PERFECT COMPETITION IN A BILATERAL MONOPOLY

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Abstract

We show that if limit orders are required to vary smoothly, then strategic (Nash) equilibria of the double auction mechanism yield competitive (Walras) allocations. It is not necessary to have competitors on any side of any market: smooth trading is a substitute for price wars. In particular, Nash equilibria are Walrasian even in a bilateral monopoly.

Keywords: Limit orders, double auction, Nash equilibria, Walras equilibria, perfect competition, bilateral monopoly, mechanism design

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1 Introduction

As is well-known Walrasian economics is built upon the Hypothesis of Perfect Competition, which can be taken as in Mas-Colell (1980) to state: "...that prices are publicly quoted and are viewed by the economic agents as exogenously given". Attempts to go beyond Walrasian economics have in particular involved giving "a theoretical explanation of the Hypothesis itself" (Mas-Colell (1980)). Among these the most remarkable are without doubt the 19th century contributions of Bertrand, Cournot and Edgeworth (for an overview, see Stigler (1965)). The Cournot approach was explored intensively, in a general equilibrium framework, in the symposium issue entitled "Non-cooperative Approaches to the Theory of Perfect Competition" (Journal of Economic Theory, Vol. 22 (1980)).
The features common to most of the symposium articles are:

(a) The strategies employed by the agents are of the Cournot type, i.e., consist in quoting quantities.

(b) The (insignificant) size of any agent relative to the market is the key explanatory variable for the tendency of strategic behavior to approximate perfect competition and, in its wake, to lead to Walrasian outcomes (Mas-Colell (1980), p.122).

The extension of pure quantity strategies from Cournot's partial equilibrium model of oligopoly to a general equilibrium framework, however, does raise questions. Underlying the Cournot model is a demand curve for the particular market under consideration which enables the suppliers to relate quantities, via prices, to expected receipts. If such a close relationship is not provided by the market, then it seems more natural to us that an agent will no longer confine himself to quoting quantities, i.e., to pure buy-or-sell market orders. To protect himself against "market uncertainty - or illiquidity, or manipulation by other agents [1]", he will also quote prices limiting the execution of those orders, consenting to sell q units of commodity j only if its price is p or more, or buy $\tilde{q}$ units only if its price is $\tilde{p}$ or less. By sending multiple

[1] to quote from Mertens (2003)

orders of this kind an agent can approximate any monotone demand or supply curve in a market by a step function, as was done in Dubey (1982, 1994). Here we go further and give each agent full manoeuvrability. He places a continuum of infinitesimal limit-price orders, which in effect enables him to send any monotone, continuous demand or supply curve for each commodity. The upshot is a striking result: provided only that all commodity markets are "active" (i.e. there is positive trade in them), and no matter how thin they are, strategic (Nash) equilibria (SE) coincide - in outcome space - with competitive (Walras) equilibria (CE). Our result thus provides a rationale, based on strategic competition, for Walrasian outcomes even in the case of a bilateral monopoly. This brings it in sharp contrast to Dubey (1982, 1994), where it was necessary to have competition on both sides of each market (in the sense of there being at least two active buyers and two active sellers for each commodity) in order to conclude that SE are CE . (Always CE are SE without much ado in both models). The models in Dubey (1982, 1994) have little to say in the setting of a bilateral monopoly, where they allow for a continuum of active non-Walrasian SE. (Indeed, the set of SE allocations coincides with the set of all individually rational allocations). In our model this continuum disappears, leaving only the CE behind. Thus full manoeuvrability of limit-price orders is tantamount to perfect competition. Even in the presence of monopolists who have cornered several markets and eliminated any vestige of competition from them, every active SE is Walrasian in our model. This is exactly the important scenario left out in Dubey (1982, 1994).

The models in Dubey (1982, 1994) rely on competition that is "cut-throat" in the spirit of Betrand. Any agent can take over a whole chunk of some buy (sell) order from another by quoting an infinitesimally higher (lower) price. Our model is not based on the possibility of such takeovers. Instead it requires that agents' behavior be "smooth", with commodities bought (sold) in infinitesimal increments of continuously non-increasing (non-decreasing) prices. The key point of our paper is that such smooth trading is a substitute for cut-throat price wars, and also gives rise to Walrasian outcomes. A monopolist may be in sole command of his own resource, but nevertheless he will be reduced to behaving as if he had cut-throat rivals, once smooth trading sets in. A related phenomenon[2] was analyzed in Coase (1972) (and following Coase (1972), a long line of literature, see e.g. Bulow (1982), Gaskins (1974), Schmalensee (1979)). There, too, a monopolist was shown to forfeit his power, but this happened in the setting of durable goods which could be sold sequentially over time to infinitely patient customers. In our model the monopolist loses power even with perishable goods which are traded at one instant of time. But we do need, unlike Coase, strategic behavior on both sides of the market as well as convex preferences.

It must be emphasized that our model is based on decentralized markets. Each commodity j is traded against fiat money ("unit of account"), and orders sent to the markets $k \neq j$ for other commodities k, do not affect how market j functions. Thus we do not allow an agent to link his buy-order for a commodity to whether the sell-order for another commodity goes through.[3] The only connection between different commodity markets is the budget-constraint of agents, requiring them to cover purchases out of their sales receipts. Our model is therefore an order-of-magnitude simpler than that of Mertens (2003), where cross-market limit orders are permitted. In spite of this paucity of our strategy-space compared to Mertens (2003), we exactly implement [4] CE via our mechanism (modulo activity in markets). In contrast, SE form a large superset [5] of CE in Mertens (2003) (though, we hasten to add, the implementation of CE was never the aim there, rather it was to well-define a mechanism that allowed for a rich menu of cross-market limit-orders).

For better perspective, we consider two somewhat contrasting versions of our model. In the first version agents act under the optimistic illusion that they can exert perfect price discrimination: sell to others, starting at the highest quoted price (or buy, starting at the lowest). In the second version we turn to a standard market game, akin to that of Dubey (1982, 1994).

[2] We thank John Geanakoplos for this reference.

[3] That would be like allowing agents to submit demand functions based on the whole price vector.

[4] Indeed, our result may be interpreted in terms of the "mechanism-design" literature (see Section 4).

[5] For instance, the SE of Shapley's "windows model" (see Sahi and Yao (1989)) are also SE in Mertens' model.

Here each agent is grimly realistic and realizes that he will be able to buy (sell) only after higher-priced buyers (lower-priced sellers) have been serviced at the market, and that the prices he gets are apropos his own quotations, not the best going.[6]

Though the two versions are built on quite different behaviorial hypotheses, we find their equilibria lead to the same outcomes, namely Walrasian.

Our model shares some of the weaknesses of the Walrasian models. In particular, since it is based on the static concept of a strategic equilibrium, our model does not address the question of what dynamic forces bring the equilibrium about and ensure that individual strategic plans become jointly compatible. But it goes beyond the Walrasian notion in at least three important ways:

(a) It is not assumed that the economic agents face perfectly elastic supply and demand curves.

(b) Prices are not quoted from outside but set by the agents themselves. Each agent, operating in a market, realizes and exerts his ability to influence price.

(c) Strategies of the individuals (i.e. supply and demand curves submitted to the market) need not be based on their true characteristics (preferences and endowments).

2 The First Version: Optimistic Conjectures and Equilibrium Points

Let $N = \{1, \ldots, n\}$ be the set of agents who trade in k commodities. Each agent $i \in N$ has an initial endowment $e^i \in \mathbb{R}^k_+ \setminus \{0\}$ and a preference relation $\succsim_i$ on $\mathbb{R}^k_+$ that is convex, continuous and monotonic (in the sense that $x \geq y$, $x \neq y$ implies $x \succ_i y$). We assume that $\sum_{i \in N} e^i \gg 0$, i.e. every

[6]We could make the same assumption also in the first market model. However we would lose economic insight, as to what happens to the consumers' and producers' surplus, when agents behave like monopolists, trying to exert perfect price discrimination .

named commodity is present in the aggregate.

An agent may enter a market either as a buyer or a seller, and submit to each of the k commodity markets a marginal demand or supply curve. Formally, let

$$M^+ = \{f : \mathbb{R}_+ \to \mathbb{R}_{++} |\ f \text{ is continuous and non-decreasing}\}$$
$$M^- = \{f : \mathbb{R}_+ \to \mathbb{R}_{++} |\ f \text{ is continuous and non-increasing}\}.$$

Then a strategic choice σ^i of agent i is given by

$$\sigma^i = (d_1^i, s_1^i; \dots; d_k^i, s_k^i | d_j^i \in M^-, s_j^i \in M^+, \text{ for } j = 1, \dots, k).$$

In the interpretation $d_j^i(q_j^i)$ is the price at which agent i is willing to buy an infinitesimal, incremental unit of commodity j, once his level of purchases has reached q_j^i. The supply curve has an analogous meaning. Denote $\sigma \equiv (\sigma_1, \dots, \sigma_n)$ and let S_j^σ, D_j^σ be the aggregate supply, demand curves.

We suppose that agent i acts under the optimistic conjecture that he can exert perfect price discrimination, i.e., that he can sell (buy) starting at the highest (lowest) prices quoted by the buyers (sellers). This means that agent i calculates his receipts (or expenditures) on the market j as the integral, starting from 0, under the curve D_j^σ (or S_j^σ). The generally non-convex budget-set $B^i(\sigma)$ for $\sigma = (\sigma^1, \dots, \sigma^n)$, is then obtained by the requiring that (perceived) expenditures do not exceed (perceived) receipts, i.e.,

$$B^i(\sigma) = \{e^i + t \mid t \in \mathbb{R}^k, e^i + t \in \mathbb{R}^k_+, \sum_{j=1}^{k} E_j^\sigma(t_j) \le \sum_{j=1}^{k} R_j^\sigma(t_j)\}$$

where

$$E_j^\sigma(q) = \int_0^q S_j^\sigma \quad \text{if } q > 0,\ 0 \text{ otherwise,}$$
$$R_j^\sigma(q) = \int_0^{|q|} D_j^\sigma \quad \text{if } q < 0,\ 0 \text{ otherwise.}$$

(Note that $t_j^i > 0$ ($t_j^i < 0$) means that i buys (sells) j .)

The collection of strategic choices σ will be called an equilibrium point (EP) if there exist trade vectors $t^1, \ldots, t^n$ in $\mathbb{R}^k$ such that

(i) $$e^i + t^i \text{ is } \succsim_i \text{-optimal on } B^i(\sigma) \text{ for } i = 1, \ldots, n$$

(ii) $$\sum_{i=1}^{n} t_j^i = 0 \text{ for } j = 1, \ldots, k$$

(iii) $$\sum_{i:t_j^i > o} t_j^i = sup\{q_j \mid S_j^\sigma(q_j) \leq D_j^\sigma(q_j)\} \text{ for } j = 1, \ldots, k$$

Conditions (i) and (ii) require that agents optimize and that markets clear. Condition (iii) says that no trade can be enforced, i.e., it stops when the (marginal) supply price for the first time exceeds the demand price; and, at the same time, in equilibrium all trades compatible with the submitted strategies are actually carried out.

An EP will be called active if there is positive trade in each market.

First let us establish that at an active EP all trade $T_j := \sum_{i:t_j^i > 0} t_j^i$ in any commodity j takes place at one price, p_j.

Lemma 1. *The curves S_j^σ and D_j^σ coincide and are constant on $[0, T_j]$ at any EP .*

Proof. For any j, let $G_j := \{i : t_j^i > 0\}, \; H_j := \{i : t_j^i < 0\}$ Then

$$
\begin{aligned}
(1) \qquad \sum_{i\in H_j} R_j^\sigma(t_j^i) &= \sum_{i\in H_j} \int_0^{|t_j^i|} D_j^\sigma \\
&\geq \int_0^{T_j} D_j^\sigma \\
&\geq D_j^\sigma(T_j)\cdot T_j \\
&\geq S_j^\sigma(T_j)\cdot T_j \\
&\geq \int_0^{T_j} S_j^\sigma \\
&\geq \sum_{i\in G_j} \int_0^{t_j^i} S_j^\sigma \\
&= \sum_{i\in G_j} E_j^\sigma(t_j^i).
\end{aligned}
$$

The third inequality follows from (iii); the other four follow from monotonicity of the supply and demand functions.

Hence

$$(2) \qquad \sum_{i=1}^{n} R_j^\sigma(t_j^i) \geq \sum_{i=1}^{n} E_j^\sigma(t_j^i) \quad \text{for } j = 1,\ldots,k.$$

From the monotonicity of preferences, and the fact that each agent has optimized, we have

$$(3) \qquad \sum_{j=1}^{k} R_j^\sigma(t_j^i) = \sum_{j=1}^{k} E_j^\sigma(t_j^i) \quad \text{for } i = 1,\ldots,n.$$

(2) and (3) together imply:

$$(4) \qquad \sum_{i=1}^{n} R_j^\sigma(t_j^i) = \sum_{i=1}^{n} E_j^\sigma(t_j^i) \quad \text{for } j = 1,\ldots,k.$$

From (4) it follows that all the inequalities in (1) must, in fact, be equalities. Therefore

$$(5) \qquad S_j^\sigma(T_j) = D_j^\sigma(T_j) =: p_j$$

and

$$(6) \qquad \int_0^{T_j} D_j^\sigma = p_j T_j = \int_0^{T_j} S_j^\sigma.$$

Since by (iii), $D_j^\sigma \geq S_j^\sigma$ on $[0, T_j]$ we get, from (6), and the monotonicity of D and S

$$(7) \qquad D_j^\sigma = S_j^\sigma \text{ on } [0, T_j].$$

□

In view of the Lemma 1 we can talk not only of the allocation but also the prices produced at an active EP . These are the constant values of S_j^σ, D_j^σ on $[0, T_j]$ for $j = 1, \ldots, k$. Note that these prices are positive by assumption.

Proposition 1. *The prices and allocation at an active equilibrium point are Walrasian.*

Proof. Let σ be an EP with trades $t^1, \ldots, t^n$ and prices p . We need to show that, for each i , $e^i + t^i$ is $\succsim_i$ -optimal on the set

$$B^i(p) := \{e^i + t : t \in \mathbb{R}^k, e^i + t \in \mathbb{R}^k_+,\ p.t = 0\}.$$

W.l.o.g. fix $i = 1$, put

$$
\begin{aligned}
J_1 &:= \{j : t_j^1 > 0\} \\
J_2 &:= \{j : t_j^1 < 0\} \\
J_3 &:= \{j : t_j^1 = 0\} \\
T_j &:= \sum_{i : t_j^i > 0} t_j^i \\
\delta_j &:= min[\{|t_j^1| : j \in J_1 \cup J_2\},\ \{T_j : j \in J_3\}] \\
N_j &:= \{\alpha \in \mathbb{R} : |t_j^1 - \alpha| < \delta_j\} \\
F_j &:= E_j - R_j
\end{aligned}
$$

(Since the EP is active, $\delta_j > 0$). Now we claim, for $j = 1, \ldots, k$:

(8) F_j is continuously differentiable and strictly increasing on N_j and its derivative at t_j^1 is p_j.

This follows from the continuity and strict positivity of S_j and D_j, and from Lemma 1 which implies:

(9) $$F_j(q) \text{ coincides with } E_j(q) = p_j q \ \text{ if } j \in J_1,\ 0 \le q \le t_j^1$$

(10) $$F_j(q) \text{ coincides with } -R_j(q) = p_j q \ \text{ if } j \in J_2,\ t_j^1 \le q \le 0$$

(11) $$F_j(q) = p_j q \ \text{ if } j \in J_3,\ q \in N_j.$$

W.l.o.g. fix commodity $j = 1$. Since $F_1, \ldots, F_k$ are all strictly increasing and $\sum_{j=1}^{k} F_j(t_j^1) = 0$, and $F(t_j^1) > 0 \quad (< 0)$ if $j \in J_1$ $(j \in J_2)$, it follows that there is a neighborhood V of $(t_2^1, \ldots, t_k^1)$ in $N_2 \times \ldots \times N_k$ such that if $(t_2, \ldots, t_k) \in V$ then there is a unique t_1 which satisfies the equation $F_1(t_1) + \ldots + F_k(t_k) = 0$. Thus we have an implicit function $G(t_2, \ldots, t_k) = -F_2(t_2) - \ldots - F_k(t_k)$ defined on V which is clearly continuously differentiable. Finally the point $t^1 = (t_1^1, \ldots, t_k^1)$ belongs by construction to the hypersurface

$M = \{(G(t_2,\ldots,t_k), t_2,\ldots,t_k) : (t_2,\ldots,t_k) \in V\}$ and, by (8), the tangent plane H to M at this point has normal p .

Since we are at an EP, e^1+t^1 is $\gtrsim_1$-optimal on $(e^1+M)\cap \mathbb{R}^k_+$. Suppose that there is some $x \in H_+ := (e^1+t^1+H)\cap \mathbb{R}^k_+$ such that $x \succ_1 e^1+t^1$. By continuity of $\succsim_1$ we can find a neighborhood Z of x (in $\mathbb{R}^k_+$) with the property: $y \in Z \Rightarrow y \succ_1 e^1+t^1$. But since M is a smooth surface there exists a point y^* in Z, such that the line segment between y^* and e^1+t^1 pierces e^1+M at some point $z^* \in (e^1+M)\cap \mathbb{R}^k_+$ (see Fig.1). By convexity of $\gtrsim_1$, we have $z^* \succ_1 e^1+t^1$, contradicting that e^1+t^1 is $\gtrsim_1$-optimal on $(e^1+M)\cap \mathbb{R}^k_+$. We conclude that e^1+t^1 is $\gtrsim_1$-optimal on H_+ . But we have $e^1 \in H_+$ (simply set trades to be zero, i.e., pick $-t^1$ in H). Therefore, in fact, $H_+ = B^1(p)$. Since the choice of $i=1$ was arbitrary, the proposition follows. □

.... Insert Figure 1 approximately here!.................

Proposition 2. *If the trades $t^1,\ldots,t^n$ and prices $p \gg 0$ are Walrasian, then they can be achieved at an EP*

Proof. For any i let

$$
\begin{aligned}
J^i_1 &= \{j : t^i_j > 0\}\\
J^i_2 &= \{j : t^i_j < 0\}\\
J^i_3 &= \{j : t^i_j = 0\}\\
f^i_j &= \text{any strictly decreasing function with } f^i_j(t^i_j) = p_j\\
g^i_j &= \text{any strictly increasing function with } g^i_j(t^i_j) = p_j
\end{aligned}
$$

and consider

$$
s^i_j(x) = \begin{cases} 0 & \text{if } j \in J^i_1 \cup J^i_3 \\ max\{p_j, g^i_j(x)\} & \text{if } j \in J^i_2\} \end{cases}
$$

$$
d^i_j(x) = \begin{cases} 0 & \text{if } j \in J^i_2 \cup J^i_3 \\ min\{p_j, f^i_j(x)\} & \text{if } j \in J^i_1\} \end{cases}
$$

Then it is readily checked that these strategies constitute a EP and produce the trades $t^1,\ldots,t^n$ at prices p. □

3 The Second Version: Strategic Market Game and Nash Equilibria

The previous results can be expressed in the form of Nash Equilibria of a (generalized [7]) strategic game. The game is defined as in Dubey (1982, 1994), except that strategies are not step functions, but rather continuously differentiable (i.e., we now take M^+ and M^- to consist of weakly monotonic C^1 functions). For simplicity of exposition, we also suppose that an agent enters a commodity market either as a buyer or as a seller, not both. (This can be dropped as in Dubey (1982)). Given the strategy-selection $\sigma = (\sigma^1, \ldots, \sigma^n)$ by the agents, let D_j^σ and S_j^σ denote the aggregate (strategic) demand and supply curves. If S_j^σ lies above D_j^σ, then no trade takes place, i.e., $t_j^i(\sigma) = 0$ for $i = 1, \ldots, n$. Otherwise set

$$T_j(\sigma) = \begin{cases} sup\{q \in \mathbb{R}_+ : D_j(q) \geq S_j(q)\} & \text{if the sup is finite} \\ \\ \text{some arbitrary positive number M, otherwise.} \end{cases}$$

.... Insert Figure 2 approximately here!.................

Define $p_j(\sigma)$ to be the intersection price of S_j^σ and D_j^σ if the sup is finite in the definition of $T_j(\sigma)$ (see Fig. 2), or an arbitrary point in the interval $[\sup_{q\in\mathbb{R}_+} S^\sigma(q), \inf_{q\in\mathbb{R}_+} D^\sigma(q)]$ otherwise (see Fig. 3).

.... Insert Figure 3 approximately here!.................

The individual trades $t_j^i(\sigma)$ are determined as follows. In the event that the sup in $T_j(\sigma)$ is finite, all purchases (sales) of quantities quoted above (below) $p_j(\sigma)$ occur, with arbitrary rationing, e.g., proportional on the margin", i.e., quantities that are in excess (either on the supply or the demand side) and that are quoted at the price $p_j(\sigma)$ are rationed. In the event that

[7] See Remark 2, however, on how to replace the generalized game by a proper game

the sup in $T_j(\sigma)$ is not finite, all purchases (sales) of quantities quoted above (below) $D_j^\sigma(M)$ ($S_j^\sigma(M)$) occur, with arbitrary rationing of quantities quoted for purchase at $D_j^\sigma(M)$ or for sales at $S_j^\sigma(M)$.

For any strategy choice $\widetilde{\sigma}^i$ of agent i denote by $(\sigma|\widetilde{\sigma}^i)$ the n-tuple $(\sigma^1,\ldots,\sigma^{i-1},\widetilde{\sigma}^i,\sigma^{i+1},\ldots,\sigma^n)$ and let

$$\widehat{E}_j^i(q,\widetilde{\sigma}^i) = \int_0^q \widetilde{d}_j^i \quad \text{if } q > 0, = 0 \text{ otherwise}$$

$$\widehat{R}_j^i(q,\widetilde{\sigma}^i) = \int_0^{|q|} \widetilde{s}_j^i \quad \text{if } q < 0, = 0 \text{ otherwise}$$

where $\widetilde{\sigma}^i = (\widetilde{d}_1^i, \widetilde{s}_1^i, \ldots \widetilde{d}_j^i, \widetilde{s}_j^i, \ldots \widetilde{d}_k^i, \widetilde{s}_k^i)$.

(Now we adopt the convention that a buyer (seller) pays (receives) the area under his own demand (supply) curve.)

Thus, for any σ, we automatically have

$$\sum_{i=1}^n t^i(\sigma) = 0$$

i.e. markets always clear. However, when i considers a deviation from $\widetilde{\sigma}^i$, it may happen that $e_j^i + t_j^i(\sigma|\widetilde{\sigma}^i) < 0$ for some j, i.e., the trader i is called upon to sell more of commodity j than he has. Furthermore, he may go bankrupt, i.e.

$$\sum_{j=1}^k \widehat{E}_j^i(t^j(\sigma|\widetilde{\sigma}^i),\widetilde{\sigma}^i) > \sum_{j=1}^k \widehat{R}_j^i(t^j(\sigma|\widetilde{\sigma}^i),\widetilde{\sigma}^i)\}.$$

The mechanism is not at fault on either count. It is blind to the private characteristics, as well as to the strategic manipulations, of the individual agents. In each market the signals submitted are resolved into trades and payments via the mechanism's publicly known rule. It is for the individual to ensure that his signal leads to trades he can honor, and that across markets he balances his budget. This motivates the definition of $\Sigma^i(\sigma)$ below of the following strategy sets for each player i :

$$\Sigma^i(\sigma) = \{\widetilde{\sigma}^i : e^i + t^i(\sigma|\widetilde{\sigma}^i) \in \mathbb{R}^k_+, \sum_{j=1}^{k} \widehat{E}^i_j(t^j(\sigma|\widetilde{\sigma}^i), \widetilde{\sigma}^i) \leq \sum_{j=1}^{k} \widehat{R}^i_j(t^j(\sigma|\widetilde{\sigma}^i), \widetilde{\sigma}^i)\}.$$

Thus $\sum^i(\sigma)$ is the set of strategies of i that lead to feasible trades for him when others' strategies are fixed at σ. Define σ to be a strategic (Nash) equilibrium (SE) if

(i) $$\sum_{i=1}^{n} t^i(\sigma) = 0$$

(ii) $$\sigma^i \in \Sigma^i(\sigma) \text{ for } i = 1, \ldots, n$$

(iii) $$e^i + t^i(\sigma) \succsim_i e^i + t^i(\sigma|\widetilde{\sigma}^i), \text{ for all } \widetilde{\sigma}^i \in \Sigma^i(\sigma),\ i = 1, \ldots, n.$$

Then propositions 1, 2 remain true with SE substituted for EP . To see this, first note that

$$\sum_{i\in H_j} \int_0^{|d^i_j|} d^i_j = \int_0^{T_j} D_j, \qquad \sum_{i\in G_j} \int_0^{t^i_j} s^i_j = \int_0^{T_j} S_j$$

and then reread the proof of the Lemma 1 with $\widehat{E}_j,\ \widehat{R}_j$ in place of R_j, E_j (in effect reversing the chain of inequalities). This proves that, at any SE, all trade takes place at one price. The rest of the proof proceeds as before, after noting the following changes. Consider T_j, the total trade in commodity j at the SE under consideration, and the (unique) price p_j at which T_j is traded in the SE. Also denote the aggregate demand and supply curves for commodity j by D_j and S_j . Now it is clear that an agent can (via unilateral deviations in strategy at the SE)

(i) buy up to T_j at the price p_j (simply by quoting to buy more, in the event that he is being rationed)

(ii) buy $x > T_j$, at the price $S_j(x)$ (by quoting the flat curve whose price is $S_j(x)$).

Thus his expenditure for buying x is $E_j(x) = xS_j(x)$.
Similarly his receipts from selling x is $R_j(x) = xD_j(x)$. Both E_j and R_j are C^1, since S_j and D_j are C^1 by hypothesis. Now the proofs hold exactly as before.

Remark 1 : (Proof against Pretension) We could enhance the strategy-set of an agent by allowing him to pretend to be any finite number of agents as he wishes, aggregating the trades he obtains via his proxies, provided the aggregate trade is feasible for him. This, as can easily be verified, would not disturb the equilibrium (i.e., he could not get higher utility by such a manoeuver).

Remark 2 : (Proper Game) It is also possible to describe a proper game by introducing default penalties if $\sum_j \widehat{E}_j^i > \sum_j \widehat{R}_j^i$ or if $e_j^i + t_j(\sigma|\widetilde{\sigma}^i) < 0$, as is done in Dubey (1982, 1994). Moreover, the penalties can be brought on by a rule for confiscating commodities in amounts commensurate with the size of the default, as discussed there. (They can be trivially be brought on by confiscating the entire consumption, in effect inflicting a huge penalty on agents who violate their feasibility constraints. For then each agent will simply prefer not to trade and to consume his initial endowment.)

Remark 3 : (Inactive markets) For any subset $L \subset \{1, \dots, k\}$ of commodities, one can define Walras equilibrium modulo L, by restricting trade to only commodities in L and restricting the preferences to this subspace of trades. Then our analysis shows that an SE yields Walras equilibrium modulo the set of commodities whose markets are active at the SE.

Remark 4 : (Equilibrium Refinement) Imagine a "market maker" who endeavours to trigger trade at market j by offering to buy (and, sell) up to $\epsilon > 0$ units of commodity j at some common price p_j and to buy (sell) more at smoothly decreasing (increasing) prices. We shall call this an "ϵ-perturbation" of market j. Treating each market maker as a strategic dummy, and postulating that he creates the commodities and the money that the mechanism calls upon him to deliver, the game is well defined even after some markets are ϵ-perturbed. We shall say that an NE is refined it there exist ϵ-perturbations of its inactive markets that do not disturb the NE. It is then trivial to verify (using the convexity of preferences) that the prices and allocations of refined NE coincide with the CE of the underlying economy, i.e., refinement eliminates the need for the extended Walras equilibria of Remark 3. (Note that refined NE would be unaffected if we required ϵ-perturbations of all the markets; the market-maker could simply trade ϵ with himself at the NE price formed at each active market).

Other, more sophisticated, versions of refinement can be thought of. One could consider the NE of the game in which all markets are ϵ-perturbed, and take the limit of these NE as $\epsilon \to 0$. With some additional constraints on the perturbations, this should lead to the same refined NE, but we will not pursue the inquiry here.

4 Mechanism Design

The strategic market game of Section 3 can be interpreted in the context of mechanism design (see Postlewaite (1985)), once we observe that agents' strategy-sets are invariant of their preferences.[8] To bring our strategy-to-outcome map in line with that literature, let us define (pro forma) the outcome to be no-trade if any agent winds up being infeasible (i.e. either violates his budget constraint or is called upon to sell more of any commodity than he has). This yields a "strategic outcome function" as in Postlewaite (1985). Next, given any economy, define the set of "extended Walras equilibria" to be the union of Walras equilibria modulo L, as L varies over all possible subsets of commodities. Then, in the terminology of Postlewaite (1985), Remark 3 implies that the "Nash performance correspondence" of our strategic outcome function exactly implements the extended Walras correspondence [9];

[8] In our model even more is true. The game is truly "anonymous" in the sense of Dubey, Mas-Colell, and Shubik (1980): all agents have the same message space, and the trade that the market assigns to any agent (prior to a feasibility check on him) depends in an identical way on his message and the distribution (indeed aggregate) of all messages.

[9] Given a CE modulo L, one can construct an NE on the L-subeconomy to match the CE as in the proof of Proposition 2. Then define strategies on the set $\{1, \ldots, k\} \setminus L$ of inactive markets as follows. Let some agents quote supply curves starting at exorbitantly high prices, while others quote demand curves starting at ridiculously low prices (scaling the CE prices up, if necessary, to make this feasible). Provided that the marginal rates

while Remark 4 implies that the "refined Nash performance correspondence" exactly implements the Walras correspondence.

Of course, others have presented mechanisms which implement the Walras correspondence (see, e.g., Hurwicz (1979). Hurwicz, Maskin, and Postlewaite (1980), Schmeidler (1980), all of whom, incidentally, require at least three agents and bypass the case of a bilateral monopoly). It is not our intention just to add to this list. We were instead inspired by the fact that the "double auction" - which underlies our mechanism - has a long and rich history, not only in academia, but in real market processes (see Friedman and Rust (1993) for an excellent survey). Our analysis reveals that a "smoothened" version of the double auction will make for efficiency and help to break monopoly power. It thereby implies that, if the "price-jumps" permitted in the bidders' strategies are reduced by mandate (of the auction-designer), every such reduction will tend to come with efficiency gains. To that extent, we hope that our analysis will also be of some interest to applied economists who are concerned with the general properties of double auctions.

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of substitution between commodities are bounded, we obtain a full-fledged NE which replicates the CE. This, in conjunction with Remark 3, shows the exact implementation.

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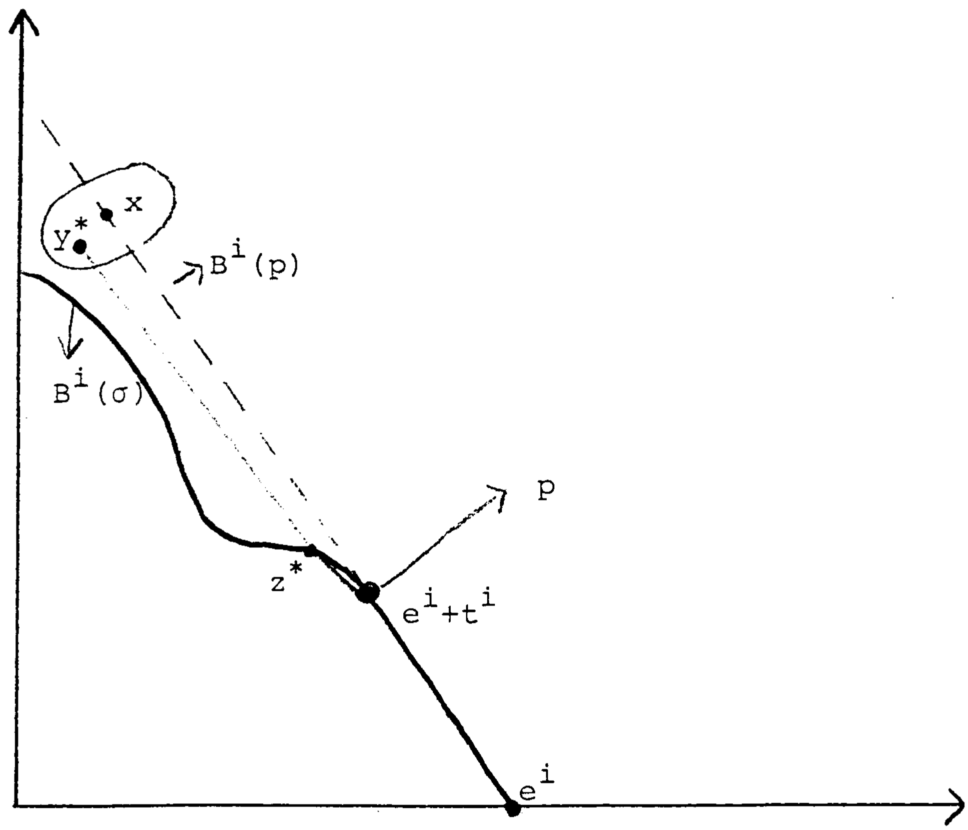

Fig. 1

Fig. 2

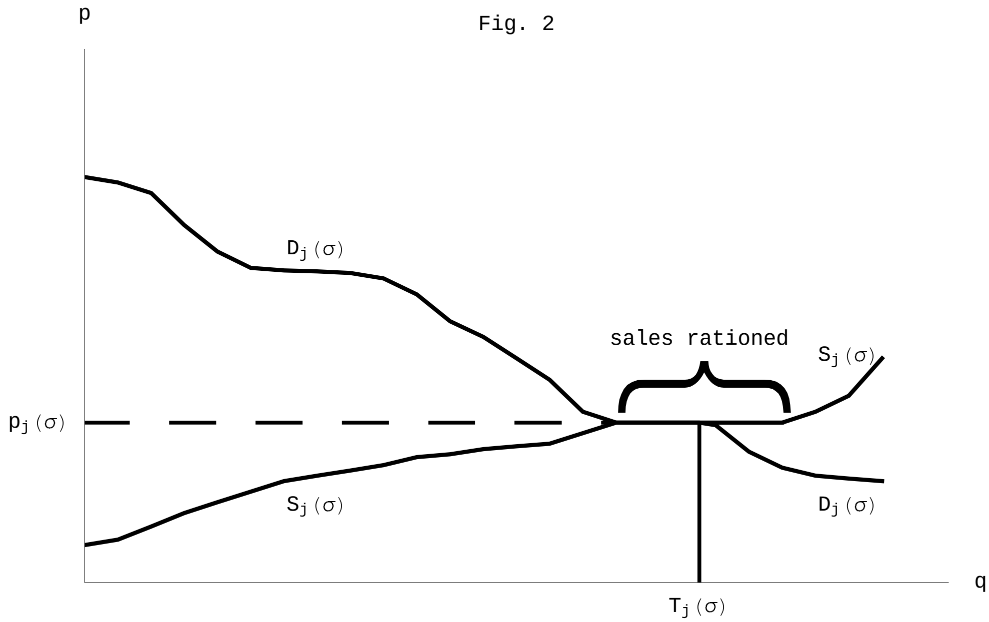
p
$D_j(\sigma)$
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Fig. 3