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ON SYMMETRIC BIMATRIX GAMES

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ON SYMMETRIC BIMATRIX GAMES *

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ABSTRACT: This paper is devoted to formulating the concept of a symmetric bimatrix game and to developing results analogous to those for symmetric matrix games. We show that, given any bimatrix game or two-person non-zero-sum game, there exist two equivalent symmetric games, with the property that symmetric equilibrium strategies for the symmetrized games yield equilibrium strategies for the original game. An adaptation of Nash's proof of the existence of equilibrium strategies for any bimatrix game is used to show that any symmetric bimatrix game with entries which are real numbers does admit of a symmetric equilibrium point. Finally, the question of whether this result on the existence of symmetric strategies holds if the matrices and strategy vectors are in an arbitrary ordered field is answered in the affirmative through the use of a metamathematical argument. The argument is sufficiently general to encompass many other results in game theory.

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INTRODUCTION

The first part of this paper is devoted to formulating the concept of a symmetric bimatrix game, and to developing results analogous to those for symmetric matrix games. It is rather surprising that, although symmetric matrix games have been extensively examined [see, for example, 1], there seems to have been almost no formal attention paid to the bimatrix case.

Briefly, a symmetric bimatrix game is a game in which the two payoff matrices are square, and each is the transpose of the other. For any bimatrix game, we give two methods of "symmetrizing the game", with the property that any symmetric equilibrium point for either of the symmetrized games yields an equilibrium point for the original game. Further, by an easy adaptation of Nash's proof of the existence of equilibrium strategies for general bimatrix games [2], we show that any symmetric bimatrix game with entries which are real numbers admits symmetric equilibrium strategies. This result is stated in the form of a simple property about any square matrix.

Finally, in Section 3, we reach the second principal point of interest of the paper. We raise the question of whether our theorem on the existence of symmetric strategies holds if the matrices and strategy vectors are required to be in an arbitrary ordered field. Traditionally, such questions have always arisen for the basic theorems on linear inequalities and games, whenever the concept of continuity has been invoked to provide proofs even though it plays no role in the statements of these theorems, and, in each instance, "rational proofs" have been found subsequently, which prove the validity of the results in the context of arbitrary ordered fields.

The present instance is no exception, for recently J. Howson and C. E. Lemke have developed an ingenious algorithm for finding symmetric equilibrium points of symmetric games, valid over any ordered field, and thus proving their existence as a by-product.

But there is a metamathematical route that can also be followed to establish existence, which we describe in Section 3, and which is applicable to many situations of the kind we consider here. We essentially show that proving the theorem in the real case is sufficient to establish it generally.

1. SYMMETRIZING A BIMATRIX GAME

A bimatrix game is defined by two real p by q payoff matrices, $A = (a_{ij})$ and $B = (b_{ij})$: if player 1 chooses $i \in \{1, \ldots, p\}$ and player 2 chooses $j \in \{1, \ldots, q\}$, then 1 receives a_{ij} and 2 receives b_{ij}. We shall denote mixed strategies for 1 and 2 by u and v, probability vectors of dimension p and q, respectively. An equilibrium point for this bimatrix game is a pair of mixed strategies $(\overline{u}, \overline{v})$ such that there exist real numbers λ and μ, with

$$(1.1) \qquad \sum_{j=1}^{q} a_{ij} \bar{v}_j \leq \lambda \qquad \text{for } i = 1, \ldots, p;$$

$$(1.2) \qquad \sum_{i=1}^{p} \bar{u}_i \left(\sum_{j=1}^{q} a_{ij} \bar{v}_j\right) = \lambda ;$$

$$(1.3) \qquad \sum_{i=1}^{p} \bar{u}_i b_{ij} \leq \mu \qquad \text{for } j = 1, \ldots, q;$$

$$(1.4) \qquad \sum_{j=1}^{q} \left(\sum_{i=1}^{p} \bar{u}_i b_{ij}\right) \bar{v}_j = \mu .$$

A symmetric bimatrix game is one in which the payoff matrices A and B are square, and $B = A^T$. We shall use the letter C to denote the payoff matrix for the first player in a symmetric bimatrix game. A symmetric equilibrium point for a symmetric bimatrix game is an equilibrium point in which the mixed strategies for the two players are the same. Letting $\bar{x}$ denote the symmetric equilibrium strategy; ν, the common value at this equilibrium point to both players; and n, the order of C; conditions (1.1)-(1.4) reduce to

$$(1.5) \qquad \sum_{j=1}^{n} c_{ij} \bar{x}_j \leq \nu \qquad \text{for } i = 1, \ldots, n$$

with

$$(1.6) \qquad \sum_{i=1}^{n} \bar{x}_i \sum_{j=1}^{n} c_{ij} \bar{x}_j = \nu .$$

An alternate way of stating (1.6) is

$$(1.6a) \qquad \sum_{j=1}^{n} c_{ij} \bar{x}_j < \nu \implies \bar{x}_i = 0 .$$

The first method of symmetrizing an arbitrary bimatrix game is related to that given by Gale, Kuhn, and Tucker in [1] for symmetrizing matrix games. It requires that each of the entries in A and B be positive. We may arrange this simply by adding a suitable positive constant to all matrix entries. This has the effect of increasing λ and μ by the same amount, but does not change $\bar{u}$ and $\bar{v}$. We shall assume this has been done, so that we have $\lambda, \mu > 0$.

Given the matrices A and B, we form the payoff matrices C and C^T of a symmetric bimatrix game:

$$(1.7) \qquad C = \left(\begin{array}{c|c} 0 & A \\ \hline B^T & 0 \end{array}\right) \qquad\qquad C^T = \left(\begin{array}{c|c} 0 & B \\ \hline A^T & 0 \end{array}\right)$$

C is a square matrix of order $n = p + q$.

We then have the following correspondence between the equilibrium points of the original game and the symmetric equilibrium points of the derived symmetric game:

1.1 If $\bar{u}, \bar{v}, \lambda, \mu$ satisfy (1.1)-(1.4) for the original bimatrix game, then

$$\bar{x}_i = \begin{cases} \frac{\lambda}{\lambda+\mu} \bar{u}_i & i = 1, \ldots, p \\ \frac{\mu}{\lambda+\mu} \bar{v}_{i-p} & i = p+1, \ldots, n \end{cases}$$

and $\nu = \frac{\lambda\mu}{\lambda+\mu}$

satisfy conditions (1.5) and (1.6), for the symmetrized game (1.7).

1.2 If $\bar{x}, \nu$ satisfy (1.5) and (1.6) for the symmetrized game (1.7), then

$$\bar{u}_i = \bar{x}_i \Big/ \sum_{i=1}^{p} \bar{x}_i \qquad i = 1, \ldots, p;$$

$$\bar{v}_j = \bar{x}_{j+p} \Big/ \sum_{j=1}^{q} \bar{x}_{j+p} \qquad j = 1, \ldots, q;$$

$$\lambda = \nu \Big/ \sum_{j=1}^{q} \bar{x}_{j+p}$$

$$\mu = \nu \Big/ \sum_{i=1}^{p} \bar{x}_i$$

satisfy conditions (1.1)-(1.4) for the original bimatrix game.

The proof of the first assertion obtains via straightforward substitutions. In the second assertion, the positivity of A and B ensure that $\sum_{i=1}^{p} \bar{x}_i$ and $\sum_{j=1}^{q} \bar{x}_{j+p}$ are both positive. The remainder of the proof then follows directly.

The second method of symmetrization is analogous to that proposed by von Neumann for matrix games [1]. Given payoff matrices A and B, we form a square matrix C of order pq as follows:

$$(1.8) \qquad c_{ij|k\ell} = a_{i\ell} + b_{kj}, \qquad i, k = 1, \ldots, p; \; j, \ell = 1, \ldots, q.$$

(The interpretation of this symmetrization remains that given in [1], viz., that of holding two plays of the original game simultaneously, with the two players assuming opposite roles in each play.)

1.3 If $\bar{u}$, $\bar{v}$, λ, μ satisfy (1.1)-(1.4) for the original game, then

$$\bar{x}_{ij} = \bar{u}_i \bar{v}_j \qquad i = 1, \ldots, p; \; j = 1, \ldots, q;$$

and

$$\nu = \lambda + \mu$$

satisfy conditions (1.5) and (1.6), for the symmetrized game (1.8).

1.4 If $\bar{x}$, ν satisfy (1.5) and (1.6) for the symmetrized game (1.8), then

$$\bar{u}_i = \sum_{j=1}^{q} \bar{x}_{ij} \qquad i = 1, \ldots, p;$$

$$\bar{v}_j = \sum_{i=1}^{p} \bar{x}_{ij} \qquad j = 1, \ldots, q;$$

$$\lambda = \max_i \sum_{j=1}^{q} a_{ij} \bar{v}_j$$

$$\mu = \max_j \sum_{i=1}^{p} \bar{u}_i b_{ij}$$

satisfy conditions (1.1)-(1.4) for the original game.

Again the method of proof involves some straightforward substitutions.

Using either of the results 1.2 or 1.4, we have thus reduced the question of the existence of an equilibrium point for an arbitrary bimatrix game to the question of the existence of a symmetric equilibrium point for a symmetric bimatrix game.

2. EXISTENCE OF A SYMMETRIC EQUILIBRIUM POINT FOR A SYMMETRIC BIMATRIX GAME

We shall now show that any symmetric bimatrix game has a symmetric equilibrium point.

Theorem 2.1: Let C denote a (square) payoff matrix for player 1 and C^T denote the payoff matrix for player 2 in a symmetric bimatrix game. Then there exists a probability vector $\bar{x}$ and a real number ν satisfying

$$(1.5) \qquad \sum_{j=1}^{n} c_{ij}\bar{x}_j \leqq \nu \qquad i = 1, \ldots, n,$$

and

$$(1.6) \qquad \sum_{i=1}^{n} \bar{x}_i \left(\sum_{j=1}^{n} c_{ij}\bar{x}_j \right) = \nu.$$

Proof: Given C and a probability vector x, we define

$$c_i(x) = \mathrm{Max}\left(0, \sum_{j=1}^{n} c_{ij} x_j - \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} c_{ij} x_j\right)\right) \quad \text{for } i = 1, \ldots, n.$$

Let Δ_{n-1} = the (n-1)-dimensional simplex consisting of all n-dimensional probability vectors. Consider the continuous mapping, $f: \Delta_{n-1} \longrightarrow \Delta_{n-1}$, defined by

$$x'_i = \frac{x_i + c_i}{1 + \sum_i c_i} \qquad \text{for } i = 1, \ldots, n.$$

We use the following:

<u>Lemma:</u> If $\bar{x}$ is a fixed point under f, then $\bar{x}$ satisfies (1.5) and (1.6), with

$$\nu = \sum_{i=1}^{n} \bar{x}_i \left(\sum_{j=1}^{n} c_{ij} \bar{x}_j \right).$$

<u>Proof:</u> Let $\bar{x}$ be a fixed point under f. Since

$$\sum_{i=1}^{n} \bar{x}_i \left(\sum_{j=1}^{n} c_{ij} \bar{x}_j \right)$$

is a convex combination of the

$$c_i = \sum_{j=1}^{n} c_{ij} \bar{x}_j \qquad \text{for which } \bar{x}_i > 0,$$

we must have

$$\sum_{i=1}^{n} \bar{x}_i \left(\sum_{j=1}^{n} c_{ij} \bar{x}_j \right) \geq \sum_{j=1}^{n} c_{ij} \bar{x}_j \quad \text{for some } i \text{ with } \bar{x}_i > 0.$$

Hence, $c_i = 0$ for this i. But since $\bar{x}_i$ is not changed under the mapping by f, $\sum_{i=1}^{n} c_i = 0$, i.e., $c_i = 0$ for all i. But this means

$$\sum_{j=1}^{n} c_{ij} \bar{x}_j \leq \sum_{i=1}^{n} \bar{x}_i \left(\sum_{j=1}^{n} c_{ij} \bar{x}_j \right) = \nu \qquad \text{for } i = 1, \ldots, n,$$

i. e., (1.5) and (1. 6) are satisfied.

We complete the argument by invoking the Brouwer Fixed Point Theorem, which requires that the mapping f have at least one fixed point, thus ensuring the existence of at least one symmetric equilibrium point for the given bimatrix game.

3. A METAMATHEMATICAL PRINCIPLE

We shall now show that the theorem of Section 2 and, by the same, argument, Nash's theorem and other results of Game Theory, such as the Minimax Theorem, by virtue of their syntactical (external) form alone, are equally valid in all ordered fields. In part, it turns out to be natural to show that all these theorems remain true within an even more general framework, which will be described presently in more detail. Roughly speaking, we shall show that the theorems in question hold (with the appropriate verbal modifications) for all ordered abelian groups with ordered fields as operator rings.

Let F be an ordered field, and let A be an ordered abelian group with respect to the operation of addition, +, which includes more than one element, and which possesses the following additional properties.

3.1 A admits F as an operator ring. That is to say, there exists a mapping (binary operation) from F X A into A, written as $\lambda a=b$, $\lambda \in F$, $a, b \in A$, such that for all $\lambda, \mu \in F, a, b \in A$, $\lambda(\mu a) = (\lambda\mu)a$; $(\lambda+\mu)a = \lambda a + \mu a$; $\lambda(a+b) = \lambda a + \lambda b$; $1 \cdot a = a$ (where 1 is the unit element of F); and, finally, if $\lambda > 0$ and $a > 0$ then $\lambda a > 0$.

We may suppose, for convenience, that F and A are disjoint, although A may be order-isomorphic to the additive group of F. The system made up of the field F and the group A will be denoted by (F;A).

We are going to construct a formal language, L, whose sentences, with a natural interpretation, refer to a system (F;A) as described. L contains the following atomic symbols, (with some comments added immediately for ease of understanding).

Individual constants "of the first kind", $\alpha, \beta_i, \gamma, \ldots$ (small Greek letters near the beginning of the alphabet, with or without subscripts) in one-to-one correspondence with the elements of the field F;

individual constants "of the second kind", a, b, c, ... to denote elements of the group A;

variables "of the first kind", χ, $\emptyset$, ψ, ... relating to F, and variables "of the second kind", x, y, z, ... relating to A;

the relations E (x, y) (for x=y), S (x, y, z) (for x+y=z), Q (x, y) (for x<y), all relating to elements of A, and R (χ, y, z) (for χy=z) which will appear in the well-formed formulae only if the symbol in the first place is related to F and the symbols in the other two places are related to A;

the usual connectives, $\sim$, $\wedge$, v, $\supset$, $\equiv$, and brackets, [,], and quantifiers, ($\forall$) and ($\exists$). Only variables of the second kind, x, y, z, ... will appear within the quantifiers.

Atomic formulae in L are obtained from the relations E, S, or Q by filling their places with variables or constants of the second kind, and from R by filling the first place with a variable or constant of the first kind and the remaining places with variables or constants of the second kind.

Well-formed formulae (wff) are obtained from the atomic formulae by the use of connectives, in the usual way, and by quantification with respect to variables of the second kind. A sentence is a wff in which all variables of the second kind (if any) are quantified.

For example, the following are sentences:

3.2 $(\forall u)(\forall v)(\forall w)(\forall x)(\forall y)(\forall z)[R(\alpha, u, v) \wedge R(\alpha, w, x) \wedge S(u, w, y) \wedge S(v, x, z) \supset R(\alpha, y, z)]$

3.3 $(\forall w)(\forall x)(\forall y)(\forall z)[R(\alpha, w, x) \wedge R(\beta, w, y) \supset R(\gamma, w, z)]$

A sentence X of L is interpreted in the usual way with respect to a particular system (F;A), given a one-to-one correspondence between the individual constants of the first kind which occur in X and some of the elements of F and between the individual constants of the second kind in X and some of the elements of A. If α corresponds to ("denotes") any element of F, then 3.2 holds in (F;A), while 3.3 holds in that system provided γ denotes the sum of the two elements of F which are denoted by α and β. Indeed, the totality of these sentences then expresses the distributivity of the operation λ a=b with respect to the elements of A on one hand and with respect to the elements of F on the other hand. Putting it in a different way, 3.2 and 3.3 are axiom schemes which express the distributive properties of R.

Notice that we have admitted as sentences well-formed formulae which are unquantified with respect to the variables of the first kind which occur in them. Such sentences will be regarded in the interpretation as open sentences, i.e., they will be interpreted as if they were quantified universally with respect to their variables of the first kind. Thus, if we replace α in 3.2 by the variable ψ, then we obtain a single axiom which is equivalent to the axiom scheme 3.2. The language L has no facilities which would enable us to apply a similar procedure to 3.3.

Now suppose in particular that $(F;A) = (F_o;A_o)$ where F_o is the ordered field of real numbers and A_o is a copy of the additive group of real numbers, with $\alpha \in F_o$ mapped on $a \in A_o$. For any $\lambda \in F_o$, the result of the operation λa is defined in the natural way, i.e., as the image in A_o of the product $\lambda \alpha$ in F_o. Then the principal result of this section is as follows:

3.4 Theorem. Let X be a sentence in L which does not contain any individual constants and which holds in $(F_o;A_o)$. Then X holds also in any other ordered abelian group with an ordered field as ring of operators, (F;A).

It is not difficult to prove 3.4 for the case that F is real-closed and A is obtained from F just like A_o was obtained from F_o. That is to say, A is a copy of the additive group of F and if $a \in A$ is the image of an element α of F, then we define that λa is, for any $\lambda \in F$, the image of $\lambda \alpha$ in A. In this case, suppose that X satisfies the assumptions of 3.4. We introduce an expression X' which is obtained from X by replacing the letter R everywhere by the letter P (to be interpreted as multiplication, $P(\chi, \emptyset, \psi)$ means $\chi\emptyset = \psi$) and by substituting variables of the second kind everywhere by variables of the first kind, without clashing with the variables which are already present in X. X' may be regarded as a sentence of the ordinary lower predicate calculus. In view of the definition of λa in (F;A), it will be seen that X holds in (F;A) if and only if X' holds in F and so, in particular, X holds in $(F_o;A_o)$ if and only if X' holds in F_o. Now, by Tarski's theorem on the completeness of the elementary theory of real-closed fields [3], X' holds either in both F_o and F or in neither one of these fields. But X holds in $(F_o;A_o)$ by assumption and so X' holds in F_o, and hence in F, and finally, X in turn holds in (F;A).

In order to prove 3.4 for the general case, we shall make use of a different method of reduction. Given (F;A), we introduce a structure A* as follows:

A* coincides with A as an ordered abelian group. Moreover, for every $\lambda \in F$ there is a one-place operation (function) $f_\lambda(x)$ which is defined on A and takes values in A such that $f_\lambda(a) = b$ if and only if $\lambda a = b$ in (F;A).

To describe A* we use a language L* of the lower predicate calculus with individual constants a, b, c, ... individual variables $x, y, z, \ldots$, relations $E(x,y)$, $Q(x,y)$, $S(x,y,z)$, for equality, order, and addition, as before and two-place relations $D_\lambda(x,y)$ where the subscript λ varies over the elements of F. In the interpretation, $D_\lambda(a,b)$ shall hold in A^* if and only if $f_\lambda(a) = b$, i.e. if and only if $\lambda a = b$ in $(F;A)$. In addition, L^* shall include the same connectives, quantifiers and brackets as before. From the atomic symbols we obtain wff, in particular sentences (this time, without free variables) in the usual way. A sentence X of L^* can be interpreted in A^* provided the individual constants of X, if any, denote (correspond to, or are identified with) certain elements of A^*.

Among the sentences of L^* which hold in A^* we may find the following, all formulated in terms of the relations E, S, P, D_λ, and in terms of the individual constant 0.

3.5 A set of sentences which states that $E(x,y)$ is a relation of equivalence with substitutivity, e.g. for all $\lambda \in F$,

$$(\forall x)(\forall y)(\forall z)(\forall w)\,[D_\lambda(x,y) \wedge E(x,z) \wedge E(y,w) \supset D_\lambda(z,w)];$$

3.6 a set of sentences which state that the structure described is an ordered abelian group containing at least two different elements;

3.7 the following sentences which express properties of $D_\lambda(x,y)$ for every $\lambda \in F$,

$$(\forall x)(\exists y)(\forall z)\,[D_\lambda(x,y)\wedge[D_\lambda(x,z)\supset E(y,z)]]$$

$$(\forall x)(\forall y)\,[D_\lambda(x,y)\wedge Q(0,x)\supset Q(0,y)]$$

$$(\forall x)\,D_1(x,x);$$

for any three elements $\lambda, \mu, \nu \in F$ such that $\lambda\mu = \nu$,

$$(\forall x)(\forall y)(\forall z)\,[D_\lambda(x,y)\wedge D_\mu(y,z)\supset D_\nu(x,z)];$$

for any $\lambda \in F$,

$$(\forall u)(\forall v)(\forall w)(\forall x)(\forall y)(\forall z)\,[D_\lambda(u,v)\wedge D_\lambda(w,x)\wedge S(u,w,y)\wedge S(v,x,z)\supset D_\lambda(y,z)];$$

for any $\lambda, \mu, \nu \in F$ such that $\lambda + \mu = \nu$,

$$(\forall w)(\forall x)(\forall y)(\forall z)\,[D_\lambda(w,x)\wedge D_\mu(w,y)\wedge S(x,y,z)\supset D_\nu(w,z)].$$

For a given ordered field F, let K_F be the set of sentences detailed in 3.5, 3.6, 3.7. We are going to prove

3.8 <u>Theorem.</u> K_F is model-complete.

<u>Proof of 3.8.</u> We shall apply the model-completeness test of [3]. For given F, let $M = A^*$ be a model of K_F and let

$$X = (\exists x_1)\ldots(\exists x_n)\,Q(x_1,\ldots,x_n)$$

be a primitive sentence (an existential sentence whose quantifiers are followed by a conjunction of atomic formulae and/or of the negations of such formulae) which is derived in M. Suppose that X holds in some extension M' of M which is a model of K_F then we have to show that X holds also in M. In ordinary language, let Σ

be a finite system of equations and inequalities and of negations of equations or inequalities, i.e. Σ consists of conditions of the form

$$3.9 \quad r = s, \quad r \neq s, \quad r < s, \quad r \geq s$$

$$r + s = t \quad r + s \neq t \quad \lambda r = s \quad \lambda r \neq s \quad ,$$

where $\lambda \in F$ and r, s, t stand either for definite elements of M or for the "unknowns" $x_1, \ldots, x_n$. Suppose that Σ possesses a solution $x_i = a_i$ in M'. Then we have to show that Σ is solvable already in M.

We may simplify Σ by deleting all expressions of the form $r + s \neq t$ and by introducing instead, in each case, a new unknown x_i and the conditions $r + s = x_i$ and $x_i \neq t$. Similarly, we may replace $\lambda r \neq s$ by $\lambda r = x_i$ and $x_i \neq s$. Finally, if $r \neq s$ in M' then either $r < s$ or $r > s$ holds in M' and if $r \geq s$ then either $r > s$ or $r = s$. We may then select the alternative which actually holds in M', for $x_i = a_i$ and replace $r \neq s$, or $r \geq s$, by it. If the modified system, Σ', can be shown to hold in M, so does the original Σ. Thus, we may suppose from now on that Σ' contains only expressions of the form

$$3.10 \quad r = s, \quad r < s, \quad r + s = t \quad \lambda r = s$$

Now suppose that M does not contain any elements which satisfy Σ, and let $x_1, \ldots, x_k$ be the unknowns which appear in Σ', $k \geq n$, and which are satisfied by a set $x_i = a_i$ in M'. Let $M^* =$

$M(a_1, \ldots, a_k)$ be the model of K_F which is generated by the adjunction of $a_1, \ldots, a_k$ to M, $M^* \subseteq M'$. M^* consists of all elements of M' which can be written in the form,

$$m + \lambda_1 a_1 + \ldots + \lambda_k a_k \text{ with } m \in M, \lambda_i \in F, i = 1, \ldots, k.$$

Then the quotient group M^*/M is of linear rank $j \leq k$ or, as we shall say, M^* is of linear rank j modulo M. Since M^* contains $a_1, \ldots, a_k$, Σ' is solvable in M^*. By rearranging the a_i if necessary we may assume that $a_1, \ldots, a_i$ are linearly independent modulo M (i.e. in M^*/M) while $a_{i+1}, \ldots, a_k$ depend on $a_1, \ldots, a_i$ modulo M. Put

$$M_o = M, \quad M_i = M_{i-1}(a_i), \; i = 1, \ldots, j,$$

so that M_i consists of all elements $m + \lambda a_i$ with $m \in M_{i-1}$, $\lambda \in F$. Since Σ' possesses a solution in $M_j = M^*$ but not in $M_o = M$ there exists a smallest integer ℓ, $1 \leq \ell \leq j$ such that Σ' possesses a solution in M_ℓ but not in $M_{\ell-1}$. Let $x_i = b_i$, $i = 1, \ldots, k$ be such a solution. Then there exist uniquely determined $m_i \in M_{\ell-1}, \lambda_i \in F$ such that

$$3.11 \qquad b_i = m_i + \lambda_i a_\ell \;, \; i = 1, \ldots, k.$$

Put

$$3.12 \qquad x_i = m_i + \lambda_i x \;, \; i = 1, \ldots, k$$

where x is a new variable. If we substitute the right-hand sides of

3.11 for the x_i in the equations of the form $r = s$, $r + s = t$, and $\lambda r = s$ which appear in Σ' then we must get identities in x since otherwise a_ℓ would depend linearly on $M_{\ell-1}$ and hence, belong to it. Thus if we make the same substitutions in the inequalities $r < s$ which belong to Σ' then any value $x = c$ which satisfy the resulting conditions will yield values of the x_i which satisfy Σ'. The conditions in question are of the form

$$3.13 \qquad m + \lambda x < n + \mu x\,, \quad m, n \in M_{\ell-1}, \; \lambda, \mu \in F$$

and we know that they are satisfied in M_ℓ by $x = a_\ell$. We claim that these inequalities must then be satisfied already by some $x = c$ in $M_{\ell-1}$. Indeed, if $\lambda = \mu$ then the inequality under consideration reduces to $m < n$, which is satisfied independently of the value of x. If $\lambda < \mu$ then we may replace the inequality by $x < (\mu - \lambda)^{-1}(n-m)$ and if $\mu < \lambda$ then the inequality is equivalent to $x > (\mu - \lambda)^{-1}(n-m)$. Accordingly, the inequalities of Σ' are equivalent to a system

$$3.14 \qquad \begin{array}{ll} x < c_i & i = 1, \ldots, p \\ x > c_i & i = p+1, \ldots, q \end{array}$$

where one or the other, or both, of the two sets may be empty. If the last-mentioned alternative occurs, we may choose for $x = c$ an arbitrary element of $M_{\ell-1}$. If neither set is empty, we must have

$$\min_{i \leq p} c_i > \max_{i \geq p+1} c_i$$

since 3.14 is solvable in M_ℓ and we may then choose

$$x = c = \frac{1}{2}\left(\min_{i \leq p} c_i + \max_{i \geq p+1} c_i \right)$$

which is in $M_{\ell-1}$. If the first set is empty but not the second, we may satisfy 3.14 in $M_{\ell-1}$ by

$$x = c = \max_{i \geq p+1} c_i + c'$$

where c' is an arbitrary positive element of $M_{\ell-1}$. Finally, if the second set is empty but not the first, we put

$$x = c = \min_{i \leq p} c_i - c'$$

where c' is again any positive element of $M_{\ell-1}$. Accordingly, we may in all cases satisfy 3.14, and hence Σ', already in $M_{\ell-1}$. This contradicts the definition of M_ℓ and proves 3.8.

3.15 Theorem. K_F is complete.

Proof of 3.15. According to [3], a set is complete provided it is model-complete and possesses a prime model. A structure M_o is a prime model for a set of sentences K if M_o is a model of K and if every model of K possesses a substructure (partial structure) which is isomorphic to M_o. For $K = K_F$, such a structure M_o is given by the ordered additive group of F in which the function $f_\lambda(a) = b$ is defined by ordinary multiplication $\lambda a = b$. Indeed if M is any model of K_F and a is a positive element of M then the subgroup M' of M which consists of all elements λa, with $\lambda \in F$, is isomorphic to M_o not only with respect to addition and order but also with respect to the functions $f_\lambda(x)$, under the correspondence $\lambda \leftrightarrow \lambda a$. This proves 3.15.

We are now in a position to complete the proof of the main theorem, 3.4. Given a system $(F;A)$ and a sentence X which holds in $(F_o;A_o)$ as assumed in 3.4, let $\bar{F}$ be the real closure of F. Consider the system $(\bar{F};\bar{A})$ where $\bar{A}$ is a copy of the additive group of $\bar{F}$ and the operation λa for $\lambda \in \bar{F}$, $a \in \bar{A}$ is explained by means of the multiplication in $\bar{F}$, similarly as before. By the part of 3.4 which was proved at the beginning, we know that X holds also in $(\bar{F};\bar{A})$. Let $X = Q(\chi_1, \ldots, \chi_n)$ where $\chi_1, \ldots, \chi_n$ are the variables of the first kind which appear in X. Then $Q(\alpha_1, \ldots, \alpha_n)$ holds in $(\bar{F};\bar{A})$ for any $\alpha_1, \ldots, \alpha_n \in \bar{F}$. It follows in particular that $Q(\alpha_1, \ldots, \alpha_n)$ holds for any $\alpha_1, \ldots, \alpha_n \in F$ and hence, that X holds in $(F, \bar{A})$. We now "translate" the sentence X which was formulated in L into a sentence X^* in L^* by replacing the atomic formulae of the form $R(\lambda, x, y)$ which occur in X by the corresponding $D_\lambda(x, y)$. Let $\bar{A}^*$ be the structure which is obtained from $\bar{A}$ by the introduction of the functions $f_\lambda(x)$ for all $\lambda \in F$, as explained previously. Then X^* holds in $\bar{A}^*$ and hence, by 3.15, holds also in any other ordered abelian group with F as operator ring. In particular, X^* holds in A^* and so X holds in $(F;A)$. This proves 3.4.

We still have to show that the result of Section 2 of the present paper (and similar results, like Nash's theorem or the Minimax theorem) can be brought within the scope of 3.4. It is indeed not possible to represent the theorem by a single sentence in L since the order of the matrix involved varies over the positive integers. Accordingly, we first split the assertion of the theorem into a sequence of statements for the individual integers n. We then regard the elements of the matrix as elements of the field F, and the components of the vectors whose existence is asserted as elements of the group A. We still have to eliminate the constant 1 which appears as a group element since no such constant is supposed to occur in the sentence X. This can be done by observing that the theorem remains true if we replace 1 by any other positive element (of the group). For given n, the statement of the theorem then takes the following form:

3.16 "For any ϕ_{ij} in F, $i, j = 1, \ldots, n$, and for any positive y in A there exist $x_1, \ldots, x_n$ in A such that $x_i \geq 0$, $i = 1, \ldots, n$, and $x_1 + \ldots + x_n = y$ and if $x_j > 0$, $1 \leq j \leq n$ then $\phi_{j1} x_1 + \phi_{j2} x_2 + \ldots + \phi_{jn} x_n \geq \phi_{k1} x_1 + \phi_{k2} x_2 + \ldots + \phi_{kn} x_n$ for $k = 1, 2, \ldots, n$."

It is now not difficult to express 3.16 as a sentence of the language L. Similar remarks apply to other results of game theory such as the Minimax theorem and Nash's theorem.

The argument may be shortened if we restrict ourselves to the assumption that the coefficients of the matrices and vectors in question belong to the same ordered field. In this case, some of the metamathematical reasoning may be replaced by algebraic considerations which show that there exists a solution whose coordinates depend rationally on the coefficients of the matrix.

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