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Abstract

We propose a simultaneous descending price auction mechanism to sell multiple heterogeneous items to a number of buyers. Buyers have different known private valuation on each of the items, and each buyer wants at most one item. We show that if sellers follow a descending price offer procedure and the buyers follow a greedy strategy for accepting the offers, the auction mechanism results in a nearly efficient allocation with the final selling prices converging close to a competitive equilibrium price. The seller's descending price offer strategy is close to a Nash equilibrium. We also discuss the equilibrium strategy for the buyers. We show the buyers are better off waiting and there is a maximum limit (corresponding to the minimum competitive equilibrium price) till which they can wait without running into risk of losing an item. Further, we prove that if the buyers wait within this limit, the prices can be brought arbitrarily close to a uniquely defined competitive equilibrium price.

Keywords: Competitive equilibrium; Dominant strategy; Dutch auction; English auction; Nash equilibrium

JEL Classification Number: D44

1 Introduction

Commerce on the Internet has increased in the last couple of years. The mechanisms for doing commerce have also received a lot of attention by researchers. Auction is becoming one of the most popular forms of trading on the Internet, as it allows competitive price discovery and fair and efficient allocation of resources. This has sparked renewed interest in the theory of auctions.

Most of the auctions on the Internet are currently conducted in isolation from one another leading to fragmented markets and inefficient allocations. The theory of multi-item auctions (Bansal and Garg, 2000) can be used to integrate these fragmented markets into a single unified market, thereby resulting in efficient allocations, and more competitive price discovery.

Consider a set of n buyers having known private valuations on a set of m items. Assume, that each buyer is interested in getting at most one item from this set of m items. How should the items be assigned to the buyers? What is a fair and competitive price of these items? What market mechanism can be designed to discover these prices and assignment? What is the impact of different strategies of buyers and sellers on the outcomes of these mechanisms? The theory of multi-item auctions has answers to some of these questions.

The earliest study on this problem was done by Koopmans and Beckmann (1957) who established the existence of Walrasian equilibrium for the above assignment problem. Later, Shapley and Shubik (1972) studied the properties of the core of the above problem and showed that the core forms a complete lattice. As a result, there exists a unique minimum and a unique maximum competitive price vector that forms a Walrasian equilibrium for the assignment problem. Recently, Gul and Stacchetti (1999) have extended these results to the general case when instead of single unit demand, the buyer demands are *gross substitutes* over sets of items.

Under the same conditions, Leonard (1983) has considered a sealed bid mechanism for allocating items to bidders. He showed that if the bidders are charged prices for items equal to their minimum competitive equilibrium prices and are assigned items according to the minimum competitive equilibrium assignment, then the resulting allocation is efficient. In this case, the resulting mechanism becomes incentive compatible (i.e. bidders have no incentive to misrepresent their valuations). Compare this with the “second price” auction mechanism of Vickrey (1961) for sealed bid auction of a single item. Here, the item is assigned to the bidder with the highest bid, but at a price equal to the second highest bid. Leonard’s mechanism can be thought as a generalization of the “second price” auction mechanism to the case of multiple heterogeneous items.

Crawford and Knoer (1981) have studied a generalized version of this problem in the context of firms and workers where workers have to be assigned to firms. They described a mechanism (called salary adjustment process) that converges to the equilibrium assignment and competitive prices (salaries). Kelso and Crawford (1982) extended this to the case where the bidders have more general demand (called *gross substitute*). The salary adjustment process requires that the individual valuations be an integral multiple of the discrete bid increment and bidders participate in the mechanism in a manner that is significantly different than the conventional auction mechanisms.

Demange et al. (1986) have described a mechanism based on traditional ascending English open-cry auctions. In this mechanism, each item is auctioned independently as an ascending open-cry auction, but all the auctions open and close at the same times. The bidders place bids on items that give them maximum

surplus. Demange et al. showed that this mechanism leads to a final allocation that can be made arbitrarily close to the minimum price equilibrium. Further, Bertsekas (1992) showed that such an auction leads to nearly efficient allocation. Miyake (1998) has studied the strategic behavior of bidders in this mechanism and showed that under certain assumption on bidding behavior, the simple greedy truth-telling strategy is a dominant strategy. Bansal and Garg (2000) later showed that under no restrictions on bidding behavior, the greedy strategy ceases to be a dominant strategy but still constitutes a Nash equilibrium. Therefore, the progressive auction mechanism of Demange et al. (1986) may be seen as a natural generalization of ascending English open-cry auctions for the case of multiple items.

While second price sealed bid auctions and ascending English auctions have been generalized to the case of multiple heterogeneous items, descending price Dutch auction have not yet been. Many times sellers prefer to auction their items using a descending price Dutch auction for a variety of reasons. Most of the auctions on the Internet are ascending-price in nature. Since, Dutch auctions are inherently faster than the English auctions, this can save costs of the customers, on time spent. For example, some ascending-price auctions on the Internet go on for weeks, but in case of a descending-price auction, the buyer can end the *game* much more rapidly. On the other hand, descending-price auctions need a natural upper price to start whereas ascending-price auctions do not require this. But the model which we consider in this paper, the sellers have reserve prices on items which can be used as a natural starting price in descending-price auctions.

In this paper we propose a multi-item generalization of descending price Dutch auctions and discuss its properties. In a Dutch auction, a seller initially announces a high price for his item and then reduces it progressively until he finds a buyer ready to buy his item. In the multi-item generalization, the sellers offer their items to each buyer starting with a high price initially. The sellers progressively reduce their prices by a small amount (say ϵ) until they find a buyer who tentatively accepts their offer. A buyer accepts an offer tentatively only if it gives her a positive surplus. The auction rules permit a buyer to reject an accepted offer and switch to a different offer. A buyer switches to a new offer if it gives her a surplus better than the surplus on the current offer. The auction ends when no more offers are made and then the buyers and sellers are obliged to honor their current commitments and offers.

We show that the multi-item generalization of Dutch auctions, results in a nearly efficient allocation. We discuss the buyers' and sellers' strategic behavior in this auction. We show that if all the sellers follow the above offer procedure then, a seller cannot increase his surplus significantly by unilaterally following a different strategy. The suggested offer procedure approaches close to a Nash equilibrium for the sellers as ϵ approaches zero. For the buyers, committing to an item as soon as its surplus becomes positive has strategic disadvantages. The buyers can derive a better surplus if they wait for the prices to fall further. However, the buyers run into the risk of losing the item if they wait long. We show that the maximum achievable surplus for a buyer corresponds to her surplus in minimum competitive price vector. If a buyer waits for her surplus to become equal to the maximum achievable (minus $m\epsilon$), she is guaranteed to get an item irrespective of how long other buyers are waiting. On the other hand, if all the other buyers do not wait after their surplus becomes equal to their maximum achievable surplus, then a buyer is guaranteed to lose her item in case she waits longer after her surplus becomes equal to the maximum achievable surplus (plus $m\epsilon$). So the strategy of waiting till a buyer gets her maximum achievable surplus approaches close to a Nash Equilibrium for buyers. These results on strategic behavior are closely related to those of Demange and Gale (1985) and Demange et al. (1986). While Demange and Gale(1985) consider strategy structure of

a continuous matching market in a sealed-bid auction scenario (where buyers submit their valuations to a referee), we consider a progressive auction scenario where the prices are successively brought down by finite decrements. Therefore, the strategic behavior results of (Demange and Gale, 1985) are only approximately valid in the discrete market. We provide the bounds within which the payoffs and prices may vary in the discrete market. Demange et al. (1986) discuss progressive auctions with finite bid-increments, but they only consider ascending price auctions and do not consider strategic behavior of agents.

We show that the final prices in the auction mechanism converges close to the maximum competitive equilibrium price if the buyers follow the safest strategy of committing to an item as soon as they get a positive surplus on it. If the buyers wait till their surplus becomes equal to their maximum achievable surplus, then the prices converge close to the minimum competitive equilibrium price. In general, the prices converge (nearly) to a uniquely defined competitive equilibrium price even if buyers decide to wait suitably and we characterize this price.

Our results on prices may be seen as a generalization of results of (Shapley and Shubik, 1972), who show that maximum competitive equilibrium prices are unique. We extend this notion while taking the buyers strategies into account (seller's in case of ascending price auctions). We show that maximum competitive equilibrium prices corresponding to a buyers surplus vector is unique and the auction mechanism converges close to these prices.

Our results indicate that for the same set of buyers, sellers and valuation functions, our auction may produce different prices. This variation in prices is due to the use of finite bid decrements (ϵ) and the random nature of tie-breaking procedure in the buyers' and sellers' strategies. We provide bounds on such price variations. We also conducted simulations to see the variation of prices. In a real-world setting, a buyer will have value on small subset of available items and the number of buyers will be more than the number of items. Under these settings, our simulations show that for about 100 buyers and 80 items, 75% of the items have price variations less than 10ϵ . As the number of buyers are increased (and the number of sellers increased proportionately too), this percentage increases, i.e. with larger market sizes, the number of items having price variations less than 10ϵ increase. In general, the price variations are of the order of $\frac{m}{5}\epsilon$, where m is the number of items. This is well below the theoretical bound of $2m\epsilon$ provided in theorems 6 and 7.

The rest of the paper is organized as follows. In Section 2 we begin with describing our proposed descending price auction mechanism, the seller's offer procedure and the buyer's response. In Section 3 we discuss the optimality properties of this mechanism. In Section 4, we discuss the strategic behaviors of buyers and sellers in our mechanism. In Section 5, we investigate the equilibrium properties of this mechanism. We comment on multi-item ascending English auctions in Section 6. In section 7 we report some simulation results of our auction and we conclude in section 8.

2 The Auction Mechanism

We propose a multi-item generalization of descending price Dutch auctions. In the multi-item generalization, all the buyers enter the auction at the same time. The sellers may however join the auction any time. We call this *Simultaneous Descending Price Auction Mechanism (SDPAM)*.

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m\}$ be the set of m items available for sale by m different sellers and $\mathcal{B} = \{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ be the set of n buyers. Assume that each buyer is interested in buying at most one item

from \mathcal{A} . Also, let $v_{\beta_i \alpha_j}$ denote the valuation of buyer β_i on item α_j . If a buyer β_i is not interested in buying an item α_j , then $v_{\beta_i \alpha_j} = 0$. Let $v_{\alpha_1}, v_{\alpha_2}, \dots, v_{\alpha_m}$ be the valuation of sellers on their respective items and $r_{\alpha_1}, r_{\alpha_2}, \dots, r_{\alpha_m}$ be their respective reserve prices.

2.1 Auction Rules

The auction is conducted as follows. A seller may *offer* his item to any buyer at any price. The buyer can either tentatively accept or reject the offer. If a buyer tentatively accepts an offer, the seller and the buyer are said to be committed to each other. Similarly other sellers may also offer their items to buyers, including those buyers who are already committed to other sellers. In case a buyer who is already committed receives an offer and prefers it to her existing commitment, she can *switch* from earlier offer (by rejecting it) to the new offer (by tentatively accepting it). The buyer remains committed in the process whereas the seller of earlier offer becomes uncommitted. An offer is binding for the seller. Unlike the buyers, a seller cannot withdraw his offer unless it is rejected by the buyer. A seller may offer his items to buyers only if he is not committed (i.e. all his earlier offers have been eventually rejected).

An uncommitted seller may withdraw his item from the auction at any point of time if he is unable to sell it at the desired price. The auction closes when no more offers are made i.e. every seller is either committed or has withdrawn his item from the auction. After the auction closing, the committed buyers are required to buy their item from their respective committed sellers at the last offer price.

2.2 Offer Procedure

We now describe a simple offer strategy called the *Decreasing Price Offer (DPO)* strategy that is a generalization of seller's behavior in Dutch auctions.

Each seller carries out his offer to the buyers independently. He starts with a high price and lowers it by the minimum price decrement (say ϵ) if there is no demand on his item. At each price, the seller takes the item to each of the buyers one by one. The order in which he makes offers to the buyers is arbitrary and does not affect the results of this paper. As soon as a buyer accepts his offer, he becomes committed and he stops offering his item. If the committed buyer switches from his item, he again starts offering the item at the current price to rest of the buyers. If no one accepts his offer at the current price, he lowers the price by ϵ and makes the offers again. The seller withdraws the item from auction if its price reaches the reserve price and no buyer commits to the item.

Alternatively, the seller may publicly announce a price for his item which amounts to making an offer to every buyer in the system. Any buyer who does not respond to the offer is assumed to have rejected the offer. The buyer who intends to tentatively accept the offer is expected to respond to the offer within a fixed time period. The seller may arbitrarily choose and notify one of the buyers who respond to his offer. The buyer who gets the seller's notification gets committed to him. In case the buyer was committed earlier, she uncommits from her earlier seller by sending him a notification. The uncommitted seller announces the same price again to make sure that no buyer is interested in his offer. The seller lowers the price by ϵ and announce the price again if no one accepts his earlier offer. The seller withdraws the item from auction if its price reaches the reserve price and no buyer commits to the item.

A seller may set a reserve price that is different from his valuation on the item. If a seller wants a

minimum surplus of t , he will set a reserve price that is t more than his valuation. We denote the strategy followed by seller of item α as $DPO(t)$ if $r_\alpha = v_\alpha + t$. If a seller sets a reserve price equal to his valuation, then the corresponding DPO strategy is $DPO(0)$. So DPO refers to a class of strategies with different minimum surplus amounts.

2.3 Buyer's Response

A buyer β_i is said to have a surplus of $v_{\beta_i, \alpha_j} - p_t(\alpha_j)$ at time instant t on item α_j , where $p_t(\alpha_j)$ is the price of the item α_j at that time instant. Let us associate with buyer $\beta_i \in \mathcal{B}$ a *starting surplus amount* \hat{s}_{β_i} . If a buyer is not committed to any item, she accepts any offer that provide at least \hat{s}_{β_i} surplus. However, if a buyer is committed to an offer, then she accepts an offer on an item while rejecting the earlier offer, if the new offer gives her more surplus. If a buyer does not reject the earlier offer, she gets committed to more than one offer at a time. If the auction ends at this instant, she will have to buy both the items, which she will never like to do in our model. Note that the only information available to a buyer at a given instant is the current price vector of the items (which is a public information) and her own valuation of these items (which is private information). Therefore buyers cannot accurately predict the ending of auctions. Also, there is no strategic advantage in holding onto two items at a time, as it only serves to stop the prices of items from falling further.

As can be seen, by following this strategy, the buyer always increases her surplus by positive amounts with every offer she accepts. We call this strategy, *Local Surplus Increasing (LSI)* strategy. More specifically, if a buyer β_i has a starting surplus amount of \hat{s}_{β_i} , we call her strategy, $LSI(\hat{s}_{\beta_i})$. If $\hat{s}_{\beta_i} = 0$, then the corresponding strategy of the buyer is $LSI(0)$. So when we say simply LSI, we refer to a class of strategies with any starting surplus amount possible.

According to the auction rules, a buyer cannot be committed to more than one item at any time instant. With LSI strategy, as soon as a buyer gets an offer with better surplus, she uncommits from her item and accepts the new offer. In case the offer has lower surplus she rejects the offer and waits for the prices of items to fall. Therefore, she is guaranteed to get a surplus equal to her starting surplus amount if she gets an item.

2.4 Total Global Surplus

Let us denote the final price vector in the mechanism by $\hat{\mathbf{p}}$. So, item α_i has a final price of \hat{p}_{α_i} . Let buyer β_j be the winner of α_i . So, the surplus of the buyer on the item α_i is $v_{\beta_j, \alpha_i} - \hat{p}_{\alpha_i}$. Surplus of the seller on this item is $\hat{p}_{\alpha_i} - v_{\alpha_i}$. Therefore, surplus of the system due to this item is $v_{\beta_j, \alpha_i} - v_{\alpha_i}$. If we denote winner of item α_i as w_i , we can define the total global surplus as $S_G = \sum_{i=1}^m (v_{w_i, \alpha_i} - v_{\alpha_i})$ (assume $v_{w_i, \alpha_i} = v_{\alpha_i}$, when α_i is not assigned). Thus, the total global surplus of the system is independent of the final price vector. It only depends on the final assignment of items to buyers.

3 Optimality Properties

In this section we discuss about the optimality properties of SDPAM. We will denote the final price of α_i ($\alpha_i \in \mathcal{A}$) in SDPAM as \hat{p}_{α_i} . We assume buyers always follow LSI strategy and sellers always follow DPO

strategy in SDPAM. Since price decrements are finite, positive and fixed, final prices cannot be less than zero, the auction mechanism terminates.

Proposition 1 *If a buyer β_i following an LSI strategy gets an item α_i in SDPAM and does not get the item α_j , then $v_{\beta_i\alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i\alpha_j} - \hat{p}_{\alpha_j} - \epsilon$.*

Proof: Consider an instant when the price of α_j is about to drop from $\hat{p}_{\alpha_j} + \epsilon$ to \hat{p}_{α_j} . Since the price is dropping (as sellers follow DPO strategy), the item must have been offered to β_i at the price $\hat{p}_{\alpha_j} + \epsilon$ and she would have rejected the offer. This means either she had a surplus less than \hat{s}_{β_i} on α_j at that price or she was committed to an item that was giving her higher surplus than α_j . In the first case, the proposition is true (since $v_{\beta_i\alpha_i} - \hat{p}_{\alpha_i} \geq \hat{s}_{\beta_i} > v_{\beta_i\alpha_j} - (\hat{p}_{\alpha_j} + \epsilon)$). In the second case, let us assume she was committed to α_k at that instant with a price p_k . But finally she switched to α_i . Since offers get only better in terms of surpluses to buyers if LSI strategy is followed, we can write the following:

$$v_{\beta_i\alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i\alpha_k} - p_k.$$

Also, since β_i did not accept the offer of α_j at the price $\hat{p}_{\alpha_j} + \epsilon$, we have the following:

$$v_{\beta_i\alpha_k} - p_k \geq v_{\beta_i\alpha_j} - \hat{p}_{\alpha_j} - \epsilon.$$

Adding last two inequalities we have $v_{\beta_i\alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i\alpha_j} - \hat{p}_{\alpha_j} - \epsilon$. □

Proposition 2 *If a buyer β_1 following an LSI strategy has a surplus of at least $\hat{s}_{\beta_1} + \epsilon$ on any item, she should get an item in SDPAM.*

Proof: Consider the item α_1 on which the buyer has surplus greater than or equal to $\hat{s}_{\beta_1} + \epsilon$. Let the price of α_1 be p_1 . Consider the instant when the price of α_1 was $p_1 + \epsilon$. At this instant, β_1 should have a surplus of at least \hat{s}_{β_1} on α_1 . Since price of α_1 drops and β_1 follows LSI once her surplus reaches \hat{s}_{β_1} , she should have been offered the item α_1 at price $p_1 + \epsilon$ (by DPO strategy of sellers). She would have eventually rejected the offer of α_1 since its price drops further. This means she should be committed to some other item. By the rules of SDPAM, once a buyer commits to an item she is guaranteed to get some item. Hence the proposition. □

Theorem 1 (Efficiency) *Let S_{OPT} denote the maximum total global surplus under any optimal allocation and S_{SYS} denote the total global surplus for the system in SDPAM. If all buyers follow LSI(0) strategy and all sellers follow DPO(0) strategy, then $S_{OPT} - S_{SYS} < m\epsilon$.*

Proof: For simplicity assume that if an item α_i is not sold there is a dummy buyer who values α_i at v_{α_i} and wins it at v_{α_i} ($v_{\alpha_i} = r_{\alpha_i}$, as sellers follow DPO(0) strategy). We can divide the set of winning buyers into two main categories, \mathcal{B}_{OPT} and \mathcal{B}_{SYS} , where \mathcal{B}_{OPT} represents the set of buyers who win an item in the optimal allocation and \mathcal{B}_{SYS} represents the set of buyers who win an item in the SDPAM following LSI(0) strategy. Let us denote the item won by buyer β_i in optimal allocation by w_{β_i} and in SDPAM by \bar{w}_{β_i} . We can write the following two equations:

$$S_{OPT} = \sum_{\beta_i \in \mathcal{B}_{OPT}} (v_{\beta_i w_{\beta_i}} - v_{w_{\beta_i}}), \quad (1)$$

$$S_{SYS} = \sum_{\beta_i \in \mathcal{B}_{SYS}} (v_{\beta_i \bar{w}_{\beta_i}} - v_{\bar{w}_{\beta_i}}). \quad (2)$$

Subtracting (2) from (1), the valuation of sellers on items will cancel out (since every item is sold because of dummy buyers). So we can write the following equation:

$$S_{OPT} - S_{SYS} = \sum_{\beta_i \in \mathcal{B}_{OPT}} v_{\beta_i w_{\beta_i}} - \sum_{\beta_i \in \mathcal{B}_{SYS}} v_{\beta_i \bar{w}_{\beta_i}}.$$

Further, \mathcal{B}_{OPT} and \mathcal{B}_{SYS} be divided into 3 disjoint sets of buyers. \mathcal{B}_{BOTH} be the set of buyers defined as $\tilde{\mathcal{B}}_{OPT} \cap \tilde{\mathcal{B}}_{SYS}$. $\tilde{\mathcal{B}} = \mathcal{B}_{OPT} - \mathcal{B}_{SYS}$ and $\hat{\mathcal{B}} = \mathcal{B}_{SYS} - \mathcal{B}_{OPT}$. So, the above equation reduces to the following form:

$$S_{OPT} - S_{SYS} = \sum_{\beta_i \in \mathcal{B}_{BOTH}} (v_{\beta_i w_{\beta_i}} - v_{\beta_i \bar{w}_{\beta_i}}) + \sum_{\beta_i \in \tilde{\mathcal{B}}} v_{\beta_i w_{\beta_i}} - \sum_{\beta_i \in \hat{\mathcal{B}}} v_{\beta_i \bar{w}_{\beta_i}}. \quad (3)$$

Let $|\mathcal{B}_{BOTH}| = q$. So $|\tilde{\mathcal{B}}| = |\hat{\mathcal{B}}| = t$ (say). If $w_{\beta_i} = \bar{w}_{\beta_i}$ then $v_{\beta_i w_{\beta_i}} - v_{\beta_i \bar{w}_{\beta_i}} = 0$. Otherwise from proposition 1 we have:

$$v_{\beta_i w_{\beta_i}} - v_{\beta_i \bar{w}_{\beta_i}} < \hat{p}_{w_{\beta_i}} - \hat{p}_{\bar{w}_{\beta_i}} + \epsilon.$$

Summing it over all $\beta_i \in \mathcal{B}_{BOTH}$ we get:

$$\sum_{\beta_i \in \mathcal{B}_{BOTH}} (v_{\beta_i w_{\beta_i}} - v_{\beta_i \bar{w}_{\beta_i}}) < \sum_{\beta_i \in \mathcal{B}_{BOTH}} (\hat{p}_{w_{\beta_i}} - \hat{p}_{\bar{w}_{\beta_i}}) + q\epsilon. \quad (4)$$

Consider $\beta_i \in \tilde{\mathcal{B}}$. From definition of $\tilde{\mathcal{B}}$, β_i does not get any item in SDPAM. Using proposition 2 and the fact that $\hat{s}_{\beta_i} = 0$ we have:

$$v_{\beta_i w_{\beta_i}} - \hat{p}_{w_{\beta_i}} < \epsilon. \quad (5)$$

Adding it for all t elements in $\tilde{\mathcal{B}}$, we get:

$$\sum_{\beta_i \in \tilde{\mathcal{B}}} v_{\beta_i w_{\beta_i}} < \sum_{\beta_i \in \tilde{\mathcal{B}}} \hat{p}_{w_{\beta_i}} + t\epsilon. \quad (6)$$

Since all the buyers in SDPAM who get some item have non-negative surplus on that item, we can write $v_{\beta_i \bar{w}_{\beta_i}} - \hat{p}_{\bar{w}_{\beta_i}} \geq 0$, where $\beta_i \in \hat{\mathcal{B}}$. Summing it over all the elements in $\hat{\mathcal{B}}$, we have the following:

$$\sum_{\beta_i \in \hat{\mathcal{B}}} v_{\beta_i \bar{w}_{\beta_i}} \geq \sum_{\beta_i \in \hat{\mathcal{B}}} \hat{p}_{\bar{w}_{\beta_i}}. \quad (7)$$

Combining (3), (4), (6) and (7) we get:

$$S_{OPT} - S_{SYS} < \sum_{\beta_i \in \mathcal{B}_{BOTH}} (\hat{p}_{w_{\beta_i}} - \hat{p}_{\bar{w}_{\beta_i}}) + q\epsilon + \sum_{\beta_i \in \tilde{\mathcal{B}}} \hat{p}_{w_{\beta_i}} + t\epsilon - \sum_{\beta_i \in \hat{\mathcal{B}}} \hat{p}_{\bar{w}_{\beta_i}}.$$

Since all items are always sold, the prices cancel out and this implies $S_{OPT} - S_{SYS} < (q + t)\epsilon$ or, $S_{OPT} - S_{SYS} < m\epsilon$. \square

The consequence of this theorem is that if ϵ is kept really small with buyers following LSI(0) strategy and sellers making offers by following DPO(0) strategy, our mechanism will lead to nearly efficient allocation. Note that the total inefficiency introduced due to finite bid decrement (ϵ) is $m\epsilon$. Therefore average inefficiency per item is only of the order of ϵ . Therefore the system is scalable from the efficiency point of view.

4 Strategic Behavior

Incentive properties of a mechanism are as important as the optimality properties. We showed that SDPAM with buyers following LSI(0) strategy and sellers following DPO(0) strategy converges close to an efficient allocation if ϵ is kept small. But whether there exists any incentive for buyers to follow LSI(0) and sellers to follow DPO(0) is a critical question. This section will try to address this question.

Since the price decrements are finite and fixed, the order in which the offers are made to the buyers (or if the offers are made publicly, then the order in which buyers respond to offers) may make a difference in the final allocation and small difference in the prices and surpluses. We show that this difference is a linear function of ϵ and can be made arbitrarily small. To discuss this, concepts like Nash equilibrium and dominant strategies need to be adapted. We therefore suggest the notion of δ -Nash equilibrium and δ -dominant strategy which can be used to approximate Nash equilibrium and dominant strategies within a small neighborhood. Let $u_i(\sigma, \sigma_{-i})$ represent the utility of player i when it adopts the strategy σ and all the other players adopt the strategy σ_{-i} .

Definition 1 (δ -Nash equilibrium) A strategy profile σ^* constitutes a δ -Nash Equilibrium if, for every player i ,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \delta,$$

$\forall \sigma_i \in S_i$, where S_i is the set of strategies player i can adopt.

Definition 2 (δ -Dominant strategy) A strategy σ_i^* is a δ -dominant strategy for player i if

$$u_i(\sigma_i^*, \underline{\sigma}_{-i}) \geq u_i(\sigma_i, \underline{\sigma}_{-i}) - \delta,$$

$\forall \sigma_i \in S_i$ and $\forall \underline{\sigma}_{-i} \in \underline{S}_{-i}$, where S_i is the set of strategies that player i can adopt and \underline{S}_{-i} is the set containing the profile of all the strategies that all the players other than player i can adopt.

We now define the concept of competitive equilibrium. Consider a buyer β_i . Define the demand set of buyer β_i at a price vector \mathbf{p} , $D_{\beta_i}(\mathbf{p})$, as the set containing all the items for which β_i has maximum non-negative surplus. Mathematically:

$$D_{\beta_i}(\mathbf{p}) = \begin{cases} \phi & \text{if } \max_{\alpha_k \in \mathcal{A}} [v_{\beta_i \alpha_k} - p_{\alpha_k}] < 0, \\ \alpha_j | v_{\beta_i \alpha_j} - p_{\alpha_j} = \max_{\alpha_k \in \mathcal{A}} [v_{\beta_i \alpha_k} - p_{\alpha_k}] & \text{otherwise} \end{cases}$$

Now we define competitive equilibrium prices as follows.

Definition 3 A price vector \mathbf{p} is a competitive equilibrium price if there exists an assignment μ of item to buyers such that following conditions hold:

1. A buyer who is assigned in μ , gets item from her demand set.
2. Every buyer with a positive surplus on an item gets an item. Buyers having negative surplus on all the items do not get any item.
3. No two buyers get the same item.
4. An assigned item has a price greater than or equal to its seller's valuation. An unassigned item has a price equal to its seller's valuation.

The assignment μ is called the competitive equilibrium assignment at \mathbf{p} and the tuple (μ, \mathbf{p}) is called a competitive equilibrium. Similar definition is also provided by Demange et al. (1986). Note that there can be multiple competitive equilibrium price vectors. Shapley and Shubik (1972) show that the set of all competitive equilibrium price vectors are contained in the core and for unit demands the core forms a complete lattice. As a result, there exist a unique minimum and a unique maximum competitive equilibrium price vector. Gul and Stacchetti (1999) have extended these results to the case of general “gross substitute” demands.

Let \mathbf{p}^{min} denote the minimum competitive equilibrium price vector and \mathbf{p}^{max} denote the maximum competitive equilibrium price vector. Let us call the surplus obtained by bidder $\beta_i \in \mathcal{B}$ in minimum competitive equilibrium as her maximum achievable surplus and denote it by $s_{\beta_i}^{max}$. Let \mathbf{s}^{max} denote the vector of bidder surpluses in minimum competitive equilibrium.

4.1 Seller's Strategy

To get a better price for his item, a seller may decrease the price of his item by large amounts, or may even increase it after decreasing it initially. The following theorem demonstrates the futility of this.

Theorem 2 *Let $\hat{\mathbf{p}}$ be the final price vector in SDPAM when all the buyers follow LSI($\hat{\mathbf{s}}$) strategies with $0 \leq \hat{s}_{\beta_i} \forall \beta_i \in \mathcal{B}$ and all the sellers follow DPO(t) ($t \geq 0$) strategies. If the seller of item α_k was to follow a different strategy and offer his item to a buyer at a price greater than $\hat{p}_{\alpha_k} + 2m\epsilon$, then his offer will eventually be rejected.*

Proof: Assume, for contradiction, that the seller's offer is not rejected. Denote the final prices of SDPAM, when seller of item α_k follows a different strategy, by vector $\bar{\mathbf{p}}$. We call the mechanism *deviated SDPAM* when seller of item α_k deviates but other sellers follow DPO strategies, and call it SDPAM when no one deviates. Buyers always follow LSI strategies. Now consider the following important lemma.

Lemma 1 *Let \mathcal{I} be a set of items won by a set of buyers \mathcal{J} in deviated SDPAM and \mathcal{I} contains the item whose seller deviates such that $\forall \alpha_i \in \mathcal{I}, \bar{p}_{\alpha_i} - \hat{p}_{\alpha_i} > \delta$ ($\delta \geq 2\epsilon$). Then there exists an item $\alpha_j \notin \mathcal{I}$ such that $\bar{p}_{\alpha_j} - \hat{p}_{\alpha_j} > \delta - 2\epsilon$.*

Proof: Since all items in \mathcal{I} have final prices at least 2ϵ above final prices in SDPAM, every buyer β_i in \mathcal{J} will have at least $2\epsilon + \hat{s}_{\beta_i}$ surplus in SDPAM. So from proposition 2, each buyer in \mathcal{J} will win some item in SDPAM. There are two cases to consider.

Case 1: There exists a buyer β_1 in \mathcal{J} who wins an item $\alpha_2 \notin \mathcal{I}$ in SDPAM. Let β_1 win $\alpha_1 \in \mathcal{I}$ in deviated SDPAM. Applying proposition 1 on buyer β_1 in SDPAM we get:

$$v_{\beta_1\alpha_2} - \hat{p}_{\alpha_2} > v_{\beta_1\alpha_1} - \hat{p}_{\alpha_1} - \epsilon.$$

Consider the instant when price of α_2 was $\bar{p}_{\alpha_2} + \epsilon$ in deviated SDPAM (since seller of α_2 does not deviate, α_2 would have been offered at $\bar{p}_{\alpha_2} + \epsilon$). Since β_1 gets α_1 in deviated SDPAM, she should have rejected the offer of α_2 at $\bar{p}_{\alpha_2} + \epsilon$. As β_1 follows LSI strategy, the following equation holds:

$$v_{\beta_1\alpha_1} - \bar{p}_{\alpha_1} > v_{\beta_1\alpha_2} - \bar{p}_{\alpha_2} - \epsilon.$$

Adding the above two inequalities and using the fact that $\bar{p}_{\alpha_1} - \hat{p}_{\alpha_1} > \delta$, we get, $\bar{p}_{\alpha_2} - \hat{p}_{\alpha_2} + 2\epsilon > \bar{p}_{\alpha_1} - \hat{p}_{\alpha_1} > \delta$. This gives $\bar{p}_{\alpha_2} - \hat{p}_{\alpha_2} > \delta - 2\epsilon$.

Case 2: Each buyer in \mathcal{J} wins an item from \mathcal{I} in SDPAM. Since $|\mathcal{I}| = |\mathcal{J}|$, every item in \mathcal{I} is sold in SDPAM.

Now, consider the last item α_j of \mathcal{I} to get committed to a buyer $\beta_i \in \mathcal{J}$ in SDPAM. Just before α_j was offered to β_i , at price \hat{p}_{α_j} , β_i must be committed to an item α_i outside \mathcal{I} . This is so because, if β_i was committed to an item from \mathcal{I} then her switch to item α_j would have freed an item from \mathcal{I} which would have been sold in SDPAM eventually. Therefore α_j couldn't have been the last item in \mathcal{I} to get committed. The other possibility is that β_i was not committed when α_j was offered to her. Let $\alpha_k \in \mathcal{I}$ be the item won by β_i in deviated SDPAM. This would mean that β_i rejected the offer of α_k at price $\hat{p}_{\alpha_k} + \epsilon$ in SDPAM, which is impossible since, $v_{\beta_i\alpha_k} - (\hat{p}_{\alpha_k} + \epsilon) > v_{\beta_i\alpha_k} - (\bar{p}_{\alpha_k} - \delta + \epsilon) > v_{\beta_i\alpha_k} - \bar{p}_{\alpha_k} \geq \hat{s}_{\beta_i}$.

Let p_i be the price at which β_i accepted the offer of α_i in SDPAM. Since β_i preferred α_i at p_i over α_k at $\hat{p}_{\alpha_k} + \epsilon$ we have:

$$v_{\beta_i\alpha_i} - p_i > v_{\beta_i\alpha_k} - \hat{p}_{\alpha_k} - \epsilon.$$

Since price of α_i can only fall we can rewrite the above inequality as:

$$v_{\beta_i\alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i\alpha_k} - \hat{p}_{\alpha_k} - \epsilon.$$

Consider the instant when price of α_i was $\bar{p}_{\alpha_i} + \epsilon$ in deviated SDPAM (since seller of α_i does not deviate, α_i would have been offered at $\bar{p}_{\alpha_i} + \epsilon$). Since β_i gets α_k in deviated SDPAM, she should have rejected the offer of α_i at $\bar{p}_{\alpha_i} + \epsilon$. As β_i follows LSI strategy, the following equation holds:

$$v_{\beta_i\alpha_k} - \bar{p}_{\alpha_k} > v_{\beta_i\alpha_i} - \bar{p}_{\alpha_i} - \epsilon.$$

Adding above two inequalities and using the fact that $\hat{p}_{\alpha_k} < \bar{p}_{\alpha_k} - \delta$ we get:

$$\bar{p}_{\alpha_i} - \hat{p}_{\alpha_i} + 2\epsilon > \bar{p}_{\alpha_k} - \hat{p}_{\alpha_k} > \delta.$$

This gives $\bar{p}_{\alpha_i} - \hat{p}_{\alpha_i} > \delta - 2\epsilon$.

So in both cases we have found an item $\alpha_2 \notin \mathcal{I}$ such that $\bar{p}_{\alpha_2} - \hat{p}_{\alpha_2} > \delta - 2\epsilon$. Hence the lemma is proved.

□

The proof of the theorem is based on repetitively applying this lemma. Since the seller's offer is not rejected, α_k will be sold at a price $\bar{p}_{\alpha_k} > \hat{p}_{\alpha_k} + 2m\epsilon$ in deviated SDPAM. Initially let $\mathcal{I} = \{\alpha_k\}$, $\delta = 2m\epsilon$

and apply Lemma 1 to get an item $\alpha_2 \notin \mathcal{I}$ such that $\bar{p}_{\alpha_2} - \hat{p}_{\alpha_1} > 2(m-1)\epsilon$. Now we can have $\mathcal{I} = \{\alpha_k, \alpha_2\}$ and apply lemma 1 again. We can continue this process and keep discovering new items till we discover $\bar{p}_{\alpha_m} - \hat{p}_{\alpha_m} > 2\epsilon$. But we can still apply Lemma 1 after this. As can be seen, we have already exhausted all the items in \mathcal{A} . Thus we reach at a contradiction. Hence our assumption must be incorrect. Therefore the seller's offer will eventually be rejected. This proves the theorem. \square

Therefore, a seller may not increase his surplus significantly by following a different strategy. However, while following DPO strategy, a seller may set a reserve price that is different than his valuation to potentially get more surplus. Clearly, there is no strategic advantage in setting the reserve price lower than the valuation since a zero surplus with unsold item is strictly better than a negative surplus with sold item. On the other hand, setting the reserve price higher than the valuation lets the seller withdraw the item from auction earlier. This also does not lead to any strategic advantage as zero surplus with unsold item is strictly worse than a non negative surplus and sold item. Therefore, it is best to set the reserve price equal to the true valuation.

If the buyers follow LSI strategy, then the DPO(0) strategy constitutes a $2m\epsilon$ -Nash equilibrium for the sellers. If the set of all possible seller strategies are restricted to the class of DPO strategies, then the DPO(0) strategy becomes a $2m\epsilon$ -dominant strategy for the sellers. Since DPO(0) is a $2m\epsilon$ dominant strategy, it is assumed in rest of the paper that sellers follow DPO(0).

The uncertainty of $2m\epsilon$ remains as the price decrements are finite, the mechanism allows the flexibility of sellers joining in at any time, offering their items in any order to buyers and breaking ties arbitrarily. If such flexibility is removed by redesigning the mechanism on the lines of *exact auction mechanism* of Demange et al. 1986, it no longer remains attractive and practical for a decentralized implementation on the Internet.

4.2 Buyer's Strategy

In the LSI(0) strategy, the buyer commits to an item as soon as her surplus reaches zero. After this she accepts any offer that gives her strictly better surplus. It is easy to see that a buyer can potentially get a better surplus if she waits for prices of items to fall further and then start committing to items. However, the chance of her winning an item decreases with more waiting. For example, consider a single item with two buyers, the winning buyer is better off waiting till the price reaches the valuation of the losing buyer, but not any longer. So, how long can a buyer exactly wait and still be assured of an item?

The next two theorems give an idea about the extent to which a buyer can wait ($\hat{s}_{\beta_i}, \forall \beta_i \in \mathcal{B}$), given that the sellers follow the DPO(0) strategy. Assume that all the sellers follow only DPO(0) strategy and any buyer $\beta_i \in \mathcal{B}$ follows LSI(\hat{s}_{β_i}) strategy, where \hat{s}_{β_i} is the starting surplus amount of buyer β_i .

Theorem 3 *If $\hat{s}_{\beta_1} > s_{\beta_1}^{max} + m\epsilon$ and $\hat{s}_{-\beta_1} \leq s_{-\beta_1}^{max}$, where $-\beta_1 \in \mathcal{B} - \{\beta_1\}$, then β_1 will not get any item in SDPAM.*

Proof: Assume, for contradiction, that β_1 wins an item α_1 in SDPAM. To prove this theorem we make use of the following important lemma.

Lemma 2 *Let \mathcal{I} be a set of items and \mathbf{p} be any competitive equilibrium price. Let the buyers (except possibly the winners of \mathcal{I}) follow LSI($\hat{\mathbf{s}}$) strategies with $\hat{s}_{\beta} \leq \max(\max_{\alpha} [v_{\beta\alpha} - p_{\alpha}], 0)$ and all the sellers follow DPO(t) strategy with $t \geq 0$. If $\forall \alpha_j \in \mathcal{I}, \hat{p}_{\alpha_j} < p_{\alpha_j} - \delta$, where $\delta \geq \epsilon$, then $\exists \alpha_i \notin \mathcal{I}$ such that $\hat{p}_{\alpha_i} < p_{\alpha_i} - \delta + \epsilon$.*

Proof: All the items in \mathcal{I} will be assigned in the competitive equilibrium assignment at \mathbf{p} . To see this, consider an item $\alpha_j \in \mathcal{I}$. $p_{\alpha_j} > \hat{p}_{\alpha_j} + \delta$. Therefore $p_{\alpha_j} > \hat{p}_{\alpha_j} \geq r_{\alpha_j} = v_{\alpha_j} + t \geq v_{\alpha_j}$. So, $p_{\alpha_j} > v_{\alpha_j}$. From definition of competitive equilibrium, item α_j must be assigned.

Let \mathcal{J} be the set of buyers who are assigned items of \mathcal{I} in the competitive equilibrium assignment. All the buyers in \mathcal{J} must get an item in SDPAM. To see this, consider a buyer β_i who does not get an item from \mathcal{I} in SDPAM. Since prices of all the items in \mathcal{I} are below the competitive equilibrium price by at least ϵ , β_i must have at least ϵ surplus on some item in \mathcal{I} in SDPAM. So, from proposition 2 and the fact that β_i follows an LSI strategy, she should get some item in SDPAM. Now consider the following two cases.

Case 1: There is a buyer $\beta_i \in \mathcal{J}$ who wins an item $\alpha_i \notin \mathcal{I}$ in SDPAM. Let α_j be her competitive equilibrium assignment. Now, consider the following proposition.

Proposition 3 *If β_i wins α_i in SDPAM and is has α_j in his demand set at some price vector \mathbf{p} , then $p_{\alpha_i} - \hat{p}_{\alpha_i} + \epsilon \geq p_{\alpha_j} - \hat{p}_{\alpha_j}$, where $\hat{\mathbf{p}}$ is the final price vector in SDPAM.*

Proof: From proposition 1 we have:

$$v_{\beta_i \alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i \alpha_j} - \hat{p}_{\alpha_j} - \epsilon.$$

Also, from the definition of competitive equilibrium we have:

$$v_{\beta_i \alpha_j} - p_{\alpha_j} \geq v_{\beta_i \alpha_i} - p_{\alpha_i}.$$

Adding above two inequalities, we get $p_{\alpha_i} - \hat{p}_{\alpha_i} + \epsilon > p_{\alpha_j} - \hat{p}_{\alpha_j}$. □

Using proposition 3 and using the fact that $\hat{p}_{\alpha_j} < p_{\alpha_j} - \delta$ we get, $p_{\alpha_i} - \hat{p}_{\alpha_i} + \epsilon > p_{\alpha_j} - \hat{p}_{\alpha_j} > \delta$. This gives $p_{\alpha_i} - \hat{p}_{\alpha_i} > \delta - \epsilon$.

Case 2: Every buyer in \mathcal{J} wins an item from \mathcal{I} in SDPAM. Since every item in \mathcal{I} is assigned in the competitive equilibrium, so $|\mathcal{I}| = |\mathcal{J}|$ and therefore, every item in \mathcal{I} is sold in SDPAM.

Now, consider the last item α_j of \mathcal{I} to get committed to a buyer $\beta_i \in \mathcal{J}$ in SDPAM. Just before α_j was offered to β_i , at price \hat{p}_{α_j} , β_i must be committed to an item α_i outside \mathcal{I} . This is so because, if β_i was committed to an item from \mathcal{I} then her switch to item α_j would have freed an item from \mathcal{I} which would have been sold eventually. Therefore α_j couldn't have been the last item in \mathcal{I} to get committed. The other possibility is that β_i was not committed when α_j was offered to her. Let $\alpha_k \in \mathcal{I}$ be the competitive assignment of β_i . This would mean that β_i rejected the offer of α_k at price $\hat{p}_{\alpha_k} + \epsilon$, which is impossible since, $v_{\beta_i \alpha_k} - (\hat{p}_{\alpha_k} + \epsilon) > v_{\beta_i \alpha_k} - (p_{\alpha_k} - \delta + \epsilon) \geq \hat{s}_{\beta_i} + \delta - \epsilon \geq \hat{s}_{\beta_i}$.

Let p_i be the price at which β_i accepted the offer of α_i . Since β_i preferred α_i at p_i over α_k at $\hat{p}_{\alpha_k} + \epsilon$ we have:

$$v_{\beta_i \alpha_i} - p_i > v_{\beta_i \alpha_k} - \hat{p}_{\alpha_k} - \epsilon.$$

Since price of α_i can only fall we can rewrite the above inequality as:

$$v_{\beta_i \alpha_i} - \hat{p}_{\alpha_i} > v_{\beta_i \alpha_k} - \hat{p}_{\alpha_k} - \epsilon.$$

From the property of competitive equilibrium, we have:

$$v_{\beta_i \alpha_k} - p_{\alpha_k} \geq v_{\beta_i \alpha_i} - p_{\alpha_i}.$$

Adding above two inequalities and using the fact that $\hat{p}_{\alpha_k} < p_{\alpha_k} - \delta$ we get:

$$p_{\alpha_i} - \hat{p}_{\alpha_i} + \epsilon > p_{\alpha_k} - \hat{p}_{\alpha_k} > \delta.$$

This gives $p_{\alpha_i} - \hat{p}_{\alpha_i} > \delta - \epsilon$. □

The proof of the theorem is based on substituting \mathbf{p} with the minimum competitive equilibrium price vector in Lemma 2 and repeatedly applying it. Consider the set $\mathcal{I} = \{\alpha_1\}$. Since \mathbf{p}^{min} corresponds to the minimum competitive equilibrium price, $\max_{\alpha} [v_{\beta\alpha} - p_{\alpha}^{min}] = s_{\beta}^{max} \geq \hat{s}_{\beta}$. This condition however is not satisfied by β_1 , winner of α_1 . But by Lemma 2, the winners of \mathcal{I} need not satisfy this condition. Now, $\hat{s}_{\beta_1} \geq s_{\beta_1}^{max} + m\epsilon$ and β_1 wins the item α_1 . So, $\hat{p}_{\alpha_1} < p_{\alpha_1}^{min} - m\epsilon$. Set $\delta = m\epsilon$ and apply Lemma 2, to find another item α_2 such that $\hat{p}_{\alpha_2} < p_{\alpha_2}^{min} - (m-1)\epsilon$. Now \mathcal{I} can be expanded to contain α_2 and δ can be set to $(m-1)\epsilon$ and Lemma 2 can be applied again to get another item α_3 . The process can continue till we have found α_m such that $p_{\alpha_m}^{min} - \hat{p}_{\alpha_m} > \epsilon$. So applying Lemma 2 again we will discover another new item. But this is a contradiction as we have already discovered all the m items in \mathcal{A} . Hence our assumption that β_1 wins an item α_1 must be incorrect. Therefore β_1 does not win any item. Hence the theorem. □

Theorem 4 *If $\hat{s}_{\beta_1} \leq s_{\beta_1}^{max} - m\epsilon$ and β_1 is assigned some item in a minimum competitive equilibrium, then β_1 will get some item in SDPAM irrespective of the starting surplus amounts of other buyers.*

Proof: Assume, for contradiction that buyer β_1 does not win any item in SDPAM. Consider the minimum competitive equilibrium price, \mathbf{p}^{min} , and the corresponding equilibrium assignment. Let α_1 be the item she is assigned in this assignment. Since β_1 does not get α_1 in SDPAM, someone else would have committed to it before the price reached $p_{\alpha_1} + m\epsilon$ (since β_1 had a surplus of $s_{\beta_1}^{max} - m\epsilon$). So $\hat{p}_{\alpha_1} > p_{\alpha_1}^{min} + m\epsilon$. Let β_2 win α_1 in SDPAM. It is easy to see that β_2 has more than $m\epsilon$ surplus on α_1 at minimum competitive equilibrium price. So she should get some item in minimum competitive equilibrium. Let that item be α_2 . From the definition of competitive equilibrium we have:

$$v_{\beta_2\alpha_2} - p_{\alpha_2}^{min} \geq v_{\beta_2\alpha_1} - p_{\alpha_1}^{min}.$$

Also we have the following inequality due to property of SDPAM (proposition 1):

$$v_{\beta_2\alpha_1} - \hat{p}_{\alpha_1} > v_{\beta_2\alpha_2} - \hat{p}_{\alpha_2} - \epsilon.$$

Adding the above two inequalities and using $\hat{p}_{\alpha_1} > p_{\alpha_1}^{min} + m\epsilon$, we have, $\hat{p}_{\alpha_2} - p_{\alpha_2}^{min} + \epsilon > \hat{p}_{\alpha_1} - p_{\alpha_1}^{min} > m\epsilon$.

So we have $\hat{p}_{\alpha_2} - p_{\alpha_2}^{min} > (m-1)\epsilon$. Since the prices are above minimum competitive equilibrium someone should have won the α_2 in SDPAM. Let that buyer be β_3 . So she will have more than $(m-1)\epsilon$ surplus on α_2 in minimum competitive equilibrium. So she should get some item α_3 in minimum competitive equilibrium. Arguing as before we will have $\hat{p}_{\alpha_3} - p_{\alpha_3}^{min} > (m-2)\epsilon$. We can repeat this process till we discover α_m such that $\hat{p}_{\alpha_m} - p_{\alpha_m}^{min} > \epsilon$. Again the winner of α_m in SDPAM will have positive surplus on it in minimum competitive equilibrium. So she should be assigned some item in minimum competitive equilibrium. But we already have all the m items assigned to some buyer in minimum competitive equilibrium. This is a contradiction. So our assumption that β_1 does not win any item in SDPAM must be incorrect. Therefore β_1 wins an item in SDPAM. Hence proved. □

The above two theorems tell us that if the set of possible buyer's strategies are restricted to LSI, then waiting till buyer β_i 's surplus becomes $s_{\beta_i}^{max} - m\epsilon$ constitutes a $2m\epsilon$ -Nash equilibrium for buyers. Also, a buyer is guaranteed to get an item irrespective of what other buyers are doing, if she makes a conservative estimate of the minimum competitive equilibrium price and sets her starting surplus amount (SSA) to be less than or equal to $s_{\beta_i}^{max} - m\epsilon$. In general a buyer doesn't know his surplus in minimum competitive equilibrium (s_{β_i}) as that requires the knowledge of the the minimum competitive equilibrium price vector, which depends on the number of buyers, number of sellers and the individual valuations of each buyer. So, this waiting strategy has risk element attached to it as the buyer can at best estimate her SSA by estimating the valuations of all the buyers in the system. The risk averse buyers will typically follow the safest strategy of setting their SSA to zero whereas the risk takers will estimate their surplus in minimum competitive equilibrium and wait for the prices to fall till their surplus reaches that value. Note that if all the buyers wait for long, then it is possible that the prices fall even below the minimum competitive equilibrium price. In this case, the buyers get surplus better than their corresponding surplus in minimum competitive equilibrium. The actual prices and allocations in SDPAM mechanism depend on the profile of the buyers and on how well the buyers estimate their surplus in the minimum competitive equilibrium. However, according to Theorem 4 it is always safe to make a conservative estimate of the surplus in the minimum competitive equilibrium. In Section 5 we discuss implications of this assumption on the final prices.

Note that if all the buyers choose a starting surplus amount such that $\hat{s}_\beta \leq \max[s_\beta^{max} - m\epsilon, 0]$, the mechanism will still lead to nearly efficient allocation (Theorem 1 will still hold). To see this, consider a buyer β such that $\hat{s}_\beta > 0$. Clearly, $s_\beta^{max} \geq \hat{s}_\beta + m\epsilon > 0$. This means that bidder β is assigned some item in minimum competitive equilibrium. Hence according to Theorem 4, β will be assigned some item in SDPAM. So, a buyer β who doesn't win an item in SDPAM has $\hat{s}_\beta = 0$. Therefore inequality (5) still holds. Since, for all buyers β , $\hat{s}_\beta \geq 0$, inequality (7) holds. Inequality (4) holds because of property of the LSI strategy. So, Theorem 1 still holds.

5 Equilibrium Prices

We saw that DPO(0) is a $2m\epsilon$ -dominant strategy for sellers if the set of strategies are restricted to the DPO class of strategies. Similarly, waiting till a buyer's surplus becomes $m\epsilon$ less than her surplus in minimum competitive equilibrium, constitutes a $2m\epsilon$ -Nash Equilibrium. Since the buyers do not know the minimum competitive equilibrium price, they can only estimate their surplus in the minimum competitive equilibrium. We also saw that it is safe for buyers to make a conservative estimate of their surplus in minimum competitive equilibrium. What happens to the final prices if all the buyers make a conservative estimate of their surplus in minimum competitive equilibrium while the sellers follow DPO strategy? In this section, we try to answer this question. Firstly, we provide some useful definitions.

We denote the surplus of buyer β_i on the items in her demand set at price vector \mathbf{p} (zero if the demand set is the null set) as $s_{\beta_i}(\mathbf{p})$. Let $\mathbf{s}(\mathbf{p})$ denote the surplus vector for all the buyers in the set \mathcal{B} , at price vector \mathbf{p} . Let \hat{s}_{β_i} denote the starting surplus amount of buyer β_i and $\hat{\mathbf{s}}$ denote the vector of starting surplus amounts of all buyers. Define a "maximal competitive equilibrium price at \mathbf{s} " as follows:

Definition 4 \mathbf{p} is a maximal competitive equilibrium price vector at \mathbf{s} if it is a competitive equilibrium price such that $\mathbf{s}(\mathbf{p}) \geq \mathbf{s}$ and there does not exist another competitive equilibrium price vector $\tilde{\mathbf{p}} > \mathbf{p}$ with $\mathbf{s}(\tilde{\mathbf{p}}) \geq \mathbf{s}$.

So a maximal competitive equilibrium price is a largest competitive equilibrium price vector whose corresponding surplus vector is greater than or equal to the given surplus vector. For example, if \mathbf{s} is a vector of zeros, then maximal competitive price at \mathbf{s} is unique and equal to the maximum competitive equilibrium price (Shapley and Shubik, 1972). We generalize this result to arbitrary surplus vectors less than or equal to the maximum achievable surplus.

Theorem 5 *For any surplus vector $\mathbf{s} \leq \mathbf{s}^{max}$, the corresponding maximal competitive equilibrium price vector (denoted by $\mathbf{p}^{max}(\mathbf{s})$) exists and is unique.*

Proof: \mathbf{p}^{min} is a competitive equilibrium price and $\mathbf{s}(\mathbf{p}^{min}) = \mathbf{s}^{max} \geq \mathbf{s}$. So, there exists a maximal competitive equilibrium price at \mathbf{s} .

Assume for contradiction, \mathbf{p}^1 and \mathbf{p}^2 are two different ($\mathbf{p}^1 \neq \mathbf{p}^2$) maximal competitive equilibrium price vectors corresponding to \mathbf{s} . Let \mathbf{p}^3 be a price vector defined as $p_{\alpha_j}^3 = \max[p_{\alpha_j}^1, p_{\alpha_j}^2]$. Define $s_{\beta_i}^3 = \min[s_{\beta_i}(\mathbf{p}^1), s_{\beta_i}(\mathbf{p}^2)]$.

From Lemma in Theorem 3 of Shapley and Shubik (1972), \mathbf{p}^3 is a competitive equilibrium price vector with $\mathbf{s}(\mathbf{p}^3) = \mathbf{s}^3$. Since \mathbf{p}^1 and \mathbf{p}^2 are different price vectors, from the definition of \mathbf{p}^3 , $\mathbf{p}^3 > \mathbf{p}^1$ and $\mathbf{p}^3 > \mathbf{p}^2$. Also, $s_{\beta_i}(\mathbf{p}^3) = s_{\beta_i}^3 = \min[s_{\beta_i}(\mathbf{p}^1), s_{\beta_i}(\mathbf{p}^2)] \geq s_{\beta_i}$. Therefore \mathbf{p}^3 is a competitive equilibrium price with $\mathbf{s}(\mathbf{p}^3) \geq \mathbf{s}$ and $\mathbf{p}^3 \geq \mathbf{p}^1, \mathbf{p}^3 \geq \mathbf{p}^2$. Therefore, \mathbf{p}^1 and \mathbf{p}^2 cannot be a maximal competitive equilibrium price at \mathbf{s} . Hence our assumption that $\mathbf{p}^1 \neq \mathbf{p}^2$ must be wrong. Therefore maximal competitive equilibrium price at \mathbf{s} is unique. \square

Since the maximal competitive equilibrium price at \mathbf{s} is unique, we call it the maximum competitive equilibrium price vector at \mathbf{s} and denote it as $\mathbf{p}^{max}(\mathbf{s})$. We now state an important proposition regarding the maximum competitive equilibrium price at a given surplus.

Proposition 4 *Consider $0 \leq \mathbf{s} \leq \mathbf{s}^{max}$ and a competitive equilibrium at $\mathbf{p}^{max}(\mathbf{s})$. Consider a set of q items that are assigned to a set of q buyers such that all the q buyers have a surplus greater than \mathbf{s} . At least one of these buyers should have an item in her demand set that is not in this set of q items.*

Proof: Assume, for contradiction that all the q buyers have all items in their demand sets from the set of q items mentioned. This means we can increase the prices of all the q items by a sufficiently small amount and still have the demand sets of the q buyers unchanged (since all of them have positive surplus). Thus we will get another higher competitive equilibrium price. Since initially all the q buyers had surplus more than \mathbf{s} , by increasing the prices by sufficiently small amount, they will still have surplus more than \mathbf{s} . So we have discovered a new competitive equilibrium price \mathbf{p} such that $\mathbf{p} > \mathbf{p}^{max}(\mathbf{s})$ and $\mathbf{s}(\mathbf{p}) \geq \mathbf{s}$. This is a contradiction from the definition of maximum competitive equilibrium price. Hence, our assumption must be incorrect i.e. there must be a buyer having an item in her demand set, that is not from the set of q items considered. Hence proved. \square

The next two theorems show that if the buyers follow LSI($\hat{\mathbf{s}}$) strategies with $\hat{\mathbf{s}} \leq \mathbf{s}^{max}$ and sellers follow the DPO(0) strategy then the final prices will converge close to $\mathbf{p}^{max}(\hat{\mathbf{s}})$. There are two interesting extreme cases of these results. If all the buyers follow the LSI(0) strategy, then the final prices in SDPAM will converge close to the maximum competitive equilibrium price (\mathbf{p}^{max}), whereas if the buyers follow LSI(\mathbf{s}^{max}) strategies, then the final prices in SDPAM will converge close to the minimum competitive equilibrium prices (\mathbf{p}^{min}).

Theorem 6 (Upper Bound) *If $\hat{\mathbf{p}}$ is the final price vector in SDPAM when all buyers follow LSI($\hat{\mathbf{s}}$) strategies (with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}^{max}$) and all the sellers follow DPO(0) strategy, then $\hat{\mathbf{p}} \leq \mathbf{p}^{max}(\hat{\mathbf{s}}) + m\epsilon$.*

Proof: Consider the following important lemma.

Lemma 3 *Consider an outcome of SDPAM with buyers following LSI($\hat{\mathbf{s}}$) strategies with $\hat{\mathbf{s}} \leq \mathbf{s}^{max}$. Let $\mathbf{p} = \mathbf{p}^{max}(\hat{\mathbf{s}})$ and $\hat{\mathbf{p}}$ be the final prices in SDPAM. Let \mathcal{I} be a set of q items such that all the items in \mathcal{I} are assigned in SDPAM and if $\alpha_j \in \mathcal{I}$ then $\hat{p}_{\alpha_j} > p_{\alpha_j} + \delta$, where $\delta \geq \epsilon$. Then $\exists \alpha_i \notin \mathcal{I}$ such that $\hat{p}_{\alpha_i} > p_{\alpha_i} + \delta - \epsilon$.*

Proof: Let \mathcal{J} be the set of q buyers who win items of \mathcal{I} in SDPAM by following LSI($\hat{\mathbf{s}}$) strategy. Consider a competitive equilibrium at \mathbf{p} . Since all the buyers in \mathcal{J} get items above the competitive equilibrium price \mathbf{p} by following LSI($\hat{\mathbf{s}}$) strategy, they should have positive surplus (as $\hat{\mathbf{s}}$ is non-negative) in the competitive equilibrium. So, from the definition of competitive equilibrium, they should be assigned some item from their demand set. There are two cases:

Case 1: $\exists \beta_i \in \mathcal{J}$ whose competitive equilibrium assignment $\alpha_i \notin \mathcal{I}$. Let β_i win α_j in SDPAM. Using proposition 3 and using the fact that $\hat{p}_{\alpha_j} - p_{\alpha_j} > \delta$ we have, $\hat{p}_{\alpha_i} - p_{\alpha_i} + \epsilon > \hat{p}_{\alpha_j} - p_{\alpha_j} > \delta$. This gives $\hat{p}_{\alpha_i} - p_{\alpha_i} > \delta - \epsilon$.

Case 2: All buyers in \mathcal{J} get assigned from \mathcal{I} in the competitive equilibrium assignment. Since $\delta > 0$, all buyers in \mathcal{J} have more than $\hat{\mathbf{s}}$ surplus in competitive equilibrium assignment. Applying proposition 4 at $\mathbf{s} = \hat{\mathbf{s}}$ we get $\exists \alpha_i \notin \mathcal{I}$ which is in the demand set of $\beta_i \in \mathcal{J}$. Let β_i win $\alpha_j \in \mathcal{I}$ in SDPAM. From proposition 3 and using the fact that $\hat{p}_{\alpha_j} - p_{\alpha_j} > \delta$, we get, $\hat{p}_{\alpha_i} - p_{\alpha_i} + \epsilon > \hat{p}_{\alpha_j} - p_{\alpha_j} > \delta$. This gives us $\hat{p}_{\alpha_i} - p_{\alpha_i} > \delta - \epsilon$.

□

The proof of the theorem is based on repetitively applying Lemma 3 and using the fact that sellers follow DPO(0) strategy. Firstly, if final price of any item in SDPAM is above $\mathbf{p}^{max}(\hat{\mathbf{s}})$, then it should also be greater than its seller's valuation. As sellers are following DPO(0) strategy, the item should be assigned to some buyer. Let us assume for contradiction that there exists an item α_1 such that $\hat{p}_{\alpha_1} - p_{\alpha_1}^{max}(\hat{\mathbf{s}}) > m\epsilon$. So in Lemma 3 we can substitute $\delta = m\epsilon$, $\mathcal{I} = \{\alpha_1\}$ and this will give us an item $\alpha_2 \notin \mathcal{I}$ and $\hat{p}_{\alpha_2} - p_{\alpha_2}^{max}(\hat{\mathbf{s}}) \geq \delta - \epsilon$. Then we can include α_2 in \mathcal{I} and substitute $\delta = (m - 1)\epsilon$ and again apply Lemma 3. This process can continue till we find α_m such that $\hat{p}_{\alpha_m} - p_{\alpha_m}^{max}(\hat{\mathbf{s}}) \geq \epsilon$. But we can still apply Lemma 3 which will give us another item. But as can be seen, we have already exhausted all the items in \mathcal{A} . Thus we reach a contradiction. Hence, our assumption that there exists an item α_1 such that $\hat{p}_{\alpha_1} - p_{\alpha_1}^{max}(\hat{\mathbf{s}}) > m\epsilon$ must be incorrect. Therefore, for all items α_i , $\hat{p}_{\alpha_i} - p_{\alpha_i}^{max}(\hat{\mathbf{s}}) \leq m\epsilon$. □

Theorem 7 (Lower Bound) *If $\hat{\mathbf{p}}$ is the final price vector in SDPAM when all buyers follow LSI($\hat{\mathbf{s}}$) strategies with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}^{max}$ and all sellers follow DPO(0) strategies, then $\hat{\mathbf{p}} \geq \mathbf{p}^{max}(\hat{\mathbf{s}}) - m\epsilon$.*

Proof: The proof of the theorem follows by applying Lemma 2 repetitively. Assume, for contradiction that there is an item α_1 such that $\hat{p}_{\alpha_1} < p_{\alpha_1}^{max}(\hat{\mathbf{s}}) - m\epsilon$. Set the competitive equilibrium vector, \mathbf{p} , in Lemma 2 to be the maximum competitive equilibrium price at $\hat{\mathbf{s}}$, $\mathbf{p}^{max}(\hat{\mathbf{s}})$. From the definition of maximum competitive equilibrium price vector at $\hat{\mathbf{s}}$, $\hat{s}_\beta \leq \max(\max_{\alpha} [v_{\beta\alpha} - p_{\alpha}], 0) \forall \beta \in \mathcal{B}$. Set $\mathcal{I} = \{\alpha_1\}$ and $\delta = m\epsilon$ and apply Lemma 2 to get an item $\alpha_2 \notin \mathcal{I}$ such that $\hat{p}_{\alpha_2} < p_{\alpha_2} - (m - 1)\epsilon$. Again \mathcal{I} can be expanded to contain α_2 and δ can be set to $(m - 1)\epsilon$. This process can be continued till we find α_m such that $\hat{p}_{\alpha_m} < p_{\alpha_m} - \epsilon$. We can

still apply Lemma 2 to give another item. But as can be seen we have exhausted all the items in \mathcal{A} . Thus we reach a contradiction. Hence, our assumption that $\hat{p}_{\alpha_1} < p_{\alpha_1}^{max}(\hat{\mathbf{s}}) - m\epsilon$ must be incorrect. Therefore, $\hat{p}_{\alpha_i} \geq p_{\alpha_i}^{max}(\hat{\mathbf{s}}) - m\epsilon \forall \alpha_i \in \mathcal{A}$. \square

Theorems 6 and 7 establish that if the sellers follow DPO strategy and buyers follow LSI class of strategies with starting surplus amount set to (non-negative) conservative estimates of their maximum achievable surplus, the final prices in the mechanism converge close to the unique maximum competitive equilibrium price vector corresponding to the starting surplus amount vector.

6 Multi-item Ascending English Auctions

Demange et al. (1986) have described an ascending price version of multi-item auctions. The buyer's strategic behavior in these auctions has been studied by Miyake (1998) and Bansal and Garg (2000). However, the seller's strategic behavior has not received much attention. It is possible to obtain results similar to Theorem 3 and 4 for sellers in ascending English auctions.

It is well known that final prices in multi-item ascending English auctions correspond to the minimum competitive prices (Demange et al., 1986). A seller may set a reserve price higher than his true valuation to obtain a larger surplus. It can be shown that a seller will always be able to sell his item as long as his reserve price is less than the maximum competitive equilibrium price (provided that the buyers follow a greedy surplus maximizing bidding strategy). It can also be shown that if all the other sellers set a reserve price that is less than their corresponding maximum competitive equilibrium prices, then a seller will not be able to sell his item by setting a reserve price higher than the maximum competitive equilibrium price. Therefore, setting reserve price equal to ($m\epsilon$ less than) the maximum competitive equilibrium price constitutes a $2m\epsilon$ -Nash equilibrium for the sellers in multi-item ascending English auctions.

Similar to maximum competitive equilibrium price at a given surplus vector, we can define a unique minimum competitive equilibrium price given a vector of reserve prices (less than the maximum competitive equilibrium prices). It is possible to establish counterparts of Theorems 6 and 7, for the ascending English auctions as well. So for arbitrarily small value of ϵ , with buyers following greedy bidding strategy and sellers setting a reserve price less than maximum competitive equilibrium price and more than their valuation, the prices in the ascending English auction will converge close to the minimum competitive equilibrium price corresponding to the reserve price of sellers.

7 Simulation

In DPO strategy, the sellers can offer their respective items to buyers in any random order. Similarly, if more than one item gives maximum surplus to a buyer, he can randomly choose one of the items. This randomness in the strategies of both buyers and sellers lead to price variations from one auction instance to the other for the same set of buyers and sellers. Our results account for such variations in prices. Theorems 6 and 7 show that such variations will be within $2m\epsilon$. But in practice, such variations will be well within $2m\epsilon$. We conducted simulations to validate this claim.

In, practice, we expect each buyer to be interested in only some of the items and not the whole set of

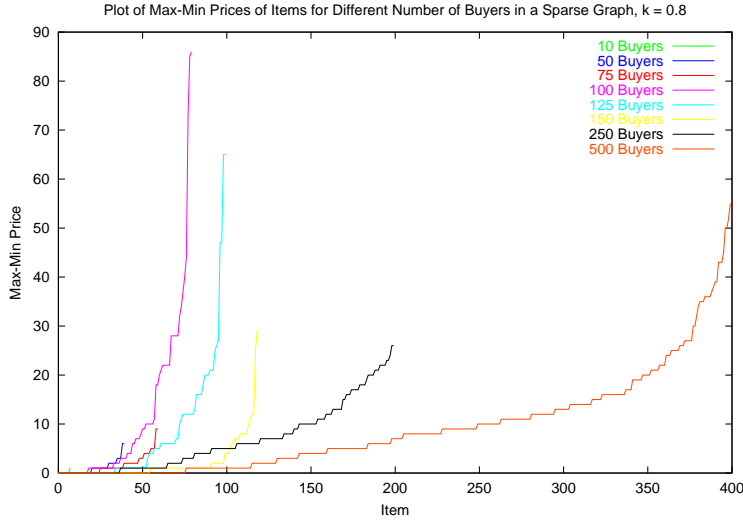


Figure 1: Effect of Market-size on Price Variations

items. To model this scenario, we generate random positive valuations for each buyer on 20% of the whole set of items and set the valuations of rest of the items to zero. Also, in practice, we expect the number of buyers to be more than the number of items sold. So, we keep the ratio (denote it as k) of number of sellers to number of buyers at 0.8. We test out price variations when number of buyers (and thus the market size varies). For a fixed number of buyer, we run the simulations 1000 times with the same valuations. Due to the randomness of offering process of sellers and accepting process of buyers, the prices vary from iteration to iteration. We found the maximum and minimum prices of each item for the 1000 iterations and plotted their difference (see Figure 1). The plotting is done after sorting the items in increasing order of (maximum - minimum) price. We take the price decrement ϵ to be 1.

From the plot of Figure 1, it can be seen as the market size increases, the number of items with price variation less than 10ϵ increase. With higher number of buyers (≥ 125), almost 75% of the items have price variations less than 10ϵ . For other items, the price variations are of the order of $\frac{m}{5}\epsilon$, where m is the number of items.

The other interesting statistics we show in Figure 2 is the plot of mean of standard deviations of all items with number of buyers. As can be seen, the standard deviations of prices is really small (< 5).

8 Conclusions

Dutch auctions have traditionally been used in specialized markets of perishable goods such as vegetable markets, flower markets etc. The price markdown mechanism typically followed in retail markets to clear inventories (especially during post-festive seasons) may be thought as a variant of the descending price mechanism where the customers (the buyers) are allowed to close the deal before the auctions end. However, the customers are still faced with a similar choice of items at different stores, and a similar dilemma of whether to wait for the prices to fall further at the risk of losing the item, or to buy at potentially higher prices. With the Internet technologies, now it has become feasible to formally carry out descending price

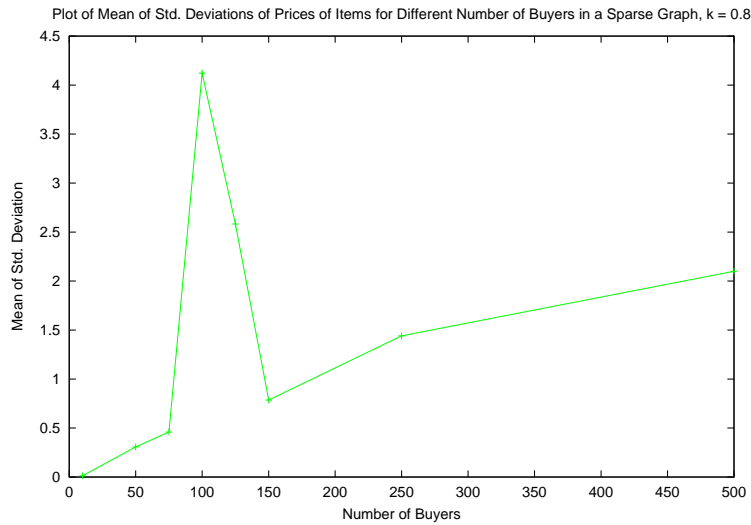


Figure 2: Effect of Market-size on Price Variations

multi-item auctions and remove the inefficiencies in the market. If the buyers are allowed to close the deals before the auctions end, they will not have to suffer the long wait as in the current (ascending price) Internet auctions. Whether such auctions will become popular on the Internet is something that remains to be seen.

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References

- Bansal, V. and Garg, R., 2000, On Simultaneous Auctions with Partial Substitutes, IBM Research Report RI00022, Available at <http://www.research.ibm.com/iac/>
- Bertsekas, D. P., 1992, Auction Algorithms for Network Flow Problems: A Tutorial Introduction, *Computational Optimization and Applications* 1, 7–66
- Crawford, V. P. and Knoer, E. M., 1981, Job Matching with Heterogeneous Firms and Workers, *Econometrica* 49(2), 437–450
- Demange, G. and Gale, D., 1985, The Strategy Structure of Two-sided Matching Markets, *Econometrica* 53, 873–888
- Demange, G., Gale, D., and Sotomayor, M., 1986, Multi-item Auctions, *Journal of Political Economy* 94(4), 863–872
- Gul, F. and Stacchetti, E., 1999, Walrasian Equilibrium with Gross Substitutes, *Journal of Economic Theory* 87(1), 95–124
- Kelso, A. S. and Crawford, V. P., 1982, Job Matching, Coalition Formation, and Gross Substitutes, *Econometrica* 50(6), 1483–1504
- Koopmans, T. C. and Beckmann, M., 1957, Assignment Problems and the Location of Economic Activities, *Econometrica* 25(1), 53–76
- Leonard, H. B., 1983, Elicitation of Honest Preferences for the Assignment of Individuals to Positions, *Journal of Political Economy* 91, 461–479
- Miyake, M., 1998, On the Incentive Properties of Multi-item Auctions, *International Journal of Game Theory* 27(1), 1–19
- Shapley, L. S. and Shubik, M., 1972, The Assignment Game I: The Core, *International Journal of Game Theory* 1(2), 111–130
- Vickrey, W., 1961, Counterspeculation, Auctions, and Competitive Sealed Tenders, *Journal of Finance* 16, 8–37