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Vipul Bansal, Rahul Garg
email: {bvipul,grahul}@in.ibm.com

IBM Research Division,
IBM India Research Lab,
Hauz Khas, New Delhi - 110016. INDIA.
Phone: +91-11-6861100, Fax: +91-11-6861555

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Vipul Bansal, Rahul Garg
{bvipul, grahul}@in.ibm.com
IBM India Research Lab.,
Hauz Khas, New Delhi - 110016, INDIA.

Abstract

Decentralized multi-item auctions offer great opportunities for integrating fragmented online auction markets into larger markets with more efficient outcomes. We extend the theory of multi-item ascending auctions of substitutes by considering any finite positive bid increment and allowing the bidders to bid asynchronously instead of bidding in a round-robin fashion. We consider a setup where the bidders' utilities over multiple items are additive and bound the maximum inefficiency in the allocation when the bidders follow a simple greedy strategy. We also obtain the limits within which the prices of individual items can vary from one outcome to another. For the special case of single unit bidder demand, we also bound the maximum surplus which a bidder can extract by unilaterally switching to some other strategy. The paper suggests an upper bound for the minimum required bid increment which would be necessary for competitive price discovery and truthful bidding in a practical online implementation.

1 Introduction

Commerce is about transfer of goods and services to entities who value them more from entities who value them less, thus creating value (surplus). There is a justification for electronic commerce if it creates additional value, either by cutting costs or doing better allocation.

Commerce on the Internet is poised to constitute a significant fraction of the entire global commerce in the years to come. Online auctions have already become popular market clearing mechanisms for electronic commerce. The reader is referred to Klemperer [10] for a review of the literature on auctions.

In general, the outcome of an auction defines the allocation (which items go to which bidders) and the prices. The allocation determines the surplus while the prices determine the sharing of the surplus between the auctioneers and the bidders. An efficient allocation is one that maximizes the total surplus in a given setting [16]. An auction outcome is the result of the interaction of bidding strategies adopted by the bidders. A profile of strategies is said to constitute a Nash equilibrium [14] if, given that all bidders adhere to the prescribed strategies, no bidder can gain by unilaterally switching to another strategy. A strategy is dominant if it maximizes a bidder's payoff independently of the strategies of the other bidders. Of particular interest are mechanisms in which truthful revelation of preferences constitutes the dominant bidding strategy.

The present day auction markets on the Internet are highly fragmented and comprise a large number of very small markets selling similar items; potentially leading to inefficient outcomes. It would be more desirable to take into account the presence of all items being sold simultaneously and the presence of all competing bidders in determining the allocation of items and their prices. Such a mechanism would provide wider choice and greater opportunities to buyers and sellers. From an economic viewpoint, larger integrated markets present with the opportunity of realizing greater efficiency and more competitive outcomes.

In this paper, we provide extensions to the theory of multi-item ascending auctions of substitutes in order to facilitate the design of a decentralized online auction mechanism. In particular we consider a system of simultaneous independent ascending auctions with discrete bid increments ϵ. We consider the case where the bidders' utilities over items are additive and bound the maximum inefficiency and price variation when the bidders follow a simple greedy strategy. For the special case of single unit bidder demand, we also provide the corresponding results which determine the incentive properties of the mechanism.

The present work could form the basis of a better informed implementation of a decentralized auction mechanism on the Internet. It serves to bridge the gap between auction mechanisms known to yield theoretically optimal results under ideal conditions and the form in which they can be actually implemented online. The proposed mechanism would allow sellers to auction their items independently using open outcry auctions, except requiring that the auctions start and stop accepting bids at the same times. It can thus provide a means for integrating fragmented auction markets on the Internet without requiring a centralized clearing house.

The market theory for multi-item auctions dates back to Shapley and Shubik [17] who considered an assignment market where each participant wants one item. A participant demands the item which, given a price vector for the items, maximizes its surplus. It was shown that if the valuations of all participants for all items are known then the model always has an equilibrium and there is a unique smallest price vector P such that P is at least as small in every component as any other equilibrium price vector. Leonard [11] and Demange and Gale [4] showed that a sealed bid multi-item auction of substitutes, where each bidder wants one item, is incentive compatible if the smallest equilibrium price vector is used as the price setting rule. This implies that submitting their true valuations is the dominant strategy for bidders. This also established the multi-item sealed bid auction as a generalization of the second price sealed bid auction of a single item [18]. However, such an auction mechanism does not lend itself to a decentralized implementation on the Internet.

An ascending auction mechanism for the assignment problem was first described by Crawford and Knoer [3] for assigning workers to firms. A generalization was provided in Kelso and Crawford [9] where multiple demand with non-additive utilities was allowed as long as the items on auction remained *gross substitutes* to the bidders. Items are considered gross substitutes if an increase in the prices of some items does not lead to a reduction in the demand of any bidder for the remaining items. Their auction mechanism with truthful bidding converges to a minimum competitive price equilibrium provided that (a) there are no ties at the equilibrium, (b) the bidding is carried out in rounds, with each bidder being present in each round, (c) the minimum required bid increment equals one, and (d) the bidders' valuations are multiples of the minimum required bid increment.

In a real life implementation, these assumptions may not be practical. One needs to explore the properties of the equilibria when bidders may bid asynchronously (they may join any time and stay inactive for some part of the auction), when ties are permitted, when minimum required bid increment may be arbitrary (say ϵ) and when bidders' valuations may not be multiples of ϵ. It needs to be established that the new equilibria are similar and the extent of the differences need to be bounded. In addition, the implications for strategic properties of the mechanism also need to be established.

Demange et al. [5] considered the special case where the bidders want only a single item and examined two mechanisms. In the *first* mechanism (*exact auction mechanism*), the auction is conducted in multiple rounds and the bidders announce honestly in each round, the items whose value to a bidder exceeds its current price by the maximum amount. They show that this mechanism converges to the minimum equilibrium price vector P even when ties are allowed. In the *second* mechanism (*approximate auction mechanism*), the bidders are allowed to place bids asynchronously (instead of rounds) and the minimum required bid increment is set to ϵ. They show that the resulting prices are not unique and each component of the price vector could deviate upto $\pm m\epsilon$ from the corresponding component in P (where m is lesser of the number of items and bidders). Bertsekas [2] further showed that the inefficiency in the resulting allocation is bounded by at most $m\epsilon$.

For the case of multiple unit bidder demand, Gul and Stacchetti [7] and [8] further examined the equilibria under greedy bidding with the gross substitutes condition of Kelso and Crawford [9]. They suggest a different version of the auction mechanism which obtains minimum competitive prices without ruling out ties. This is a multi-demand generalization of the first mechanism of Demange et al. [5] described above. However, it still retains the other problems of the mechanism of Kelso and Crawford [9]. What we need is a multi-demand generalization of the second mechanism of Demange et al. [5].

Milgrom [12] examined simultaneous ascending auctions for substitutes in the context of FCC auctions [6]. He considered a multi-round sealed bid auction for multiple items and showed that if the bidders bid honestly, then the maximum inefficiency is bounded by a quantity equal to ϵ times of the sum of the components of the final price vector (where ϵ is the minimum required bid increment expressed as a fraction of the current price). Milgrom's mechanism, however, retains the other practical problems encountered in the mechanism proposed by Gul and Stacchetti above. Also, he does not elaborate on how much can the prices vary from the corresponding minimum competitive prices.

Parkes and Ungar [15] considered a combinatorial auction which progresses in rounds. In each round, the bidders place *exclusive-or* bids on all bundles of items that maximize their current utility within $\pm\epsilon$. Such a mechanism is shown to yield an outcome which is efficient within $3m\epsilon$. While they prermit generalized utility functions, the winner determination algorithm is NP-hard and does not lend itself to a decentralized implementation.

In this paper we consider a multi-demand generalization of the second mechanism by Demange et al. [5]. Thus, we allow asynchronous bidding and permit the minimum required bid increment to be any finite positive ϵ. We also permit ties at any stage and allow the bidders' utilities over items to be any finite non-negative numbers. In this setup, we bound the maximum inefficiency in the allocation for the system and the maximum fluctuation in the prices of individual items. In this way we facilitate a well-informed online implementation of the mechanism. Our work extends (a) the result on efficiency by Bertsekas [2], and (b) the results on prices by Demange et al. [5] to the case of multiple unit demand with additive utilities. It is less general than the work of Kelso and Crawford [9] and Gul and Stacchetti [7] and [8] in the sense that the bidders' utilities are additive; but more general in the sense that it allows asynchronous bidding, permits general minimum required bid increment and permits any finite utility values.

We also briefly examine the strategic properties at equilibria in the given context. For single unit bidder demand, we bound the maximum surplus which a bidder not following the LGB strategy can obtain when all other bidders follow the LGB strategy. Our results show that the condition $\epsilon < 1/m$ (where m is lesser of the number of units of items supplied and units of items demanded) must hold for the LGB strategy to approximate a Nash Equilibrium for the system. For a larger ϵ, the incentive to deviate grows unbounded as the size of the system increases.

A known problem with the mechanisms involving multiple demand and supply is that a surplus maximizing bidder (or auctioneer) may withhold a part of its demand (or supply) to obtain lower (or higher) prices and greater overall surplus. Ausubel and Cramton [1] showed the the existence of incentive to withhold demand on the part of large bidders in auctions involving sale of multiple identical objects. They further showed that such incentives lead to inefficient outcomes.

It is important to note that the incentive to withhold demand or supply is common when an individual's demand or supply is a significant part of the aggregate demand or supply. The incentive progressively weakens as the ability of individuals to influence market prices diminish. The approach described in this paper leads to online auction markets which are integrated in an economic sense. They can therefore be expected to be much less susceptible to incentive of withholding demand or supply owing to the aggregation of demand and supply of a large number of bidders and auctioneers.

The rest of the paper is organized as follows: Section 2 formally defines the system of simultaneous independent ascending auctions (SIA). Section 3 defines a truth-telling local greedy bidding (LGB) strategy. In Section 4 we discuss the characteristics of the equilibria of SIA under LGB bidding. Section 5 discusses the implications of our work for online auctions. Finally Section 6 lists the conclusions. Important proofs are given in the Appendix.

2 Simultaneous Independent Ascending Auctions

We consider a system of multi-item ascending (open outcry) auctions. The auctions are decentralized wherein the auction for any subset of items is potentially conducted independent of other items except that the times when all the auctions start and stop accepting bids are synchronized. We consider the general case where each item from a set A of m items is on an independent open outcry auction. The auctions are characterized by a vector of reserve prices P^r. We assume here that the reserve price for an item j denotes the minimum price acceptable to its seller and is representative of its true valuation of j.

Let B be the set of n bidders in the system. For any set of items, $S \subseteq A$, a bidder i has a valuation (utility) $U_i(S)$ defined by: (a) $U_i(\phi) = 0$, (b) For a singleton $S = \{j\}$, bidder i has an independent private valuation $v_{ij} \geq 0$. (c) If $|S| > 1, U_i(S)$ equals the sum of r highest single item valuations v_{ij} over all items j in S, where $r = \min(k_i, |S|)$. The items are thus substitutes to the bidders in the sense that a bidder i wants at most k_i items. Note that unlimited bidder demand is a special case when $\forall i \in B, k_i \geq m$. It can also be easily verified that this utility function satisfies the 'gross substitutes' condition of Kelso and Crawford [9].

The system of Simultaneous Independent Ascending auctions can be represented as $SIA(V, K, P^r)$ where V is the $n \times m$ matrix of the bidders' valuations v_{ij} and K is the $n \times 1$ vector of the bidders' maximum demand k_i. To ensure that the auctions terminate in a finite number of steps, a new bid must exceed an existing bid by ϵ. The state of the system at any time is given by the vector, P of current prices (winning bids) for the m items, and their corresponding winners. When the auctions close, the bidder with the highest bid in an auction is declared the winner and pays an amount equal to its bid to get the item.

Given a current price vector P, the surplus (net utility) of a bidder i on an item j is defined to be the amount by which its valuation for the item exceeds the item price, i.e., $s_{ij} = v_{ij} - p_j$. The corresponding surplus of the seller of j equals $p_j - p^r_j$. The total surplus in auction is the sum of the seller and buyer surpluses and equals $v_{ij} - p^r_j$. The total surplus is independent of the price and depends only on the assignment of the items to bidders. The global total surplus for $SIA(V, K, P^r)$ is simply the sum of the total surplus in each of the m auctions. Given an allocation of items to bidders, the global total surplus is given by: $S = \sum_{j=1}^{m}(v_{w(j)j} - p_j^r)$ where $w(j)$ is the winner of item j.

3 Local Greedy Bidding

We consider a multi-unit generalization of the bidding strategy used in the second mechanism of by Demange et al. [5]. Consider a bidder i who wants upto k_i items in total and is currently winning on r_i items ($0 \leq r_i < k_i$). The bidder i orders the items with a surplus of ϵ or greater (from amongst the items on which it is not winning) in the decreasing order of their surplus and places bids on the first $k_i - r_i$ items. We refer to this bidding strategy by Local Greedy Bidding (LGB) strategy. If a bidder is outbid, it uses the LGB strategy to determine its fresh bids. If there are fewer than $k_i - r_i$ items which give a surplus of ϵ or greater, the bidder places fewer bids. When there are no such items, the bidder stops bidding.

Note that LGB ensures that a bidder i never has more than k_i outstanding (winning) bids at any point of time. Further, if a bidder i has a current outstanding bid on an item j with a surplus s then the bidder i can never obtain a surplus greater than s by placing a bid on any other item which it is not currently winning. By definition of LGB, this is true for a bid when the bid is just placed. From then onwards and until that bid is superceded, the prices of the remaining items on which i was not winning when it placed this bid can only rise (and so i's surplus on them can only fall).

We assume the presence of m dummy bidders in the system, one corresponding to each item in A. The dummy bidder corresponding to an item j has a valuation of p_j^r for the item j and zero for all other items. The set of bidders B is assumed to include these dummy bidders and the rest of the bidders in B are referred to as 'exogenous' bidders. At the start of the simultaneous auctions, the dummy bidders place their bids at reserve prices on the respective items prior to any bids being placed by exogenous bidders.

4 Equilibria with Local Greedy Bidding

Consider the system $SIA(V, K, P^r)$ with the bidders following the LGB strategy. Given a current price vector P, define $G_P(i)$, the greedy choice set of items of bidder i to comprise the r_i items that it is currently winning and upto $k_i - r_i$ highest surplus items from the remaining items which give it a surplus of ϵ or more. In case two or more items give equal surplus, the bidder can break ties in any way it likes. Define the competition on an item j, given the current price vector P, to be the number of bidders for whom, j is in their greedy choice set of items. More precisely, $COM_P(j) = \mid \{i | j \in G_P(i)\} \mid$. The competition in the system $SIA(V, K, P^r)$ is defined to be the sum of competitions on all the m items on auction, $COM_P = \sum_{j=1}^{m} COM_P(j)$.

Presence of dummy bidders ensures that at any time, $\forall j \in A, COM_P(j) \geq 1$. This is because any item j must have a current winner at all times. If no exogenous bidder has yet placed a bid on item j, then the current winner is simply the dummy bidder.

Lemma 1
[**Termination**] *The bidding in $SIA(V, K, P^r)$, with the bidders following the LGB strategy, terminates in a finite number of steps. Furthermore, the bidding terminates iff $\forall j \in A, COM_P(j) = 1$.*

The proof is provided in the Appendix. When the auctions commence, there are many items which may belong to the greedy choice sets of multiple bidders. As a result of this competition on a given item, the price of that item increases. This makes the surplus on it less than the surplus on some other items for some of the bidders who may be competing for this item. This causes them to move away to other items (or quit, if there are no items left which gives them a non-negative surplus). Thus, LGB shifts competition from one item to another as bidders keep hunting for items that maximize their local surplus. In the process the prices rise, leading to exit of some bidders, thereby reducing competition in the system.

If bidders follow the LGB strategy, the resulting allocation is nearly efficient, i.e., nearly maximizes the global total surplus in the system. The following theorem establishes an upper bound on the inefficiency for the system $SIA(V, K, P^r)$ under LGB bidding.

Theorem 1 *If S_{OPT} is the maximum global total surplus for the system $SIA(V, K, P^r)$ under any allocation, and S_{LGB} is the global total surplus of an outcome when bidders follow the LGB strategy, then $S_{OPT} - S_{LGB} \leq m\epsilon$.*

The proof is provided in the Appendix. This theorem implies that the average inefficiency per item for a system selling m items on simultaneous auctions is of the order of ϵ, independent of the number of units demanded or supplied in the system. Therefore, the system can scale to arbitrarily large integrated auction markets on the Internet without losing on efficiency in the process. For the special case when ϵ tends to zero, we get asymptotic efficiency as expected (for example, [7] and [8]).

Note that the prices of individual items do not affect the total surplus in the system. However, they determine the distribution of the surplus amongst bidders and auctioneers, thus influencing their behaviour and participation in the system. To obtain the results on prices, we identify the bidders who are responsible for determining the prices in the present context. Given an outcome of $SIA(V, K, P^r)$ with LGB, we define the *marginal losers* of a set of items, $C \subseteq A$, as the bidders who had placed the second highest bid on an item in C and after being outbid on that item, did not place any more bids on items in C.

Lemma 2 [Existence of a Marginal Loser] *In any outcome of $SIA(V, K, P^r)$ with LGB, for any $C \subseteq A$, either there exists at least one marginal loser or else all the items in C are sold to dummy bidders at reserve prices.*

The proof is provided in the Appendix. Lemma 2 signifies that there are bidders who just lost out on the margin. For $C = A$, it implies that it is not possible for all prices in $SIA(V, K, P^r)$ to fall simultaneously by more than ϵ otherwise a marginal loser would step in and place a bid. The marginal losers thus hold the final prices. These characteristics resemble those observed in $M + 1^{st}$ price auctions where the highest non-winning bid determines the price.

Theorem 2 *Consider the system $SIA(V, K, P^r)$ with the bidders following the LGB strategy. Let P^1 and P^2 be the final price vectors for two different outcomes of the system. Then $\forall j \in A$, $\mid p_j^1 - p_j^2 \mid \leq 3m\epsilon$.*

The proof is provided in the Appendix. Theorem 2 generalizes the earlier result on prices for single unit demand by Demange et al. [5] to the multi-unit demand scenario and shows that a very similar result holds. The existing literature ([9], [7] and [8]) suggests that for the special case when ϵ approaches zero, the prices in this mechanism should asymptotically converge to the minimum competitive prices. We have shown here that in the general case, when ϵ need not be made to approach zero, the maximum variation in the prices from one outcome to another is bounded by at most $3m\epsilon$.

We note as m increases, the uncertainty in price of each item increases. The uncertainty in the prices can be bounded by a constant provided that ϵ scales as $1/m$. We can increase m as long as ϵ can be correspondingly reduced. Increasing m implies better allocation (maximization of surplus for a larger system) but it also leads to a greater uncertainty in prices and individual surpluses.

The source of this uncertainty lies in the formation of ties and the order in which bidders may place their bids. If every bidder had an ordering for breaking ties while choosing items for placing bids whenever it encountered multiple items with equal current surplus, and if the bidders placed their bids in a round-robin manner, then, given the bidders and the items, the entire sequence of bids is determined by a deterministic algorithm, thereby implying a unique allocation and price vector. Note that different orderings would yield different allocations and price vectors within the bounds obtained in this paper. The finite bounds exist because of the finite bid increment ϵ.

Remark 1: Minimal Complete Subsystems of SIA

We show that instance of $SIA(V, K, P^r)$ can be decomposed into smaller subsystems. Define a *complete subsystem* of $SIA(V, K, P^r)$ to consist of the set of items $A' \subseteq A$ and the set of bidders $B' \subseteq B$ such that all the items for which the bidders in B' have valuations not less than reserve prices are contained in A' and all the bidders who have valuations not less than reserve prices for one or more items in A' are contained in B'. Clearly the overall system $SIA(V, K, P^r$ satisfies this property and is a complete subsystem by definition.

We note that the allocation and prices in a complete subsystem cannot be influenced by items or bidders not in the subsystem because a bidder who is not inside a complete subsystem would never place a bid on any item in that subsystem and vice versa.

A complete subsystem is defined to be a *minimal complete subsystem* if it does not contain any smaller (having lesser number of bidders or items or both) complete subsystem. A simple algorithm to decompose $SIA(V, K, P^r)$ into its minimal complete subsystems is as follows: Consider a bipartite graph $G(A, B, E)$ with the bidders and items as the two sets of vertices. An edge $(i, j) \in G$ iff $v_{ij} \geq p_j^r$. Each connected component of the graph G constitutes a minimal complete subsystem of $SIA(V, K, P^r)$. It is easy to see by the above construction that the minimal complete subsystems are mutually disjoint.

Therefore, an instance of the SIA system can be decomposed into disjoint subsystems each of which has all the properties of a full-fledged SIA system.

We note that the maximum variation in the price of an item would be bounded by the size of the minimal complete subsystem in which the item is present and need not be bounded in terms of the size, m, of the overall system. However, when all the items are very similar, the minimal complete subsystems may not be significantly smaller than the overall system.

Remark 2: Strategic Behaviour: Single Unit Demand

For the case of single unit demand and supply, the auction mechanism can be seen as a means for solving the assignment problem. For this case, Demange et al. [5] have shown that LGB leads to minimum competitive prices as ϵ approaches zero. Further, these prices are known to correspond to the prices in a Vickrey auction (due to earlier work by Leonard [11] and Demange and Gale [4]). It thus follows that LGB approximates a Nash Equilibrium for the system as the bid increment approaches zero.

It would be interesting to study the strategic behaviour of the bidders when the bid increment, ϵ is not small enough to be ignored. The Theorem below bounds the maximum surplus that a bidder not following LGB can extract when all the other bidders follow the LGB strategy.

Theorem 3 *Consider any outcome of $SIA(V, K, P^r)$ with the bidders following the LGB strategy and let s_i be the surplus of bidder i. Bidder i cannot obtain a surplus greater than $s_i + 3m\epsilon$ in any other outcome by unilaterally following some other strategy.*

The proof is provided in the Appendix. We note that ϵ must be set to a number smaller than $1/m$ if the honest bidding strategy, LGB is to approximate a Nash Equilibrium for $SIA(V, K, P^r)$. For larger ϵ, the bidders may have incentive to realize non-negligible gains by following other strategies. The incentive grows with the size m of the system.

If the set of strategies that the bidders can follow is suitably restricted, LGB can be shown to be a dominant strategy in the limit when ϵ approaches zero. Miyake [13] restricts this set to strategies which can be expressed by (a) defining a total order on the set of pairs (item, price) which satisfies the properties of monotonicity and continuity, and (b) bidding at each stage on the item of highest rank given the current price vector. He shows that under these conditions, the truth telling bidding strategy in which the total order is defined by the bidder surplus is a dominant strategy.

In a general setup where no restrictions are placed on the strategies that the bidders can choose, LGB need not be a dominant strategy. Even in the case of $m = 1$ (auction of a single item), it is known that there are equilibria that do not require truthful bidding. We present an example to illustrate why this happens in the present context.

Consider a system $SIA(V, K, P^r)$ as shown in Figure 1 with $A = (a_1, a_2)$, $B = (b_1, b_2, b_3)$, $V(b_1) = (10, 100)$, $V(b_2) = (10, 100)$, $V(b_3) = (0, 50)$ and $P^r = (0, 0)$. Assume that minimum required bid increment, $\epsilon = 1$. Bidder b_1 follows the strategy: if $0 < p_1 < 10$ then bid on a_1 else if $p_1 = 0$ and $p_2 < 100$ then bid on a_2 else do not bid. Bidder b_2 follows the strategy: place its first bid on a_1. On being outbid, if $p_2 < 100$ place a bid on a_2. Bidder b_3 uses the strategy: if $p_2 < 50$, place bid on a_2.

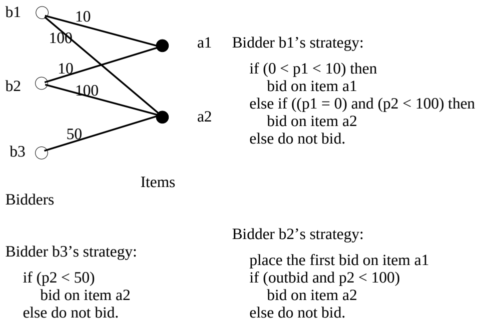

Figure 1: An example where LGB is not a dominant strategy

With these set of strategies, b_2 will either get a_1 at a price of 1 (with a surplus of 9) or a_2 at a price of 51 (surplus of 49). However, if b_2 were to follow LGB strategy, it would always get item a_2 at a price of either 90 or 91 (surpluses of 10 and 9 respectively). Therefore, b_2 may realize significantly greater surplus by following a strategy other than LGB when other bidders follow arbitrary strategies as indicated. Therefore, LGB is not a dominant strategy for the system under general conditions.

Remark 2: Strategic Behaviour: Multi-unit Demand and Supply

For multi-unit demand, the truth-telling LGB strategy may not constitute a Nash Equilibrium due to the incentive to withhold demand. The basis of such an incentive is that withholding a part of the demand by a bidder may lead to lower prices and a greater overall surplus for the bidder. Similarly, an auctioneer having multiple units to sell may withhold a part of its supply by either not putting some items on auction and / or setting higher reserve prices.

It is crucial to note that the extent of such incentives depend on the ability of individual participants to influence market prices. In a large system (n and m large), individual buyers and sellers cannot influence the market prices significantly by their decisions and hence the incentives to withhold demand or supply are reduced. For example, consider a bidder i who wants upto 10 items. Consider a case where the aggregate demand in the minimal complete SIA subsystem containing the bidder is 50 items. If the bidder withholds 50% of its demand, the total demand shrinks by 10%. Consider another case where the relevant aggregate demand is 500 items. Here a demand withholding of 50% by the bidder i leads to only 1% fall in the aggregate demand. If the price elasticity of demand is the same in both cases (say, equal to

1), then the prices fall on an average 10% in the first case but only 1% in the second case. Strictly speaking, the foregoing example is valid for a case of homogeneous items but the general argument can also be made for the case of heterogeneous goods where similar (although weaker) price linkages exist.

We saw that for the decentralized mechanism proposed in this paper, the demand and supply in the independent auctions get aggregated in an economic sense across the numerous auctions. Therefore, one can expect the incentives to withhold demand and supply to be considerably reduced as long as few large bidders or auctioneers do not dominate the market.

Remark 3: Collusion and Price Reduction

If bidders were capable of first identifying the items which they would respectively win using LGB, and then bid only on them, the resulting prices will be lower than the prices when they use LGB. To understand this, consider the case of single unit bidder demand. When the bidders use LGB, some of the second highest bids can come from amongst the winners of some other items. If each bidder chose and bid only for its items, these second highest bids which came from winners (when LGB was used) would be replaced by second highest bids coming from some bidders who did not win any item and also did not have any second highest bids when LGB was used. With LGB, all the prices exceeded the valuations of these bidders by at least ϵ. But with the new strategy, there may be some prices, which are equal to their valuations or no more than ϵ greater than their valuations.

Therefore, if bidders were to collude (or know each others valuations) then they can follow a strategy of bidding only on their respective items and get them at lower prices. Note that this also requires the cooperation of the bidders whose demands are not fully satisfied at equilibrium.

In practice this would not happen because (a) the valuations of various bidders are not known to one another and (b) the bidders whose demands are not fully satisfied would keep bidding as long as they get non-negative surplus on some items. In this respect, the mechanism is no worse off than the simple open outcry auction for multiple copies of a single item. Moreover, with finite ϵ, the bidder surpluses are non-unique upto $3m\epsilon$. Therefore, a bidder cannot easily benchmark its expected payoff for collusion purposes, thereby reducing chances for a successful gain-sharing agreement with other bidders.

5 Implications for Online Auctions

The present work extends the theory of multi-item ascending auctions for substitutes to address some issues that become important in a practical online implementation. The multi-item ascending auctions can be implemented in the form of a decentralized mechanism consisting of simultaneous independent ascending (open outcry) auctions on the Internet. We showed that the resulting markets are integrated in an economic sense (for allocation and price discovery) so that the proposed decentralized mechanism can help integrated the highly fragmented online auction markets on the Internet.

We saw that bidding using the LGB strategy in SIA leads to a nearly efficient allocation. Furthermore, the average inefficiency per item is bounded by the size of the bid increment and is independent of the size of the system. By this property, we can make the system as large as we wish and use SIA for allocating items efficiently. However, as the size of the system is increased, the uncertainty in the prices of individual items grows linearly. This may lead to incentives to deviate from the truth-telling LGB strategy. In order to bound the uncertainty in the prices and make it independent of the size of the system, the minimum required bid increment must scale inversely as the size of the system. If software agents are used for bidding, it may be feasible to make the bid increments small enough and thus realize efficient allocation as well as unique prices.

With multi-unit demand and supply, the incentives to withhold demand or supply typically lead to inefficient outcomes. Larger markets in the present context also imply that such incentives are expected to be significantly reduced as the ability of the participants to influence market prices diminish.

The proposed mechanism is well suited for a practical online implementation. It does not require a centralized clearing house and each auctioneer can conduct its own auction independently as a simple open outcry auction. For a bidder, LGB is a fairly simple strategy and needs only the valuations of the bidder and the current price vector of auctions for bidding. It allows the bidders to bid asynchronously (independent of other bidders). Very simple software agents can execute the strategy for bidding across multiple auctions on an auction site and also across multiple auction sites.

The implementation would require that the auctioneers agree on common start and end times for the auctions. They may do so because simultaneous auctions with bidders bidding across them implies more competition for each item in general. (Strictly speaking, the competition works both ways - more bidders compete for each item and more items compete for each bidder). Besides, having a common time-slot for the sale of similar items may have practical advantages in terms of focusing attention of potential buyers.

6 Conclusions

In this paper, we considered multi-item ascending auctions of substitutes in a setup where the bidders want at most a specified number of items and have additive utilities over the items. We relax some of the assumptions used in the literature which are not realistic in a practical implementation. In particular, we allow the minimum required bid increment to be any finite positive quantity and let the bidders place bids asynchronously (any bidder can bid at any time on any item). We considered a multi-demand generalization of the truth-telling greedy bidding strategy of Demange et al. [5] and derived the corresponding results for the properties of the system at equilibrium.

In the given setup, we showed that the maximum inefficiency in the allocation is bounded by $m\epsilon$ where m is the number of items sold. The average inefficiency per item is constant and does not increase as m increases. We further showed that the prices of individual items can vary from one outcome to another by at most $3m\epsilon$. We thus showed that for the prices to converge to close to minimum competitive prices, the bid increment ϵ must be less than $1/m$. For the case of single unit bidder demand, we bounded the maximum surplus which a bidder can extract by unilaterally switching to some other bidding strategy. Based on our results, we suggested an upper bound of $1/m$ on the minimum required bid increment without which the deviation of the parameters at equilibrium (from the corresponding optimal values) grows unbounded as m increases.

We argued that the multi-item ascending auctions can be implemented in the form of independent (decentralized) auctions on the Internet and would help in creation of large integrated markets. The bidding strategy could be executed by simple software agents which would also make it feasible to keep ϵ small enough.

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8 Appendix

Proof: [Lemma 1] Each new bid increases some price in the system by ϵ. Since the valuations in V are finite, and since the prices of items on auction cannot go beyond the valuations, the bidding terminates in a finite number of steps.

To show that bidding would have terminated when $\forall j \in A, COM_P(j) = 1$, note that when $COM_P(j) = 1$, then by definition of $COM_P(j)$, there exists only one bidder i s.t. $j \in G_P(i)$. Therefore, there is no bidder who would outbid bidder i on item j. Since this applies to all items j, the bidding has terminated.

To show the converse, assume for contradiction that at termination, there is an item j such that $COM_P(j) > 1$. Since there can be at most one current winner of item j, there exists a bidder i such that $j \in G_P(i)$ and i is not winning on j. In the next step, bidder i should place a bid on item j. Therefore the auction has not yet reached termination, which contradicts our assumption. But, $\forall j, COM_P(j) \geq 1$ is always true due to the presence of dummy bidders. Hence, at termination $\forall j \in A, COM_P(j) = 1$. □

Proof: [Theorem 1] Let $G = (A, B, E, W)$ be the bipartite graph where A is the set of m vertices representing the m items, B is the set of n vertices representing the n bidders. The edge $(a_i, b_j) \in E$ iff $v_{ij} \geq p_j^r$ and the weight of the edge $W(a_i, b_j)$ is equal to $v_{ij} - p_j^r$. Define, at any stage in the auction, the minimum item surplus (s_i^{min}) of bidder i to be zero if $\mid G_P(i) \mid < k_i$. Otherwise $s_i^{min} = \min(v_{ij} - p_j)$, where j is an item where bidder i currently has a winning bid. Initially, define s_i^{min} as the k_i^{th} largest non negative surplus on the m items. Let s_i^{*min} be the minimum item surplus of bidder i at auction closing.

Claim 1 *At any stage in the auction, if $\exists j$ s.t. $v_{ij} - p_j > s_i^{min} + \epsilon$ and bidder i is currently not winning item j then bidder i will place a bid on the item j.*

Proof: When the bidder i starts bidding, it will place bids on its k_i highest surplus items. From the definition of s_i^{min} the above claim is true initially. At a later stage, the only items (j) for which the condition $v_{ij} - p_j > s_i^{min} + \epsilon$ can be satisfied are the ones which bidder i was winning at some stage.

Now suppose the bidder i gets outbid on item j, and $v_{ij} - p_j > s_i^{min} + \epsilon$, then clearly item j is the maximum surplus item for bidder i and bidder i is winning less than k_i items. Therefore, bidder i will place a bid on item j. If bidder i gets outbid on more than one item (say $j \in L_i$), then only the items in L_i will be among highest surplus items and bidder i will place a bid on each of these items, unless $v_{ij} - p_j \leq s_i^{min} + \epsilon$ for an item j in the set L_i. If $v_{ij} - p_j \leq s_i^{min} + \epsilon$ for an item $j \in L_i$, bidder i will place a bid on the highest surplus item (which may be different from item j) and s_i^{min} will be reduced. Therefore, if $v_{ij} - p_j > s_i^{min} + \epsilon$, then bidder i will place a bid on the item j, always. □

Claim 2 *If bidder i does not win item j, then $v_{ij} - p_j^* \leq s_i^{*min} + \epsilon$.*

Proof: Assume for contradiction that bidder i did not win item j, and when the auction terminates $v_{ij} - p_j^* > s_i^{*min} - \epsilon$. From Claim 1, bidder i will place a bid on item j. Therefore, there is a contradiction to our assumption that the auction has terminated. Hence the proof. □

Define a feasible allocation F on the graph G as a set of edges, $F \subseteq E$ such that for each vertex $j \in A$, there is at most one edge incident on it and for each vertex $i \in B$, there are at most k_i edges incident on it. F represents an allocation of items to bidders. The global total surplus for any allocation is equal to weight $W(F)$ of the allocation (i.e. the sum of the weights of its edges). An allocation maximizing the global total surplus of the system is also a maximum weight feasible allocation (F_{OPT}) of G. Let F_{LGB} be an allocation obtained with the LGB strategy at a price vector P^*.

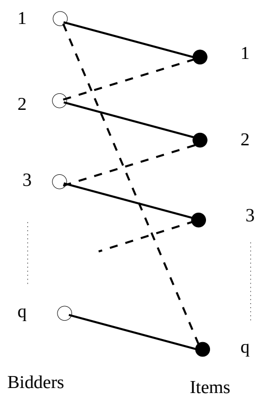

Figure 2: A cycle

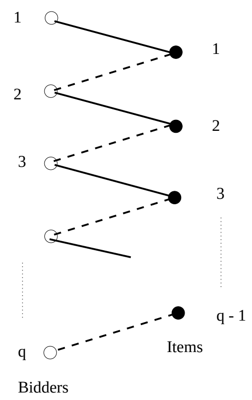

Figure 3: A path

Consider the difference graph $G_D = (A, B, E_D, W)$ where $E_D = (F_{LGB} \cup F_{OPT}) - (F_{LGB} \cap F_{OPT})$. Every item will be sold (allocated) in both the allocations (because of dummy bidders). Therefore G_D can be decomposed into disjoint cycles and maximal paths consisting of alternating edges from F_{LGB} and F_{OPT}. Since every item in both the allocations is sold exactly once, every decomposed path starts with a bidder vertex and ends at a bidder vertex. Define the weight of a subset of edges $Q \subset E_D$ in G_D as:

$$W(Q) = \sum_{e \in Q \ \& \ e \in F_{OPT}} W(e) - \sum_{e \in Q \ \& \ e \in F_{LGB}} W(e).$$

Clearly $W(F_{OPT}) = W(E_D) + W(F_{LGB})$.

Claim 3 *No alternating edge cycle Q in G_D with q bidders has a weight greater than ϵq.*

Proof: Without loss of generality, consider an alternating edge cycle Q as shown in Figure 1. The dashed edges represent the LGB allocations and the solid edges represent the optimal allocation. Since dummy bidders have valuations on only one item, none of the bidders in Q can be dummy bidders. Since bidder 1 does not win item 1 using LGB, from Claim 2 we have:

$$v_{11} - p_1^* \leq s_1^{*min} + \epsilon.$$

Bidder 1 wins item q with LGB strategy. From the definition of s^{*min} we have:

$$s_1^{*min} \leq v_{1q} - p_q^*.$$

Combine the above two equations to get:

$$v_{11} - p_1^* \leq v_{1q} - p_q^* + \epsilon.$$

Similarly for bidder i, $(i \geq 2)$ we have:

$$v_{ii} - p_i^* \leq v_{ii-1} - p_{i-1}^* + \epsilon.$$

Adding the above equations for all values of i yields:

$$W(Q) \leq \epsilon q.$$

□

Claim 4 *No alternating edge maximal path Q of G_D, with q bidders has weight greater than ϵq.*

Proof: Consider an alternating edge maximal path Q of G_D. Without loss of generality, consider the path shown in Figure 2. Clearly, bidder 1 did not get some of its k_1 demanded items with LGB strategy (otherwise the path is not a maximal path). Since bidder 1 could not satisfy its total demand with LGB strategy:

$$v_{11} \leq p_1^* + \epsilon.$$

Since bidder i $(i \neq q)$ won item $i-1$ and did not win item i with LGB strategy, we have

$$v_{ii} - p_i^* \leq s_i^{*min} + \epsilon \leq v_{ii-1} - p_{i-1}^* + \epsilon.$$

Since bidder q bids for item $q-1$ in LGB, its valuation of item $q-1$ must be greater than or equal to the price at which it was sold:

$$v_{qq-1} \geq p_q^*.$$

Note that the above equation holds even if bidder i was a dummy bidder. Adding above equations over i yields:

$$W(Q) \leq \epsilon(q-3).$$

□

Now consider alternating edge maximal paths and cycles of the graph G_D and apply claim 3 or 4 and remove them one by one from G_D. Note that Claim 4 is true even after removal of alternating edge cycles and alternating edge maximal paths from the graph G_D. Since G_D can be decomposed into disjoint alternating edge maximal paths and alternating edge cycles, it follows that $W(E_D) \leq m\epsilon$. Since $W(E_D)$ can also be written as $W(M_{OPT}) - W(M_{LGB})$, therefore $W(M_{OPT}) - W(M_{LGB}) \leq m\epsilon$. □

Proof: [Lemma 2] Order the items of C in ascending order of the time at which the winning bids on the items were received. Consider the item j which is last in this sequence and consider the system at the time when all the bids on all items in C except the final bid on this item had been received. If there already existed some bids on j at this time, then the bidder who placed the current winning bid on j is a marginal loser of set C by definition. If there did not exist any bid on j at this time, the item j received only one bid, that of its dummy bidder. Since all bids by dummy bidders precede any bid by an exogenous bidder, therefore no item in C got any bids from any exogenous bidder. $\square$

Proof: [Theorem 2] We first prove the following claim:

Claim 5 *For any set of items $S \subseteq A$, if $\forall j \in S, p_j^1 - p_j^2 \geq \delta > 3\epsilon$, then $\exists\ t \notin S$ s.t. $p_t^1 - p_t^2 \geq \delta - 3\epsilon$.*

Proof: Let bidder i be a marginal loser of the set S in the first outcome. By Lemma 2 we know that it exists. There are the following two cases:

Case 1: Bidder i wins r_1 and r_2 items of S in the first and second outcomes respectively, and $r_1 \neq r_2$.

Because of dummy bidders, each item of S is sold in both the outcomes. Since $r_1 \neq r_2$, there exists a bidder (say i') who wins more items of S in the first outcome than in the second outcome. Let item j' be an item of S that was won by bidder i' in the first outcome, but not in the second outcome. Since bidder i' wins item j' in the first outcome, from the definition of minimum item surplus, it follows that:

$$v_{i'j'} - p_{j'}^1 \geq s_{i'}^{1min}.$$

Since bidder i' did not get item j' in the second outcome, from Claim 2 it follows that:

$$v_{i'j'} - p_{j'}^2 \leq s_{i'}^{2min} + \epsilon.$$

Combine the above two inequalities to get:

$$s_{i'}^{2min} - s_{i'}^{1min} \geq \delta - \epsilon. \tag{1}$$

Since $\delta > 2\epsilon$, $s_{i'}^{2min} > \epsilon$, and therefore bidder i' gets $k_{i'}$ items in the second outcome. So, bidder i' gets an item t not in S in the second outcome that it did not get in the first outcome. From Claim 2 and the definition of minimum item surplus we get the following two inequalities.

$$s_{i'}^{1min} \geq v_{i't} - p_t^1 - \epsilon$$

$$s_{i'}^{2min} \leq v_{i't} - p_t^2.$$

Combine the above two inequalities with (1) to get:

$$p_t^1 - p_t^2 \quad \geq \quad \delta - 2\epsilon. \tag{2}$$

Case 2: Bidder i wins r items of S in both the outcomes.

By definition of marginal loser, there exists an item j in S on which bidder i placed the second highest bid in the first outcome and did not place any more bids on items in S. There can be two subcases:

(a) Bidder i does not win item j in the second outcome.

Since bidder i placed the second highest bid on item j in the first outcome,

$$v_{ij} - (p_j^1 - \epsilon) \geq s_i^{1min}.$$

Bidder i does not win item j in the second outcome. From Claim 2:

$$v_{ij} - p_j^2 \leq s_i^{2min} + \epsilon.$$

Combine the above two inequalities to get:

$$s_i^{2min} - s_i^{1min} \quad \geq \quad \delta - 2\epsilon.$$

(b) Bidder i wins item j in the second outcome.

Since bidder i wins the same number of items of S in the two outcomes, and bidder i does not win item j of S in the first outcome, there must exist an item j' in S which bidder i wins in the first outcome but not in the second outcome. Bidder i wins item j' in the first outcome, so:

$$v_{ij'} - p_{j'}^1 \geq s_i^{1min}$$

Bidder i does not win item j' in the second outcome. Using Claim 2:

$$v_{ij'} - p_{j'}^2 - \epsilon \leq s_j^{2min}$$

Combine the above two inequalities to get:

$$\begin{aligned} s_i^{2min} - s_i^{1min} &\geq p_{j'}^1 - p_{j'}^2 - \epsilon \\ \Rightarrow s_i^{2min} - s_i^{1min} &\geq \delta - 2\epsilon. \end{aligned} \tag{3}$$

Thus, in both the subcases, the minimum item surplus for bidder i increases by at least $\delta - 2\epsilon > \epsilon$ in the second outcome as given in the following inequality:

$$s_i^{2min} - s_i^{1min} \geq \delta - 2\epsilon > \epsilon. \tag{4}$$

Now there are the following two subcases.

(a) Bidder i won less than k_i items in the first outcome.

In this case, $s_i^{1min} = 0$. From Expression 4 it follows that $s_i^{2min} > \epsilon$. So, it must win k_i items in the second outcome. As a result, there exists an item t not in the set S which bidder i wins in the second outcome but not in the first outcome. So, we have the following inequalities:

$$v_{it} - p_t^2 \geq s_i^{2min}$$

$$v_{it} - p_t^1 \leq s_i^{1min} + \epsilon$$

Combine the above two inequalities with (4) to get:

$$p_t^1 - p_t^2 \quad \geq \quad \delta - 3\epsilon. \qquad (5)$$

(b) Bidder i won k_i items in the first outcome. After being outbid on the item j of S, bidder i never places a bid on any items of S. But it does place bids on items outside S. Therefore it should win an item j' outside the set S such that:

$$v_{ij'} - p_{j'}^1 = s_i^{1min} \qquad (6)$$

Consider the following two subcases:

i. Bidder i wins item j' in the second outcome.

$$v_{ij'} - p_{j'}^2 \quad \geq \quad s_i^{2min}$$

Combine the above inequality with (4) and (6) to get:

$$p_{j'}^1 - p_{j'}^2 \quad \geq \quad \delta - 2\epsilon. \qquad (7)$$

ii. Bidder i does not win item j' in the second outcome.

Since $s_i^{2min} > \epsilon$, bidder i must win k_i items in the second outcome. So, there must exist an item t not in the set S that bidder i wins in the second outcome, but not in the first outcome. So we have the following:

$$\begin{aligned} v_{it} - p_t^1 \quad &\leq \quad s_i^{1min} + \epsilon \\ v_{it} - p_t^2 \quad &\geq \quad s_i^{2min} \end{aligned}$$

Using above two inequalities and (4) it follows that:

$$p_t^1 - p_t^2 \quad \geq \quad \delta - 3\epsilon. \qquad (8)$$

Proof of Claim 5 follows from (2), (5), (7) and (8). □

To complete the proof of Theorem 2, Assume for contradiction that there is an item j such that $p_j^1 - p_j^2 > 3m\epsilon$. Consider the set $S = \{j\}$ and apply Claim 5 on set S to get an item t such that $p_t^1 - p_t^2 > 3(m-1)\epsilon$. Expand the set S repeatedly by adding a new item t in it using Claim 5. Therefore S can be expanded to cover all the items in A. This implies that price of each item in A falls by at least 3ϵ in the second outcome (i.e. $\forall j \in A, p_j^1 - p_j^2 > 3\epsilon$). At this stage, it is not possible to find another item t not in S such that $p_t^1 - p_t^2 > 0$ and Claim 5 is violated. Therefore our assumption that $\exists j$ s.t. $p_j^1 - p_j^2 > 3m\epsilon$ must be incorrect. Therefore: $\forall j \in A : p_j^1 - p_j^2 \leq 3m\epsilon$. Since the two outcomes can be interchanged without losing generality, Theorem 2 follows. □

Proof: [Theorem 3] The proof uses the following claim:

Claim 6 *Let P^1 be the final price vector for an outcome of $SIA(V, K, P^r)$ with bidders having single unit demands when all the bidders follow the LGB strategy. Consider another outcome where all bidders except for a bidder i follow LGB. If bidder i places a bid p_j on item j in the second outcome such that $p_j^1 - p_j > 3m\epsilon$, it will be outbid.*

Proof: Assume for contradiction that bidder i is not outbid. The price of item j in the second outcome is given by $p_j^2 = p_j < p_j^1 - 3m\epsilon$. Consider the set $S = \{j\}$ and expand it by applying Claim 5 to include a new item t such that $p_t^1 - p_t^2 > 3(m-1)\epsilon$. Note that Claim 5 places no restriction on the bidding strategy followed by the winners of S in the second outcome and hence it can be applied even in the presence of a bidder who may not follow LGB. Therefore the set S can be expanded by applying Claim 5 repeatedly to include all the items A and eventually get a contradiction as in the proof of Theorem 2. Therefore bidder i will eventually be outbid. □

Let P^1 be the final price vector when all bidders follow LGB. It is clear from Claim 6 that the bidder i cannot win any item j at a price less than $p_j^1 - 3m\epsilon$. If it did not win any item when it followed LGB (and so had a zero surplus on all items at the price vector P^1), it cannot get a surplus of greater than $3m\epsilon$ by winning any item using any other strategy. On the other hand, if the bidder i had won an item j when it followed LGB, it cannot increase its surplus by more than $3m\epsilon$ by winning the same item j. Further, since item j was the surplus maximizing item for bidder i at P^1, the bidder cannot increase its surplus by more than $3m\epsilon$ by winning any other item when prices do not fall by more than $3m\epsilon$. This completes the proof. □