beadset, $X\#_c X'$, by concatenating w(X) and w(X′), c being the coupling not supplied by X or by X′.

As will be seen in 3.2, 3.3, and 3.5, by changing the alphabet a beadset is essentially a word. Nevertheless, the above construction is more convenient to describe the Mandelbrot geometric beadsets because it allows the use of smaller alphabets.

2.3. *Some Markov chains whose values are beadsets.*

The following are given in addition to the data of the above paragraph:

(i) for every $a \in \underline{L}$, a mapping, g_a, from Ω_a to $\mathbf{B}$,

(ii) for every a_1, $a_2 \in \underline{L}$, $c \in \underline{C}_{a_1,a_2}$ and $(\omega_1,\omega_2) \in \Omega_{a_1} *_c \Omega_{a_2}$ a coupling, $\xi(a_1, a_2, c, \omega_1, \omega_2)$ in $\underline{C}_{b_1,b_2}$, where b_1 is the last letter of $w(g_{a_1}(\omega_1))$ and b_2 the first letter of $w(g_{a_2}(\omega_2))$.

Now let us define a transition probability, Q, on $\mathbf{B}\times\mathbf{B}$. Consider a beadset $X = (\{a_j\}_{1\le j\le\nu}, \{c_j\}_{1\le j<\nu})$. Let Ω_X be the probability space $\Omega_{a_1} *_{c_1} \Omega_{a_2} *_{c_2} \cdots *_{c_{\nu-1}} \Omega_{a_\nu}$. Let p_X be the probability on Ω_X. Let ω be a point in Ω_X; it projects at ω_j on Ω_{a_j}. Let us set $Y_j = g_{a_j}(\omega_j)$, $\xi_j = \xi(a_j, a_{j+1}, c_j, \omega_j, \omega_{j+1})$ and $g_X(\omega) = Y_1 \#_{\xi_1} Y_2 \#_{\xi_2} \cdots \#_{\xi_{\nu-1}} Y_\nu$. Thus a mapping g_X from Ω_X to $\mathbf{B}$ has been defined. Then $Q(X,\bullet)$ is the image of the probability p_X by g_X.

Here we are studying the canonical Markov chain, $\{X_n\}_{n\ge 0}$, associated with Q. As usual, P_a stands for the probability on trajectories starting from a and E_a is the associated expectation.(*)

(*)A scheme for the simultaneous substitution of letters in a word has been considered by Lindenmayer [5] and extended by Lee and Rozenberg [6] to allow interaction between substitutes of neighboring letters. The present paper uses a procedure different from that of Lee and Rozenberg. The Lee-Rozenberg scheme was randomized by Jürgensen [7] in the absence of interaction.

2.4 *Some more notations and assumptions.*

A matrix, M, indexed by $\underline{L}\times\underline{L}$ is defined: its a-th column is $\int L(g_a(\omega))dp_a(\omega)$.

M is supposed irreducible. It means that there is no way of ordering $\underline{L}$ such that M has the form

$$\begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$$ where M_{11} and M_{22} are square matrices.

Let λ be the maximum of the moduli of the eigenvalues of M. By the Perron-Frobenius theorem, λ is a simple eigenvalue of M and that other eigenvalues with moduli λ, if any, are simple too. Let e and f be eigenvectors of M and M*, associated with λ and such that $\langle f,e\rangle = 1$. It is known that every component of e and f is positive. Set $N = ef^*$.

If there exists a positive integer, n, such that any entry of M^n is positive, then M is said to be primitive and λ is its only eigenvalue of modulus λ. Facts about nonnegative matrices can be found in [8].

Two more hypotheses are made throughout this paper.

(i) $\lambda > 1$,

(ii) there exists a positive integer, ℓ, such that if $X = (\{a_j\}_{1\le j\le \ell+1}, \{c_j\}_{1\le j\le \ell})$ is a beadset of length $\ell+1$, then the projections of Ω_X onto Ω_{a_1} and $\Omega_{a_{\ell+1}}$ are independent.

The above condition if fulfilled with $\ell=1$ when the only coupling considered is the null one.

The sum of absolute values of the components of a vector, V, will be denoted $|V|$. The relation $V\ge 0$ means that every component of V is nonnegative.

Let ρ be the maximum length of words $\{g_a(\omega);\ a\in\underline{L},\ \omega\in\Omega_a\}$. Let E be the set of finite sequences of elements of $\{1,2,\ldots,\rho\}$. E is a tree in a natural way. (The root is the empty sequence.) Set $K = \{1,2,\dots,\rho\}^{N^*}$. If $(j_1, \dots, j_m) \in E$, $I(j_1,\dots,j_m)$ is the cylinder $\{\{i_n\}_{n\geq 1} \in K;\ i_1=j_1,\dots,i_m=j_m\}$. If $x \in K$, $I_m(x)$ is the set $I(j_1,\dots,j_m)$ that contains x.

With each trajectory $\{X_n\}_{n\geq 0}$ of the Markov chain is associated a function, ε, on E assuming only values 0 and 1. This function is defined by induction as follows.

$\varepsilon(j) = 1$ if $j \leq |X_1|$; otherwise, $\varepsilon(j) = 0$.

If $\varepsilon(j_1,\dots,j_m)=0$, then $\varepsilon(j_1,\dots,j_m,j)=0$ for every j.

If $\varepsilon(j_1,\dots,j_m)=1$, set

$$\nu = \sum_{i_1<j_1} \sum_{i_2,\dots,i_m} \varepsilon(i_1,\dots,i_m) + \sum_{i_2<j_2} \sum_{i_3,\dots,i_m} \varepsilon(j_1,i_2,\dots,i_m) + \dots$$

$$+ \sum_{i_{n-1}<j_{n-1}} \sum_{i_n} \varepsilon(j_1,\dots,j_{n-2},i_{n-1},i_n) + \sum_{i_n\leq j_n} \varepsilon(j_1,j_2,\dots,j_{n-1},i_n).$$

Let ν' be the number of letters to which the ν-th letter of $w(X_n)$ gives birth in $w(X_{n+1})$. Then $\varepsilon(j_1,\dots,j_m,j) = 1$ if $j\leq\nu'$; 0 otherwise.

It is clear that $\varepsilon^{-1}(1)$ is a sub-tree of E and that $|X_n| = \sum_{j_1,\dots,j_n} \varepsilon(j_1,\dots,j_n)$.

2.5. *Results.*

Theorem 1.1. For every $a \in \underline{L}$, $\lambda^{-n}L(X_n)$ tends P_a-almost surely to a limit, Z.

Theorem 1.2. For every $a \in \underline{L}$ we have $P_a(Z=0)=0$.

Theorem 1.3. If the matrix M is primitive, then $\lambda^{-n}L(X_n)$ tends to Z, P_a-almost surely. In both cases the convergence also takes place in quadratic mean.

Corollary 1. For every $a \in \underline{L}$, P_a-almost surely, $0 < \inf \lambda^{-n}|X_n| \leq \sup \lambda^{-n}|X_n| < \infty$.

Corollary 2. Let x be a beadset and $\sigma_n(x)$ be the frequency of x as a sub-beadset of X_n, ($\sigma_n(x)=0$ if $|X_n|<|x|$). Then, if M is primitive, $\sigma_n(x)$ tends to a constant, P_a-almost surely, for every $a \in L$.

The next theorem is useful to compute the Hausdorff-Besicovitch dimension in cases where beadsets are geometrically realized.

Theorem 2. There exists a random probability, μ, on K, carried by the adherence of $\varepsilon^{-1}(1)$ and such that $\lim_{n\to\infty} [\log \mu(I_n(x))]/n = -\log \lambda$ almost surely for μ-almost every x.

3. *PROOFS.*

Theorems 1.1–1.3 are proved in sections 3.1–3.4, the corollaries in section 3.5, and Theorem 2 in section 3.6.

3.1. *Proof of Theorem 1.1.*

Let $X = (\{a_j\}_{1\le j\le\nu}, \{c_j\}_{1\le j<\nu})$ be a beadset. Then

$$E_{p_X}[L(g_X)] = \int \sum_{1\le j\le\nu} L(g_{a_j}(\omega_j))dp_X(\omega) = \sum_{1\le j\le\nu} E_{p_{a_j}}[L(g_{a_j})] = ML(X).$$

Therefore,

$$E(NL(X_{n+1})\,|\,X_0,X_1,\dots,X_n) = \lambda NL(X_n).$$

Thus $\lambda^{-n}NL(X_n)$ is a martingale. Its components are nonnegative, so it converges P_{X_0}-almost surely.

3.2. *Estimation of second moments.*

First, no beadset is recurrent. Indeed, if X were, then, with probability 1, starting from $X_0=X$, there would exist $n\ge1$ such that $X_n=X$. So, for every letter, a, that may occur in the descent of X_0 (i.e., for every letter since M is irreducible), $|g_a(\omega)|$ would be 1 whatever ω may be. So M would be markovian and λ would be 1. So, P_{X_0}-almost surely, $|X_n|$ tends to infinity.

Let $\mathbf{B}_k$ be the set of beadsets of length k that occur with a nonzero probability as a sub-beadset of one of the terms of the sequence $\{X_n\}_{n\geq 0}$, when X_0 consists of a single letter (which does not matter because of the irreducibility of M).

When $k\geq 2$, set $\underline{L}_k = \mathbf{B}_k \cup \mathbf{B}_{k-1}$. With each beadset $X = (\{a_j\}_{1\leq j\leq \nu}, \{c_j\}_{1\leq j<\nu})$ of length larger than k−1 is associated a sequence, $X^{(k)}$, of $\nu-k+2$ elements of $\underline{L}_k$: if $1\leq j<\nu-k+2$, the jth element of $X^{(k)}$ is the beadset $(\{a_{j+i}\}_{0\leq i\leq k-1}, \{c_{j+i}\}_{0\leq i<k-1}) \in \mathbf{B}_k$, the last term of $X^{(k)}$ is $(\{a_{\nu-k+1+i}\}_{1\leq i\leq k-1}, \{c_{\nu-k+1+i}\}_{1\leq i<k-1}) \in \mathbf{B}_{k-1}$.

Together with beadsets on the alphabet $\underline{L}$, beadsets on the alphabet $\underline{L}_k$ are going to be considered. The latter will be referred to as k-beadsets in the sequel. But before speaking of k-beadsets, it is necessary to define the sets $\underline{C}^k_{x,x'}$ of admissible couplings when x and x′ are in $\underline{L}_k$. These sets are defined in such a way that the class of k-beadsets contains the $X^{(k)}$'s.

A probability space has been previously associated with each element of $\underline{L}_k$. If $x\in\mathbf{B}_k$ and $x'\in\underline{L}_k$, and if there exists a y in $\mathbf{B}_{k-1}$ such that x ends by y and x′ begins by y, then $\underline{C}^k_{x,x'}$ has as its only element the coupling defined by the projections of Ω_x and $\Omega_{x'}$ onto Ω_y. Otherwise, $\underline{C}^k_{x,x'}=\phi$.

Now, if $|X_0| \geq k-1$, with the chain, $\{X_n\}_{n\geq 0}$, is associated a random sequence, $\{X_n^{(k)}\}_{n\geq 0}$, of k-beadsets. It will be a Markov chain of k-beadsets with a transition defined as in 2.3, if generation mappings are defined as follows:

if $x \in \mathbf{B}_{k-1}$, then $g_x^k(\omega) = (g_x(\omega))^{(k)}$,

if $x \in \mathbf{B}_k$, then $g_x^k(\omega)$ is the k-beadset consisting in the sequence of sub-beadsets of $g_x(\omega)$, whose length is k and whose first letter is generated by the first letter of x.

With these generation mappings is associated a square matrix, M'_k, indexed by $\underline{L}_k \times \underline{L}_k$. According to the splitting, $\underline{L}_k = \mathbf{B}_k \cup \mathbf{B}_{k-1}$, of $\underline{L}_k$ the matrix M'_k has the form

$$M'_k = \begin{pmatrix} M_k & T_k \\ 0 & S_k \end{pmatrix},$$

where S_k is a column-markovian matrix.

The matrix M_k is obtained by replacing each element m_{ij} in M by a matrix whose sum of elements of each column is m_{ij}. From this it is easy to deduce that λ is an eigenvalue of M_k and that moduli of other eigenvalues are less than λ (in fact M_k is irreducible, but this is not needed in this paragraph).

If X is a beadset of length larger than k−1, let $q_{k-1}(X)$ be the beadset of length k−1 by which X ends and let $L^k(X)$ be the vector, indexed by $\mathbf{B}_k$, whose component $L^k_x(X)$ is the number of times that x appears as a sub-beadset of X.

Let P_k be the matrix of the projection of $\mathbb{R}^{\underline{L}_k}$ onto $\mathbb{R}^{\mathbf{B}_k}$. Suppose $|X_0| \geq k-1$, then

$$E[L^k(X_n)] = P_k M_k'^n \begin{pmatrix} L^k(X_0) \\ L^{k-1}(q_{k-1}(X_0)) \end{pmatrix} .$$

Set, when $x \in \underline{L}$,

$$V^1_x = E_{p_x}[L(g_x)L(g_x)^*] - E_{p_x}[L(g_x)]E_{p_x}[L(g_x)^*]$$

and, when $x \in \mathbf{B}_k$ (k>1),

$$V^k_x = E_{p_x}[L(g_{x_1})L(g_{x_k})^* + L(g_{x_k})L(g_{x_1})^*]$$
$$- E_{p_{x_1}}[L(g_{x_1})]E_{p_{x_k}}[L(g_{x_k})^*] - E_{p_{x_k}}[L(g_{x_k})]E_{p_{x_1}}[L(g_{x_1})^*]$$

(x_1 is the first letter of x, x_k the last one.) Then

$$E_{p_X}[L(g_X)L(g_X)^*] = M\, L(X)L(X)^*M^* + \sum_{x\in\underline{L}} V_x{}^1 L_x(X)$$
$$+ \sum_{1<k\leq\ell} \sum_{x\in\underline{L}_k} V_x{}^k L_x{}^k(X).$$

An expression such that $\Sigma_{x\in\underline{L}_k} V_x{}^k L_x{}^k(X)$ will be denoted $V^k \bullet L^k(X)$. Let P_1 be the identity matrix of dimension card $\underline{L}$. The matrix M is renamed M_1'

Suppose that X_0, the starting state of the chain, has a length larger than $\ell-1$ and is one of the possible descendants of a single letter. Set $\Lambda_1 = L(X_0)$ and, for $2 \le k \le \ell$,

$$\Lambda_k = \begin{pmatrix} L^k(X_0) \\ L^{k-1}(q_{k-1}(X_0)) \end{pmatrix}$$

and $C_n = E_{X_0}[L(X_n)L(X_n)^*]$. The above formula gives

$$C_n = M^n C_0 M^{*n} + \sum_{1 \le k \le \ell} \sum_{0 \le j < n} M^{(n-j-1)}(V^k \bullet P_k M_k'^{j} \Lambda_k) M^{*(n-j-1)}.$$

When M is primitive, then exactly as in [9], one gets

$$\lambda^{-2n} C_n = N[C_0 + \lambda^{-2} \sum_{1 \le k \le \ell} V^k \bullet P_k (I - \lambda^{-2} M'_k)^{-1} \Lambda_k] N^* + O(\alpha^n)$$

where $0 < \alpha < 1$.

When M is merely irreducible, one can only infer that $\lambda^{-2n} C_n$ has a finite number of limit values.

3.3. *Proof of Theorem 1.3.*

If $|X_0| \ge \ell$, according to the results of the preceding paragraph, the same proof as in [6] works: $\lambda^{-n} L(X_n)$ tends to Ze, P_{X_0}-almost surely and in quadratic mean.

If X_0 consists in a single letter, since $|X_n|$ tends almost surely to infinity, the following formula defines a finite stopping time:

$$\tau = \inf[n \ge 1;\ |X_n| \ge \ell].$$

X_τ can assume only a finite number of values. So it results from the above facts that $\lambda^{-n} L(X_n)$ converges P_{X_0}-almost surely and in quadratic mean.

3.4. *Proof of Theorem 1.2.*

As in the preceding paragraph, $\sup_n E_a(|X_n|^2)$ is finite when $a \in \underline{L}$. So the martingale $\lambda^{-n} NL(X_n)$ converges in $L^2(P_a)$ and, therefore, $P_a(Z \ne 0) > 0$.

For technical reasons, another Markov chain closely related to the previous one is needed for the work done in this section. As a matter of fact, it is necessary to keep track of the way one passes from X_n to X_{n+1}.

Let Ω be the disjoint union of the family of sets $\{\Omega_X\}_{X\in \mathbf{B}}$. Let $\mathbf{B}^\sim$ be the set $\mathbf{B}\times\Omega$. A transition probability, $Q^\sim$, is to be defined on $\mathbf{B}^\sim$. $Q^\sim((X,\zeta),\bullet)$ will not depend on ζ.

Given a beadset, X, a mapping, $\tilde{g}_X$, from Ω_X to $\mathbf{B}^\sim$ is defined as follows: $\tilde{g}_X(\omega) = (g_X(\omega),\omega)$. Then $Q^\sim((X,\zeta),\bullet)$ is the image of the probability, p_X, by $\tilde{g}_X$.

The image of $Q^\sim((X,\zeta),\bullet)$ by the projection of $\mathbf{B}^\sim$ onto $\mathbf{B}$ is $Q(X,\bullet)$. Therefore, if $\{Y_n\}_{n\geq 0} = \{X_n,\zeta_n\}_{n\geq 0}$ is the canonical Markov chain with stationary transition $Q^\sim$, then $\{X_n\}_{n\geq 0}$ is a Markov chain with stationary transition Q.

Consider the Markov chain $\{Y_n\}_{n\geq 0}$ and suppose that $X_0 = (\{a_j\}_{1\leq j\leq \nu}, \{c_j\}_{1\leq j<\nu})$. For $j = 1,2,\ldots,\nu$ sequences, $\{Y^j_n\}_{n\geq 0} = \{(X^j_n, \zeta^j_n)\}_{n\geq 0}$, of elements of $\mathbf{B}^\sim$ are defined by induction:

$X^j_0 = (\{a_j\}, \phi)$,

ζ^j_0 is arbitrarily chosen,

ζ^j_{n+1} is the projection of ζ_{n+1} on $\Omega_{X^j_n}$,

$X^j_{n+1} = g_{X^j_n}(\zeta^j_{n+1})$.

For each j, the process $\{Y^j_n\}_{n\geq 0}$ is a Markov chain with transition $Q^\sim$. The word $w(X_n)$ results from concatenation of words $\{w(X^j_n)\}_{1\leq j\leq \nu}$. So

$$|X_n| = \Sigma_{1\leq j\leq \nu}|X^j_n|.$$

Let $Z_1e, \ldots, Z_\nu e$ be the limits of $\lambda^{-n}NL(X^1_n), \ldots, \lambda^{-n}NL(X^\nu_n)$. P_{Y_0}-almost surely

$$Z = Z_1 + Z_2 + \ldots + Z_\nu.$$

The distribution of Z_j with respect to P_{X_0} is the same as the one of Z with respect to P_{a_j}. If $j_1-j_2>\ell$, then Z_{j_1} and Z_{j_2} are independent.

Set $\Lambda_a = P_a(Z=0)$ and denote $x_1,\dots,x_r$ the elements of $\underline{L}$ in such a way that $\Lambda_{x_1} \geq \Lambda_{x_2} \geq \dots \geq \Lambda_{x_r}$.

For each $i \in \{1,2,\dots,r\}$ and for every $\omega \in \Omega_{x_i}$, let $m_i(\omega)$ be the largest index of letters appearing in $w(g_{x_i}(\omega))$.

Starting from x_j, by conditioning first with respect to Y_1, one can get

$$(*) \quad \Lambda_{x_j} \leq \int \Lambda_{x_{m_j(\omega)}} \, dp_{x_j}(\omega).$$

If $j<r$, then $\sup \{ \text{ess sup } m_k;\ 1 \leq k \leq j \} > j$ (otherwise M would be reducible). So, using (*) and the fact that the Λ's are in nonincreasing order, one can show that $1 \leq j < r$ and $\Lambda_{x_1} = \dots = \Lambda_{x_j}$ imply $\Lambda_{x_1} = \dots = \Lambda_{x_{j+1}}$. Therefore $\Lambda_{x_1} = \Lambda_{x_2} = \dots = \Lambda_{x_r} = \alpha$.

It remains to be proven that $\alpha=0$.

First, one may suppose that there is an $a \in \underline{L}$ and an $\omega \in \Omega_a$ such that $p_a(\omega) \neq 0$ and $|(g_a(\omega)| > \ell$. (Indeed, if it is not so, one may consider the process at times multiples of a certain integer.)

Then, starting from a, assuming that $\zeta_1=\omega$, one has $\lambda Z = Z_1 + \dots + Z_{|g_a(\omega)|}$. At least two of these variables are independent. The probability of vanishing of each of them is α. Thus one gets $\alpha \leq (1-t)\alpha + t\alpha^2$, with $t = p_a(\omega) > 0$. This implies $\alpha=0$ or $\alpha=1$. But the expectation of Z is positive since the martingale converges in quadratic mean. So $\alpha=0$.

Remark. Set $F_a(t) = E_a(\exp -tZ)$ and $F(t) = \{F_a(t)\}_{a \in \underline{L}}$. By a similar argument, it can be shown that, for t large enough, $F(\lambda t) \leq M' F(t)$, where M' is a sub-markovian matrix. So Z has some finite moments of negative order with respect to any P_a.

3.5. *Proofs of the corollaries.*

Corollary 1 results from Theorem 1.1 and the fact that every component of e is positive.

Corollary 2 will result from application of Theorem 1.3 to k-beadsets.

First it will be shown that for every $k \geq 2$, the matrix M_k is primitive. This results from several remarks.

For each $x \in \mathbf{B}_k$, let $\tau(x)$ be the smallest integer such that there exists $a \in \underline{L}$ such that, with positive P_a-probability, $X_{\tau(x)}$ contains x as a sub-beadset. Set $\tau = \sup\{\tau(x); x \in \mathbf{B}_k\}$. Now if x is any element of $\mathbf{B}_k$, there exists $a \in \underline{L}$ such that, with positive P_a-probability, X_τ contains x as a sub-beadset.

In view of Theorem 1.3, there exists an integer m with the following property: for any $a \in \underline{L}$, with positive P_a-probability, every component of $L(X_m)$ is greater than $(\ell+1)$ card $\mathbf{B}_k$. So, with positive P_a-probability, for any $a \in \underline{L}$, $X_{m+\tau}$ contains simultaneously as sub-beadsets every element of $\mathbf{B}_k$. The same is evidently true when a is any element of $\mathbf{B}_k$. Taking the expectation of $L(X_{m+\tau})$ shows that $M_k^{m+\tau}$ is a positive matrix.

So Theorem 1.3 can be applied to chains of k-beadsets whose beads are in $\mathbf{B}_k$.

Let the oscillation of a sequence, $\{W_n\}_{n\geq 0}$, of vectors be the number

$$\lim_{m\to\infty} \sup_{n_1,n_2\geq m} |W_{n_1}-W_{n_2}|.$$

If $|X_0| \leq k-1$, then the Doob maximal theorem applied to the square integrable martingale, $\lambda^{-n}NL(X_n)$, shows that $E_{X_0}(\sup_{n\geq 0} \lambda^{-n}|X_n|)$ is finite.

Now suppose $X_0=a$. Choose an integer m and condition with respect to the value of X_m.

If $|X_m| \le k-1$, then the oscillation of $\lambda^{-n}L(X_n)$ is majorized by

$$2\lambda^{-m} \sup_{j\ge 0} \lambda^{-j} |X_{m+j}|.$$

If $|X_m| \ge k$, then if $n \ge m$, write $L^k(X_n) = L'(X_n) + L''(X_n)$, where in $L''(X_n)$ are taken in account sub-beadsets of X_n, whose length is k and whose first letter is in the descent of one of the last (k−1) letters of X_m. By Theorem 1.3, the oscillation of $\lambda^{-n}L'(X_n)$ is almost surely zero. The oscillation of $\lambda^{-n}L''(X_n)$ is majorized by $2\lambda^{-m} \sup_{j\ge 0} \lambda^{-j} |X'_j|$, where X'_j is the sub-beadset of X_{n+j} generated by the last (k−1) letters of X_m. This proves that there exists a constant, γ, such that for any $m \ge 1$, the conditional expectation, given X_m, of the oscillation of $\lambda^{-n}L^k(X_n)$ is less than $\gamma\lambda^{-m}$. This proves the almost sure convergence of $\lambda^{-n}L^k(X_n)$. A slight modification of the argument shows that the limit is almost surely a nonzero multiple of the eigenvector of M_k corresponding to λ.

3.6. *Proof of Theorem 2.*

Let $(j_1,\dots,j_n)$ be a point of E. If $\varepsilon(j_1,\dots,j_n) = 0$, set $Z(j_1,\dots,j_n) = 0$, else consider the Markov chain, $\{X_m(j_1,\dots,j_n)\}_{m\ge 0}$, whose elements are the sub-beadsets of $\{X_{n+m}\}_{m\ge 0}$ to which the letter associated with $(j_1,\dots,j_n)$ in X_n gives birth, and set

$$Z(j_1,\dots,j_n)e = \lim_{m\to\infty} \lambda^{-m} NL(X_m(j_1,\dots,j_n)).$$

Then, almost surely,

$$\sum_j Z(j) = \lambda Z \text{ and}$$

$$\sum_j Z(i_1,\dots,i_n,j) = \lambda Z(i_1,\dots,i_n).$$

So, almost surely, there is a unique measure, μ, on K such that

$$\mu(I(j_1,\dots,j_n)) = \lambda^{-n} Z^{-1} Z(j_1,\dots,j_n).$$

Now let α be a real number

$$E_a\left(\int Z[Z\lambda^n \mu(I_n(x))]^\alpha d\mu(x) \mid X_0, X_1, \dots, X_n \right) = \lambda^{-n} \Sigma_{1\le j\le |X_n|} E_{a_j}(Z^{1+\alpha})$$

where $a_1,\dots,a_{|X_n|}$ are the letters of $w(X_n)$). So

$$\sup_n E_a\left(\int Z[Z\lambda^n\mu(I_n(x))]^\alpha d\mu(x)\right)$$

is finite if $-1 \leq \alpha \leq 1$.

The Borel-Cantelli lemma gives

$$1/n^2 \leq \lambda^n\mu(I_n(x)) \leq n^2Z^{-2},$$

almost surely, μ-almost everywhere beyond a certain rank. This proves the theorem.

4. *Examples.*

4.1. If the only coupling used is the null one, then the process $L(X_n)$ is a Galton-Watson one.

One can interpret the construction in a different way. A context-free grammar is given. To generate a new word from an old one, each letter is replaced at random according to possible substitutions. Substitutions of different letters of the same word are independent, so are the different steps. A letter is replaced always according to the same law. Theorems 1.1–1.3 give some information about the almost sure structure of generated words.

4.2. The case considered in [10] can be handled by the previous formalism. $\underline{L} = \{a,b\}$: a stands for a triangle whose exit side follows immediately its entry side (according to the orientation of its boundary), b stands for the other triangles. Ω_a and Ω_b are $\{0,1\} \times \{0,1\}$ endowed with the product of the same probability on each factor ($P(0)=p$).

The only coupling between Ω_{a_1} and Ω_{a_2} to be considered is the one that is defined by the second projection of Ω_{a_1} onto $\{0,1\}$ and the first one of Ω_{a_2} onto $\{0,1\}$. Condition 2.4(ii) is fulfilled with $\ell=2$. In this case, a beadset is merely a word. Generation mappings are so defined:

$$g_a(0,0) = \text{abb},\ g_a(0,1) = \text{aaa},\ g_a(1,0) = \text{a},\quad g_a(1,1) = \text{bba},$$

$$g_b(0,0) = \text{aab},\ g_b(0,1) = \text{b},\quad g_b(1,0) = \text{bbb},\ g_b(1,1) = \text{baa}.$$

Then

$$M = \begin{pmatrix} 1+2p(1-p) & 2p^2+2(1-p)^2 \\ 2p^2+2(1-p)^2 & 1+2p(1-p) \end{pmatrix} \text{ and } \lambda = 3-2p(1-p).$$

Let K_n be the reunion of the elements of X_n and $\gamma=\cap K_n$.

γ is covered by $|X_n|$ triangles out of those obtained after n dissections of the original one. So the Hausdorff Besicovitch dimension of γ is not greater than lim inf $(1/n) \log_2 |X_n|$; that is $\log_2[3-2p(1-p]$ with probability 1.

The converse inequality results from Theorem 2. Indeed, with probability 1, the measure μ has a natural image, $\tilde{\mu}$, carried by γ. Moreover, by Theorem 2, this measure $\tilde{\mu}$ is, with probability 1, such that, for $\tilde{\mu}$-almost every x,

$$\lim \log \tilde{\mu}(T_n(x))/\log \operatorname{diam}(T_n(x)) = \log_2[3-2p(1-p)]$$

where $T_n(x)$ is the triangle of the n-th generation that contains x (the ambiguity of this definition does not matter for it can be shown that the set of litigious points does not bear any mass of $\tilde{\mu}$). According to [11], p. 144, or to [12], this implies that the dimension of γ is not less than $\log_2[3-2p(1-p)]$.

4.3. The following construction is also due to Mandelbrot [4]. A random sequence $\{X_n\}_{n>0}$ is constructed. X_n itself is a sequence of squares associated with the grid $2^{-n}Z^2$ such that consecutive elements share one side.

Boundaries of squares are oriented counter-clockwise. Let us start with Q_0, the unit square $[0,1]\times[0,1]$, and choose on its boundary entry and exit sides. One half of the entry side, say A, is selected at random (both halves are equiprobable). One half, B, of the exit side is selected independently in the same way. One of the segments, say C, joining the center of Q_0 to the middle of the edges is chosen at random (all these segments are equiprobable) independently of the previous choices. Now there is only one sequence of distinct squares, associated with $2^{-1}Z^2$, contained in Q_0 and such that

(i) two consecutive squares share one side different from C,

(ii) A is a side of the first square,

(iii) B is a side of the last square.

Each new square has an entry and an exit side. Thus the above construction can be performed for each of these new squares with the following restriction: if A is the exit side of the square Q_1 and the entry side of Q_2, then only one half of A is selected and used to perform the construction in Q_1 and Q_2. This procedure can be repeated, and the sequence $\{X_n\}$ is as defined.

Squares are of three types:

(i) the exit side follows immediately the entry side,

(ii) the exit side is opposite to the entry side,

(iii) the exit side precedes immediately the entry side.

Let U_ν be the set $\{0,1,\ldots,\nu-1\}$ endowed with the uniform probability. If a segment is oriented, choosing one of its halves is simply choosing a point in U_2.

If a segment is a side common to two squares, the orientations induced on it are opposite. So this situation is described as follows. $\underline{L} = \{a,b,c\}$ (each letter corresponds to a type of square). $\Omega_a = \Omega_b = \Omega_c = U_2 \times U_4 \times U_2$. U_4 describes the cutout.

Let a_1 and a_2 be two elements of $\underline{L}$. $\underline{C}_{a_1,a_2}$ has its only element defined by the couple (d_1,d_2) where d_1 is the projection of Ω_{a_1} on its last factor and $1-d_2$ is the projection of Ω_{a_2} on its first factor. So a beadset is simply a word. Condition 2.4(ii) is fulfilled with $\ell=2$. The functions g are defined as follows.

$$g_a(0,j,0) = \begin{cases} ab & \text{if } j\neq 0 \\ baac & \text{if } j=0 \end{cases} \qquad g_a(0,j,1) = \begin{cases} bab & \text{if } j=0,1 \\ aca & \text{if } j=2,3 \end{cases}$$

$$g_a(1,j,0) = a \qquad g_a(1,j,1) = \begin{cases} ba & \text{if } j\neq 1 \\ caab & \text{if } j=1 \end{cases}$$

$$g_b(0,j,0) = \begin{cases} \text{bac} & \text{if } j=0,1 \\ \text{acb} & \text{if } j=2,3 \end{cases} \qquad g_b(0,j,1) = \begin{cases} \text{bb} & \text{if } j\neq 3 \\ \text{acca} & \text{if } j=3 \end{cases}$$

$$g_b(1,j,0) = \begin{cases} \text{bb} & \text{if } j\neq 1 \\ \text{caac} & \text{if } j=1 \end{cases} \qquad g_b(1,j,1) = \begin{cases} \text{bca} & \text{if } j=0,3 \\ \text{cab} & \text{if } j=1,2 \end{cases}$$

$$g_c(0,j,0) = \begin{cases} \text{bc} & \text{if } j\neq 3 \\ \text{accb} & \text{if } j=3 \end{cases} \qquad g_c(0,j,1) = \text{c}$$

$$g_c(1,j,0) = \begin{cases} \text{bcb} & \text{if } j=0,3 \\ \text{cac} & \text{if } j=1,2 \end{cases} \qquad g_c(1,j,1) = \begin{cases} \text{cb} & \text{if } j=0 \\ \text{bcca} & \text{if } j\neq 0 \end{cases}$$

Then

$$M = \frac{1}{4} \begin{pmatrix} 5 & 3 & 1 \\ 3 & 5 & 3 \\ 1 & 3 & 5 \end{pmatrix} \text{ and } \lambda = \frac{11 + \sqrt{73}}{8} .$$

Let K_n be the reunion of the elements of X_n and $\gamma = \cap K_n$. Thus the Hausdorff-Besicovitch dimension of γ is $\mathrm{Log}_2(11+\sqrt{73})-3$.

4.4. The following example shows that if the assumption 2.4(ii) is dropped, $P_a(Z=0)=0$ need not hold. Set $\underline{L}=\{a\}$, $\Omega_a = \{-1,1\}$ with uniform probability, $\mathcal{C}_{a,a} = \{\text{identity}\}$, $g_a(-1) =$ a and $g_a(1) =$ aa. Then $\lambda = 3/2$ and $\lambda^{-n}|X_n| = |X_0| \Pi_{1\le j\le n} [1 + (\varepsilon_j/3)]$, where ε's are independent random variables assuming values ± 1 with equal probabilities. With probability 1, $\lambda^{-n}|X_n|$ tends to zero.

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