RI-05-009 September, 2005 Subject: Computer Science

IBM Research Report

Competing for Customers in a Social Network

Pradeep Dubey
pradeepkdubey@yahoo.com
Center for Game Theory, Stony Brook University
Cowles Foundation, Yale University, USA

Rahul Garg
grahul@in.ibm.com
IBM India Research Lab
New Delhi, INDIA

IBM Research Division,
IBM India Research Lab,
Hauz Khas, New Delhi - 110016. INDIA.
Phone: +91-11-2686-1100, Fax: +91-11-2686-1555

IBM Research Division
Almaden - Austin - Beijing - Delhi - Haifa - T.J. Watson - Tokyo - Zurich

LIMITED DISTRIBUTION NOTICE: This report has been submitted for publication outside of IBM and will probably be copyrighted if accepted for publication. It has been issued as a Research Report for early dissemination of its contents. In view of the transfer of copyright to the outside publisher, its distribution outside of IBM prior to publication should be limited to peer communications and specific requests. After outside publication, requests should be filled only by reprints or legally obtained copies of the article (e.g., payment of royalties). Copies may be requested from IBM T.J. Watson Research Center, Publications, P.O. Box 218, Yorktown Heights, NY 10598 USA (email: reports@us.ibm.com). Some reports are available on the internet at http://domino.watson.ibm.com/library/CyberDig.nsf/home

Competing for Customers in a Social Network

Pradeep Dubey
pradeepkdubey@yahoo.com
Center for Game Theory, Stony Brook University
Cowles Foundation, Yale University, USA

Rahul Garg
grahul@in.ibm.com
IBM India Research Lab
New Delhi, INDIA

Abstract

There are many situations in which a customer's proclivity to buy the product of any firm depends not only on the classical attributes of the product such as its price and quality, but also on who else is buying the same product. We model these situations as games in which firms compete for customers located in a "social network". Nash Equilibrium (NE) in pure strategies exist in general. In the quasi-linear version of the model, NE turn out to be unique and can be precisely characterized. There is a cut-off level above which high cost firms are blockaded at an NE, while the rest compete *uniformly* throughout the network. We also explore the relation between the connectivity of a customer and the money firms spend on him.

1 Introduction

Consider a situation in which firms compete for customers located in a "social network". Any customer i has, of course, a higher proclivity to buy from firm α, if α lowers its price relative to those quoted by its rivals. But another, quite independent, consideration also influences i's decision. He is keen to conform[1] to his neighbors in the network. If the bulk of them purchase firm β's product, then he is tempted to do likewise, even though β may be charging a higher price than α. Customer i's behavior thus involves a delicate balance between the "externality" exerted by his neighbors and the more classical constituents of demand — the price and the intrinsic quality of the product itself. Such externalities arise naturally in several contexts (see, e.g., [1, 4]).

The externality in demand clearly has significant impact on the strategic interaction between the firms[2]. Firm α may spend resources marketing its product to i, not because α cares about i per se as a client, but because i enjoys the position of a "hub" in the social network and so wields influence on other potential clients that are of value to α. This in turn might instigate rival firms to spend further on i, since they wish to wean i away from an excessive tilt toward α; causing α to increase its outlay on i even more, unleashing yet another round of incremental expenditures on i.

The scenario invites us to model it as a non-cooperative game between the firms[3]. We take our cue from [1, 4], which explore the optimal marketing strategy of a *single* firm, based on the "network value" of the customers. Our innovation is to introduce competition between *several* firms in this setting. The model we

[1] A typical example: if most of the people who i calls subscribe to Sprint, and if Sprint-to-Sprint calls have superior connectivity compared with Sprint-to-ATT calls, then i will have incentive to subscribe to Sprint even if its price is higher than ATT.

[2] Besides price discounts, already alluded to, one can think of special warranties offered to i or free add-ons of supplementary products, or simply the money spent on advertising to i, etc.

[3] Customers are not strategic in our model. As in [1], they are described in behavioristic terms.

present is more general than that of [1], though inspired by it. As in [1, 4], the social network, specifying the field of influence of each customer, is taken to be exogenous. Rival firms choose how much money to spend on each customer. For any profile of firms' strategies, we show that the externality effect stabilizes over the social network and leads to unambiguous customer-purchases. Each firm values the sales it winds up with, but must deduct the expenditure incurred in the process of generating it.

Our main interest is in understanding the structure of the Nash Equilibria (NE) of the game between the firms. Will they end up as regional monopolies, operating in separate parts of the network? Or will they compete fiercely throughout? Which firms will enter the fray, and which will be blockaded? And how will the money spent on a customer depend on his connectivity in the social network?

In Section 2 we describe a general non-linear model. So long as the externalities are "non-expansive", the strategy-to-outcome map (and thus the game) is well-defined. We show in Section 3 that NE exist in pure strategies under the standard convexity assumptions. Section 4 specializes to the quasi-linear case (and includes the model in [1], by setting # firms = 1). Here we prove that NE are unique and can be easily computed in polynomial time via closed-form expressions involving matrix inverses. It turns out that, provided that there are no apriori biases between firms and customers, any NE has a cut-off cost: all firms whose costs are above the cut-off are blockaded, and the rest enter the fray. Moreover there is no "regionalization" of firms in an NE: each active firm spends money on *every* customer-node of the social network. The money spent on node i is related to the connectivity of i, but the relation is somewhat subtle, though expressible in precise algebraic form. Finally in Section 5 we allow for convex costs in the quasi-linear model, and present two sets of hypotheses, under either of which NE remain unique. We conjecture that uniqueness of NE holds even without these hypotheses.

2 The General Model

There is a finite set $\mathcal{T}$ of firms and $\mathcal{I}$ of customers. Firm $\alpha \in \mathcal{T}$ can spend m_i^α dollars on customer $i \in \mathcal{I}$. Thus we may view $R_+^{\mathcal{I}}$ as the strategy set of firm α, with elements $m^\alpha \equiv (m_i^\alpha)_{i\in\mathcal{I}}$.

Consider a profile of firms' strategies $m \equiv (m^\alpha)_{\alpha\in\mathcal{T}} \in R_+^{\mathcal{I}\times\mathcal{T}}$. This induces probabilities $p(m) \equiv (p^\alpha(m))_{\alpha\in\mathcal{T}}$, where $p^\alpha(m) \equiv (p_i^\alpha(m))_{i\in\mathcal{I}} \in [0,1]^{\mathcal{I}}$. The interpretation is that $p_i^\alpha(m)$ is the probability with which customer i purchases the product of firm α.

The benefit to any particular firm α from its clientele is described by a function $U^\alpha : [0,1]^{\mathcal{I}} \to R$. On the other hand, the cost[4] to α of employing strategy m^α is $C^\alpha(m^\alpha)$. Thus α's payoff in the game is given by

$$\Pi^\alpha(m) = U^\alpha(p^\alpha(m)) - C^\alpha(m^\alpha)$$

It remains to define the map from m to $p(m)$. We do so in two steps. First, the profile m implies expenditures $m_i \equiv (m_i^\beta)_{\beta\in\mathcal{T}}$ on each customer i, creating impetus for i to purchase firm α's product with probability [5] $\gamma_i^\alpha(m_i) \in [0,1]$. Denoting $(m_i^\beta)_{\beta\in\mathcal{T}\setminus\{\alpha\}}$ by $m_i^{-\alpha}$, it stands to reason that the "marketing" impact $\gamma_i^\alpha(m_i^\alpha, m_i^{-\alpha})$ be strictly monotonic[6] in m_i^α for any fixed $m_i^{-\alpha}$. We assume this and a little bit more: γ_i^α is also concave in m_i^α

[4] A natural candidate is $C^\alpha(m^\alpha) = \sum_{i\in\mathcal{I}} m_i^\alpha$, which simply totals the money spent by α. But the money may need to be borrowed with interest rates that rise with subsequent tranches, and in this case it would be appropriate to write $C^\alpha(m^\alpha) = f(\sum_{i\in\mathcal{I}} m_i^\alpha)$ where f is a piecewise linear, convex function. Our formulation of cost is general and includes these cases.

[5] Alternatively, one could think of $\gamma_i^\alpha(m_i)$ as the *quantity* of firm α's product purchased by customer i. This dual interpretation should be kept in mind, though we will give prominence to the probabilistic interpretation in the text.

[6] A function G is strictly monotonic (or, increasing) if $x \geq y$ (component wise) and $x \neq y \Rightarrow G(x) > G(y)$. It is weakly monotonic if $x \geq y \Rightarrow G(x) \geq G(y)$.

for fixed $m_i^{-\alpha}$, reflecting the diminishing returns to α of incremental dollars spent on i. A canonical example we have in mind is $\gamma_i^\alpha(m_i) = m_i^\alpha/\overline{m}_i$ where $\overline{m}_i \equiv (\sum_{\beta\in\mathcal{I}} m_i^\beta)$ (with $\gamma_i^\alpha(0) \equiv 0$). In short, i's probability of purchase from different firms is simply set proportional to the money they spend on him[7].

There is, as was said, a quite different "externality" effect which also influences i in his decision to buy from α. To state it precisely, consider maps

$$F_i^\alpha : [0,1]^{\mathcal{I}} \to [0,1],$$

for all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$. The interpretation is that when customers buy α's product with probabilities $p^\alpha \equiv (p_j^\alpha)_{j\in\mathcal{I}} \in [0,1]^{\mathcal{I}}$, then i's proclivity to conform and also buy from α is given by the probability $F_i^\alpha(p^\alpha)$. (It is natural, though not necessary, to assume that F_i^α does not depend on the component p_i^α of p^α).

Customer i weights the two factors (i.e., the externality impact and the marketing impact) by θ_i^α and $1-\theta_i^\alpha$, where $0 \le \theta_i^\alpha < 1$. Thus, given a strategy profile m, the final probabilities of purchase $p(m) \equiv (p^\alpha(m))_{\alpha\in\mathcal{T}} \in [0,1]^{\mathcal{I}\times\mathcal{T}}$, with $p^\alpha \equiv (p_j^\alpha(m))_{j\in\mathcal{I}}$, must satisfy[8]

$$p_i^\alpha(m) = (1-\theta_i^\alpha)\gamma_i^\alpha(m) + \theta_i^\alpha F_i^\alpha(p_j^\alpha(m)_{j\in\mathcal{I}}) \tag{1}$$

for all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$.

The immediate question is whether (1) has a solution and, if so, is the solution unique? Fortunately the answer on both counts is "yes" (see Lemma 1 in the Appendix), provided the externality is non-expansive, i.e.,

$$||F_i^\alpha(p^\alpha) - F_i^\alpha(q^\alpha)|| \le ||p^\alpha - q^\alpha|| \tag{2}$$

for all $p^\alpha, q^\alpha \in [0,1]^{\mathcal{I}}$, $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$. (Here $||\ \ ||$ denotes the maximum norm.) Condition (2) in essence says that the externality does not "explode". It holds in a variety of special cases of interest, which we shall outline later.

Thus we have specified the strategy-to-outcome map $m \longmapsto p(m)$, where $p(m)$ denotes the unique solution of (2); and therewith the non-cooperative game Γ on the player-set $\mathcal{T}$ of firms.

3 Existence of Nash Equilibrium

Recall that strategy profile m is called a **Nash Equilibrium**[9] (NE) of the game Γ if

$$\Pi^\alpha(m) \ge \Pi^\alpha(\tilde{m}^\alpha, m^{-\alpha}) \ \ \forall\ \tilde{m}^\alpha \in R_+^{\mathcal{I}}$$

for all $\alpha \in \mathcal{T}$ (where $m^{-\alpha} \equiv (m^\beta)_{\beta\in\mathcal{I}\setminus\{\alpha\}}$).

It turns out that NE exist in our model under quite general conditions. First let us list the "standard" assumptions on the functions C^α, U^α, γ_i^α, F_i^α (for all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$), in addition to the requirement that the F_j^α be non-expansive (which was already needed in order to just well-define the game).

AI: The cost function $C^\alpha : R_+^{\mathcal{I}} \to R_+$ is continuous, convex and strictly monotonic.

[7] More generally, $\gamma_i^\alpha(m_i) = (m_i^\alpha/\overline{m}_i)(\overline{m}_i)^r$ where $0 \le r < 1$. We may think of $(\overline{m}_i)^r$ as the "market penetration", which rises with the total money spent. (If $\gamma_i^\alpha(m_i)$ is to be a probability, one must amend $(\overline{m}_i)^r$ to $\max\{(\overline{m}_i^r), 1\}$ or a suitably smoothed version of this function.)

[8] If we were to assume that $\sum_{\alpha\in\mathcal{T}} F_i^\alpha(p) = 1$ for all p and $\sum_{\alpha\in\mathcal{T}} \gamma_i^\alpha(m) = 1$ for all m, then it would follow that $\sum_{\alpha\in\mathcal{T}} p_i^\alpha(m) = 1$ for all $i \in \mathcal{I}$ and all m. In this case we may view the products of different firms as exact "substitutes".

[9] Throughout we confine attention to "pure" strategies.

AII: The benefit function $U^\alpha : R_+^{\mathcal{I}} \to R$ is continuous, concave and weakly monotonic.

AIII: The externality function F_i^α is non-expansive, concave and weakly monotonic.

AIV: If $m_i \equiv (m_i^\alpha, m_i^{-\alpha}) \neq 0$, then the marketing function $\gamma_i^\alpha(m_i^\alpha, m_i^{-\alpha})$ is continuous in all its variables; and concave and weakly monotonic in the variable m_i^α (for every fixed choice of $m_i^{-\alpha}$).

Our last assumption has to do with the possible discontinuity of the marketing function $\gamma_i^\alpha(m_i)$ as $m_i \to 0$, as in our canonical example $\gamma_i^\alpha(m_i) = m_i^\alpha/(\sum_{\beta \in \mathcal{T}} m_i^\beta)$. We require that there be at least two distinct firms who value customer i, so that the competition between them will ensure that $||m_i|| > 0$ in any NE. The intuition is that, if m_i is too small, either of the two firms could spend a "sliver" on i, which costs very little, but is nevertheless overwhelmingly more than other firms' expenditures on i, and thus is able to "buy out" i. To make this precise, we need some notation.

Our last assumption will only be invoked for certain pairs $(\alpha, i) \in \mathcal{T} \times \mathcal{I}$, so we label it AV$(\alpha, i)$.

AV(α, i): $\liminf_{(m_i,\delta)\to 0}[\gamma_i^\alpha(m_i^\alpha + \delta, m_i^{-\alpha}) - \gamma_i^\alpha(m_i)]/\delta = \infty$
where the lim inf is taken over the sequences $\{m_i, \delta\}$ that satisfy the conditions: $(m_i^\alpha + \delta)/m_i^\beta \to \infty$ for all $\beta \in \mathcal{T}\backslash\{\alpha\}$, and $m_i^\alpha \leq m_i^\beta$ for some $\beta \in \mathcal{T}\backslash\{\alpha\}$.

To get a sense of AV(α, i), note that one would expect[10] $\liminf \gamma_i^\alpha(m_i^\alpha + \delta, m_i^{-\alpha}) = 1$ since all $\beta \in \mathcal{T}\backslash\{\alpha\}$ have expenditures m_i^β that are vanishingly small compared to the expenditure $m_i^\alpha + \delta$ made by α; and would expect $\limsup \gamma_i^\alpha(m_i) \leq 1/2$ since α's expenditure in m_i is over-matched by at least one rival firm. Thus the bracketed term $[\ldots]$ would be of the order of $1/2\delta \to \infty$ as $\delta \to 0$. Our assumption is weaker[11], allowing for the total probability of purchase across all firms by customer i to go to zero with the aggregate expenditure m_i.

Finally, a bit of terminology. We say that *firm α values customer i* if $U^\alpha(p^\alpha)$ is strictly monotonic in the variable p_i^α.

We are ready to state our main existence result.

Theorem 1 *Assume AI, AII, AIII, AIV. Further assume, for each customer i, that either (a) γ_i^β is continuous at 0 for all $\beta \in \mathcal{T}$ or (b) there exist at least two distinct firms α and α' who value i and for which AV(α, i) and AV(α', i) hold. Then a Nash Equilibrium exists in the game Γ.*

See the Appendix.

4 The Quasi-Linear Model

4.1 The Data of the Economy

We turn to a quasi-linear version of our model, which is particularly transparent, and in which NE are not only unique but can be precisely characterized. The social network now has a concrete representation in terms of a directed, weighted graph $G = (\mathcal{I}, E, w)$. The nodes of G are identified with the set of customers $\mathcal{I}$. Each directed edge $(i, j) \in E \equiv \mathcal{I} \times \mathcal{I}$ has weights $(w_{ij}^\alpha)_{\alpha \in \mathcal{T}}$, where $w_{ij}^\alpha \geq 0$ is a measure of the influence j has on i, with regard to purchases from α. Precisely, if $p^\alpha = (p_j^\alpha)_{j \in \mathcal{I}}$ denotes the probabilities of purchases, then the

[10] e.g., in our canonical example $\gamma_i^\alpha(m_i) = m_i^\alpha/(\sum_{\beta \in \mathcal{T}} m_i^\beta)$.
[11] Thus the examples of footnote 2 also satisfies AV(α, i).

externality impact of p^α on i is $\sum_{j\in\mathcal{I}} w_{ij}^\alpha p_j^\alpha$. We assume that $\sum_{j\in\mathcal{I}} w_{ij}^\alpha \leq 1$ and $w_{ii}^\alpha = 0$, for all $i \in \mathcal{I}$ and $\alpha \in \mathcal{T}$. (One may view $(\mathcal{I}, E^\alpha, w^\alpha)$ as the social network relevant for the firm α, with $E^\alpha = \{(i,j) \in E : w_{ij}^\alpha > 0\}$).

Let us recapitulate how the externality effect determines purchases in the social network. Fix any profile $m \equiv (m^\beta)_{\beta\in\mathcal{T}} \equiv ((m_j^\beta)_{j\in\mathcal{I}})_{\beta\in\mathcal{T}}$ of firms' strategies. Recall that, for any particular firm α and customer i, $\gamma_i^\alpha(m_i)$ is the probability with which i is initially induced to buy from firm α on account of the direct marketing impact, where $m_i \equiv (m_i^\beta)_{\beta\in\mathcal{T}}$ gives the expenditures induced on i by m. Thus the profile m leads to purchase probabilities $p_i^\alpha(m)$ for all $(\alpha, i) \in \mathcal{T} \times \mathcal{I}$ which satisfy the "steady state" conditions

$$p_i^\alpha(m) = (1-\theta_i^\alpha)\gamma_i^\alpha(m_i) + \theta_i^\alpha \sum_{j\in\mathcal{I}} w_{ij}^\alpha p_j^\alpha(m) \tag{3}$$

The fact that (3) has a unique solution follows, of course from our analysis of the general model, once one observes that the map $(p_i^\beta)_{i\in\mathcal{I}}^{\beta\in\mathcal{T}} \longmapsto ((1-\theta_i^\beta)\sum_{j\in\mathcal{I}} w_{ij}^\beta p_j^\beta)_{i\in\mathcal{I}}^{\beta\in\mathcal{T}}$ is non-expansive. But a simpler argument can be given in the quasi-linear case. Indeed denote by W^α the $\mathcal{I}\times\mathcal{I}$ matrix whose ij^{th} entry is $\theta_i^\alpha w_{ij}^\alpha$; and by $\tilde{\gamma}^\alpha(m)$ the column vector with components $\tilde{\gamma}_j^\alpha(m) = (1-\theta_j^\alpha)\gamma_j^\alpha(m)$ for $j \in \mathcal{I}$. Then (3) can be rewritten as

$$(I - W^\alpha)p^\alpha(m) = \tilde{\gamma}^\alpha(m)$$

But, as is well-known, the matrix $I - W^\alpha$ is invertible since the row sums of W^α are less than 1 (on account of $\theta_i^\alpha < 1$ and $\sum_{j\in\mathcal{I}} w_{ij} \leq 1$). Hence the solution to (3) is

$$p^\alpha(m) = (I - W^\alpha)^{-1}\tilde{\gamma}^\alpha(m)$$

It still remains to specify U^α, C^α and γ_i^α. We take U^α and C^α to be linear:

$$U^\alpha(p^\alpha) = \sum_{j\in\mathcal{I}} u_j^\alpha p_j^\alpha$$
$$C^\alpha(m^\alpha) = \sum_{j\in\mathcal{I}} c_j^\alpha m_j^\alpha$$

with $u_j^\alpha \geq 0$ and $c_j^\alpha > 0$ for all $j \in \mathcal{I}$. This gives

$$\Pi^\alpha(m) = [(I - W^\alpha)^{-1}\tilde{\gamma}^\alpha(m)]^T u - (c^\alpha)^T m^\alpha \tag{4}$$

where u^α and c^α are the column vectors $(u_j^\alpha)_{j\in\mathcal{I}}$, $(c_j^\alpha)_{j\in\mathcal{I}}$ and T stands for the transpose operation. Denote $(I - W^\alpha)^{-1}$ by $Z^\alpha \equiv (z_{ij}^\alpha)_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ and define

$$v_i^\alpha \equiv \sum_{j\in\mathcal{I}} z_{ji}^\alpha u_j^\alpha (1-\theta_j^\alpha) \tag{5}$$

Then (4) may be rewritten:

$$\Pi^\alpha(m) = \sum_{i\in\mathcal{I}} (v_i^\alpha \gamma_i^\alpha(m_i) - c_i^\alpha m_i^\alpha) \tag{6}$$

Our key assumption on $\gamma_i^\alpha(m_i)$ is that it depends only on the variables m_i^α and $\overline{m}_i^{-\alpha} \equiv \sum_{\beta\in\mathcal{T}\setminus\{\alpha\}} m_i^\beta$, i.e., firm α is affected only by the aggregate[12] expenditure of its rivals.

[12] Aggregation is a form of anonymity that is common to many markets. It says, in essence, that if a firm pretends to be two entities and splits its expenditure between them, this has no effect on *other* firms. This form of "anonymity toward numbers" is tantamount to aggregation.

Assume $\gamma_i^\alpha(m_i^\alpha, \overline{m}_i^{-\alpha})$ is monotonic[13] and differentiable w.r.t. m_i^α whenever $\overline{m}_i \equiv m_i^\alpha + \overline{m}_i^{-\alpha} > 0$ and define $\lambda_i^\alpha : [0,1] \times R_{++} \to R_+$ by

$$\lambda_i^\alpha(r_i^\alpha, \overline{m}_i) \equiv \frac{\partial}{\partial x}\gamma_i^\alpha(x,y), \text{ with } x = r_i^\alpha \overline{m}_i \text{ and } y = (1 - r_i^\alpha)\overline{m}_i$$

(Thus $r_i^\alpha \equiv m_i^\alpha / \overline{m}_i$.) We suppose that λ_i^α is strictly decreasing in (i) r_i^α (for any fixed $\overline{m}_i$); and (ii) in $\overline{m}_i$ (for any fixed r_i^α). Condition (i) reflects diminishing returns on incremental dollars spent by α. Condition (ii) states the obvious fact that an incremental dollar of α counts for less when α's rivals have put in more money. Note that both the conditions are satisfied by our examples in footnote 2.

Finally we assume that for each $i \in \mathcal{I}$ there exists at least two firms α and α' which value i (i.e., $u_i^\alpha > 0$ and $u_i^{\alpha'} > 0$) and for which AV(α, i) and AV(α', i) hold.

4.2 Uniqueness of Nash Equilibrium

Theorem 2 *There exists a unique Nash Equilibrium in the quasi-linear model.*

Proof: First observe that conditions (i) and (ii) on λ_i^α together imply that $(\partial/\partial m_i^\alpha)\gamma_i^\alpha(m_i^\alpha, m_i^{-\alpha})$ is decreasing in m_i^α (for any fixed $m_i^{-\alpha}$), i.e., γ_i^α is concave in its first variable. Thus all the requirements of Theorem 1 are met and an NE exists.

We claim that if m is an NE, then $\overline{m}_i \equiv \sum_{\beta \in \mathcal{I}} m_i^\beta > 0$ for all $i \in \mathcal{I}$. Suppose, to the contrary, that $\overline{m}_i = 0$ for some i. By assumption, there exists α such that $u_i^\alpha > 0$ (and therefore $v_i^\alpha > 0$ by (5)) and such that AV(α, i) holds.

Take a sequence $\{\delta_n\}_{n=1}^\infty$ of positive numbers such that $\delta_n \to 0$. By AV(α, i), $\liminf_{n\to\infty}[\gamma_i^\alpha(\delta_n, 0) - \gamma_i^\alpha(0,0)]/\delta_n = \liminf_{n\to\infty} \gamma_i^\alpha(\delta_n, 0)/\delta_n = \infty$. Let firm α unilaterally deviate from m by spending δ_n on customer i. Then, by (6) α's change is payoff is

$$v_i^\alpha \gamma_i^\alpha(\delta_n, 0) - c_i^\alpha \delta_n$$

which becomes positive for large enough n since $\liminf_{n\to\infty} \gamma_i^\alpha(\delta_n, 0)/\delta_n = \infty$, contradicting that m is an NE. We conclude that $\overline{m}_i > 0$ for all $i \in \mathcal{I}$.

Suppose $m \equiv (m^\alpha)_{\alpha \in \mathcal{I}}$ and $\eta \equiv (\eta^\alpha)_{\alpha \in \mathcal{I}}$ are two NE. Denote $r_i^\alpha \equiv m_i^\alpha/\overline{m}_i$ and $s_i^\alpha \equiv \eta_i^\alpha/\overline{\eta}_i$ (where, recall, $\overline{m}_i \equiv \sum_{\alpha \in \mathcal{T}} m_i^\alpha$ etc.). It suffices to show that $\overline{m}_i = \overline{\eta}_i$ and $r_i^\alpha = s_i^\alpha$ for all $\alpha \in \mathcal{T}$ and all $i \in \mathcal{I}$.

The first-order conditions[14] for maximizing payoffs imply

$$v_i^\alpha \lambda_i^\alpha(r_i^\alpha, \overline{m}_i) = c_i^\alpha \text{ if } m_i^\alpha > 0 \tag{7}$$

$$v_i^\alpha \lambda_i^\alpha(r_i^\alpha, \overline{m}_i) \le c_i^\alpha \text{ if } m_i^\alpha = 0 \tag{8}$$

$$v_i^\alpha \lambda_i^\alpha(s_i^\alpha, \overline{\eta}_i) = c_i^\alpha \text{ if } \eta_i^\alpha > 0 \tag{9}$$

$$v_i^\alpha \lambda_i^\alpha(s_i^\alpha, \overline{\eta}_i) \le c_i^\alpha \text{ if } \eta_i^\alpha = 0 \tag{10}$$

Fix $i \in \mathcal{I}$ and suppose w.l.o.g. that $\overline{m}_i \le \overline{\eta}_i$.

Step 1: $s_i^\alpha \le r_i^\alpha$ for all $\alpha \in \mathcal{T}$.

[13] Note that we do *not* assume that γ_i^α is linear in m_i^α. (Linearity of γ_i^α does not hold even in our elementary canonical examples.)

[14] Since $\overline{m}_i > 0$ and $\overline{\eta}_i > 0$, and the γ_i^α are differentiable away from zero, these conditions can be invoked.

Proof: Let $s_i^\alpha > 0$ (otherwise the claim is vacuously true) and suppose, to the contrary, that $s_i^\alpha > r_i^\alpha$. Since $\overline{\eta}_i \geq \overline{m}_i$, conditions (i) and (ii) on λ_i^α imply $\lambda_i^\alpha(s_i^\alpha, \overline{\eta}_i) < \lambda_i^\alpha(r_i^\alpha, \overline{m}_i)$, so LHS of (9) < LHS of (7). But $r_i^\alpha \geq s_i^\alpha > 0$ implies that (9) and (7) hold, yielding LHS of (9) = LHS of (7), a contradiction.

Step 2: $s_i^\alpha = r_i^\alpha$ for all $\alpha \in \mathcal{T}$.

Proof: Immediate from step 1, since $\sum_{\alpha\in\mathcal{T}} s_i^\alpha = 1 = \sum_{\alpha\in\mathcal{T}} r_i^\alpha$.

Step 3: $\overline{m}_i = \overline{\eta}_i$

Proof: Suppose $\overline{\eta}_i > \overline{m}_i$ (by assumption we already have $\geq$). By step 2, and condition (ii) on λ_i^α, we have LHS of (9) < LHS of (7). Since $\sum_{\beta\in\mathcal{T}} r_i^\beta = 1$ there exists β' such that $r_i^{\beta'} > 0$. By step 2, $s_i^{\beta'} = r_i^{\beta'}$, so both (9) and (7) hold, hence LHS of (9) = LHS of (7), a contradiction. This proves step 3.

Since the choice of i was arbitrary, we have shown that $\overline{\eta}_i = \overline{m}_i$ and $r_i^\alpha = s_i^\alpha$ for all $\alpha \in \mathcal{T}$ and all $i \in \mathcal{I}$. Thus $m = \eta$, establishing the uniqueness of NE. □

4.3 Characterization of Nash Equilibrium

We focus on our canonical case: $\gamma_i^\alpha(m_i) = m_i^\alpha/\overline{m}_i$ (other closed-form expressions for the γ_i^α will lead to analogous characterizations). Fix customer i and rank all the firms in $\mathcal{T} \equiv \{1, 2, \ldots, n\}$ in order of increasing $\kappa_i^\alpha \equiv c_i^\alpha/v_i^\alpha$ (see (5) for the definition of v_i^α). For convenience denote this order $\kappa_i^1 \leq \kappa_i^2 \leq \ldots \leq \kappa_i^n$. Let

$$k = \max\left\{ l \in \{1, \ldots, n\} : (l-2)\kappa_i^l < \sum_{\alpha=1}^{l-1} \kappa_i^\alpha \right\} \tag{11}$$

In the unique NE, firms $1, \ldots, k$ will spend money on customer i as follows:

$$m_i^\alpha = \left(\frac{k-1}{\sum_{\beta=1}^k \kappa_i^\beta} \right) \left(1 - \frac{(k-1)\kappa_i^\alpha}{\sum_{\beta=1}^k \kappa_i^\beta} \right) \tag{12}$$

Firms $k+1, \ldots, n$ will put no money on customer i.

To verify this characterization, note that $\lambda_i^\alpha(r_i^\alpha, \overline{m}_i) = (1 - r_i^\alpha)/\overline{m}_i$ in our canonical case. Thus the first-order conditions (7) and (8) become

$$\frac{1 - r_i^\alpha}{\overline{m}_i} = \kappa_i^\alpha \text{ if } r_i^\alpha > 0 \tag{13}$$

$$\frac{1 - r_i^\alpha}{\overline{m}_i} \leq \kappa_i^\alpha \text{ if } r_i^\alpha = 0 \tag{14}$$

It follows at once that, if $r_i^\alpha > 0$, then $r_i^\beta > 0$ whenever $\kappa_i^\beta \leq \kappa_i^\alpha$. Hence we only need check that: (i) condition (13) holds for $1 \leq \alpha \leq k$; (ii) condition (14) holds for $k+1 \leq \alpha \leq n$; and (iii) $r_i^\alpha > 0$ for $1 \leq \alpha \leq k$.

Note that (12) implies

$$\overline{m}_i = \sum_{\beta=1}^k m_i^\beta = \frac{k-1}{\sum_{\beta=1}^k \kappa_i^\beta} \tag{15}$$

And (12) and (15) imply

$$\begin{aligned} 1 - r_i^\alpha &= 1 - \frac{m_i^\alpha}{\overline{m}_i} \\ &= \frac{(k-1)\kappa_i^\alpha}{\sum_{\beta=1}^k \kappa_i^\beta} \end{aligned}$$

Then (i) follows from the above equation and (15).

It suffices to show (ii) for $\alpha = k+1$, since LHS of (14) $= 1/\overline{m}_i$ for all $\alpha \geq k+1$ (on account of $r_i^\alpha = 0$) and since RHS of (14) rises with α.

Taking $l = k+1$, and violating the inequality in (11), we obtain

$$(k-1)\kappa_i^{k+1} \geq \sum_{\beta=1}^{k} \kappa_i^\beta$$

i.e.,

$$\frac{\sum_{\beta=1}^{k} \kappa_i^\beta}{k-1} \leq \kappa_i^{k+1}$$

which, together with (15), implies (ii).

Finally, taking $l = k$ in (11), we have

$$(k-2)\kappa_i^k \leq \sum_{\beta=1}^{k-1} \kappa_i^\beta$$

Adding κ_i^k to both sides gives

$$(k-1)\kappa_i^k < \sum_{\beta=1}^{k} \kappa_i^\beta$$

i.e.,

$$1 - \frac{(k-1)\kappa_i^k}{\sum_{\beta=1}^{k} \kappa_i^\beta} > 0$$

But, since $\kappa_i^\alpha \leq \kappa_i^k$ for $\alpha \leq k$, this yields

$$1 - \frac{(k-1)\kappa_i^\alpha}{\sum_{\beta=1}^{k} \kappa_i^\beta} > 0$$

for $1 \leq \alpha \leq k$. Then, by (12), $m_i^\alpha > 0$, i.e., $r_i^\alpha > 0$ for $1 \leq \alpha \leq k$ verifying (iii).

4.4 Impact of the Social Network on Nash Equilibrium

To simplify the analysis, let us further assume: $c_i^\alpha = c^\alpha$, $w_{ij}^\alpha = w_{ij}$ and $\theta_i^\alpha = \theta_i$ for all $\alpha \in \mathcal{T}$ and $i, j \in \mathcal{I}$. Our analysis in Section 4.2 immediately implies that $v_i^\alpha = v_i$ for all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$. Moreover we can rank the firms, independently of i, by their costs; say (after relabeling)

$$c^1 \leq c^2 \leq \ldots \leq c^n$$

At the Nash Equilibrium a subset of low-cost firms $\{1, \ldots, k\}$ will be active (see (11), while all the higher-cost forms $\{k+1, \ldots, n\}$ will be blockaded, where

$$k = \max \left\{ l \in \{1, \ldots, n\} : (l-2)c^l < \sum_{\beta=1}^{l-1} c^\beta \right\}$$

Each active firm $\alpha \in \{1, \ldots, k\}$ will spend an amount $m_i^\alpha > 0$ on all the nodes $i \in \mathcal{I}$ that is proportional to v_i. Indeed, by (12), we have

$$m_i^\alpha = \frac{v_i(k-1)}{\sum_{\beta=1}^{k} c^\beta} \left(1 - \frac{(k-1)c^\alpha}{\sum_{\beta=1}^{k} c^\beta} \right)$$

θ	v_1	v_4	v_5	v_6	v_4/v_5	v_1/v_6
0.25	0.843	1.488	1.27	0.856	1.17	0.985
0.5	0.73	1.84	1.47	0.746	1.25	0.978
0.75	0.646	2.11	1.63	0.658	1.29	0.98
0.99	0.585	2.325	1.746	0.586	1.33	0.9986

Table 1: The parameters v_i's for the social network

which also shows that $\overline{m}^\alpha > \overline{m}^\beta$ if $\alpha < \beta$, i.e., lower cost firms spend more money than their higher-cost rivals. Finally, by (15), we have

$$\overline{m}_i = \frac{v_i(k-1)}{\sum_{\beta=1}^{k} c^\beta}$$

Thus there is no "regionalization" of customer territory at NE, with firms operating in disjoint pieces of the social network. Instead, firms that are not blockaded, compete *uniformly* throughout the social network.

It might be interesting to see how v_i (and thus the total money $\overline{m}_i$ attracted by node i) varies with the connectivity of i in the graph.

For simplicity, let us further assume

$$\theta_i^\alpha = \theta, \; w_{ik'} = w_{ik} \text{ and } \sum_{j \in \mathcal{I}} w_{ij} = 1$$

for all $\alpha \in \mathcal{T}, i \in \mathcal{I}$ and k, k' such that $w_{ik} > 0, \quad w_{ik'} > 0$ (i.e., all the nodes connected to i have the same influence on i).

Example 1:

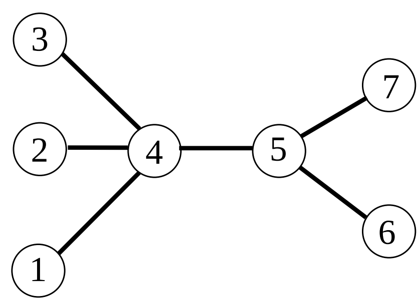

Figure 1: A sample network

(In Table 1, the v_i's are computed for different values of θ, for the social network shown in Figure 1.)

Denote the connectivity of a node i by $c(i)$. Notice that v_i is a monotonic function of $c(i)$, for any fixed θ. For low values of θ, v_i is quite evenly spread across $i \in \mathcal{I}$. But, as θ rises, nodes with high connectivity $c(i)$ gain more in their v_i at the cost of nodes with low $c(i)$; and as $\theta \to 1$, the v_i become proportional to $c(i)$. Note also that the variation of v_i with θ is not linear.

Finally it is worth noticing that, in the oversimplified setting of this section, $\sum_{i \in I} v_i$ is equal to the number of nodes in the network. This fact, in conjunction the above equations, implies that firms' profits (in the unique NE) depend only on their costs and the number of nodes in the social network.

5 Toward Uniqueness with Convex Costs

Consider the quasi-linear model of Section 4 but with differentiable and convex (instead of linear) costs. We conjecture that NE remain unique, though we are able to prove this only with additional hypotheses.

5.1 Multi-Concave Marketing Functions

We make dual (though mild) concavity and convexity assumptions on the marketing impact γ_i^α and the cost C^α.

Define

$$\mu_i^\alpha(m_i^\alpha, \overline{m}_i) \equiv \frac{\partial \gamma_i^\alpha}{\partial m_i^\alpha}(m_i^\alpha, \overline{m}_i - m_i^\alpha)$$

for $\overline{m}_i \equiv \sum_{\beta \in \mathcal{T}} m_j^\beta > 0$. Assume

AVI μ_i^α is strictly decreasing in both its variables.

In the same vein, define

$$\psi_i^\alpha(m^\alpha) \equiv \frac{\partial C^\alpha}{\partial m_i^\alpha}(m^\alpha)$$

and *suppose* that ψ_i^α is a function[15] of just the variables m_i^α and $\overline{m}^\alpha$ (e.g., $C^\alpha(m^\alpha) = f(m^\alpha) + \sum_{i \in \mathcal{I}} c_i^\alpha m_i^\alpha$ where f is differentiable, convex and monotonic). Our dual assumption is on ψ_i^α.

AVII ψ_i^α is weakly increasing in both its variables[16].

Theorem 3 *There exists a unique Nash Equilibrium in the quasi-linear model with convex costs, if assumptions (AVI) and (AVII) hold.*

Proof: (Existence follows from Theorem 1.) First note that $\overline{m}_i > 0$ for all $i \in \mathcal{I}$ at any Nash Equilibrium m. (This follows as in the proof of Theorem 2, replacing c_i^α by the partial derivative of C^α w.r.t. m_i^α evaluated at the point $(b, \ldots, b)$ where b is as in Step 1 of the proof of Theorem 2.)

Let m and η be two NE of the game with $\overline{m} \leq \overline{\eta}$ (where, recall, $\overline{m} \equiv \sum_{j \in \mathcal{I}} \sum_{\beta \in \mathcal{T}} m_j^\beta$ etc.). Define

$$\mathcal{T}_1 = \{\alpha \in \mathcal{T} : \overline{m}^\alpha \leq \overline{\eta}^\alpha\}$$
$$\mathcal{T}_2 = \mathcal{T} \backslash \mathcal{T}_1$$
$$\mathcal{I}_1 = \{j \in \mathcal{I} : \overline{m}_j \leq \overline{\eta}_j\}$$
$$\mathcal{I}_2 = \mathcal{I} \backslash \mathcal{I}_1$$

Clearly, $\mathcal{T}_1 \neq \phi$ and $\mathcal{I}_1 \neq \phi$.

Step 1: If $i \in \mathcal{I}_1$ and $\alpha \in \mathcal{T}_1$, then $\eta_i^\alpha \leq m_i^\alpha$.

Proof: Suppose, to the contrary, that $\eta_i^\alpha > m_i^\alpha \geq 0$. The first-order conditions (8) and (9) may be rewritten

$$v_i^\alpha \mu_i^\alpha(\eta_i^\alpha, \overline{\eta}_i) = \psi_i^\alpha(\eta_i^\alpha, \overline{\eta}^\alpha) \tag{16}$$
$$v_i^\alpha \mu_i^\alpha(m_i^\alpha, \overline{m}_i) \leq \psi_i^\alpha(m_i^\alpha, \overline{m}^\alpha) \tag{17}$$

[15] Without confusion, still named ψ_i^α.

[16] i.e., strictly increasing in m^α (or $\overline{m}_i$) for every fixed $\overline{m}_i$ (or, m_i^α)

Since $\alpha \in \mathcal{T}_1$, we have $\overline{m}^\alpha \leq \overline{\eta}^\alpha$; and so, by (AVII) and the assumption $\eta_i^\alpha > m_i^\alpha$, RHS of (16) $\geq$ RHS of (17). This implies that LHS of (17) $\leq$ LHS of (16), i.e.,

$$\mu_i^\alpha(\eta_i^\alpha, \overline{\eta}_i) \geq \mu_i^\alpha(m_i^\alpha, \overline{m}_i)$$

Also, since $i \in \mathcal{I}_1, \overline{m}_i \leq \overline{\eta}_i$. The last two inequalities, together with assumption (AVI), yield $\eta_i^\alpha \leq m_i^\alpha$, a contradiction. □

Step 2: If $i \in \mathcal{I}_2$ and $\alpha \in \mathcal{T}_2$, then $m_i^\alpha \leq \eta_i^\alpha$.
Proof: Analogous to Step 1.
Step 3: $\mathcal{I}_2 = \phi = \mathcal{T}_2$
Proof: Define

$$M_{kl} = \sum_{i \in \mathcal{I}_k} \sum_{\alpha \in \mathcal{T}_l} m_i^\alpha$$

and

$$N_{kl} = \sum_{i \in \mathcal{I}_k} \sum_{\alpha \in \mathcal{T}_l} \eta_i^\alpha$$

for $k \in \{1, 2\}$ and $l \in \{1, 2\}$ (with summation over the empty set understood, as usual, to be zero). By the definition of $\mathcal{T}_1$, $N_{11} + N_{21} \geq M_{11} + M_{21}$. But by Step 1, $M_{11} \geq N_{11}$. So, $N_{21} \geq M_{21}$. Also, by Step 2, $N_{22} \geq M_{22}$. Hence $N_{21} + N_{22} \geq M_{21} + M_{22}$. However, if $\mathcal{I}_2 \neq \phi$, we must have $N_{21} + N_{22} < M_{21} + M_{22}$ by the definition of $\mathcal{I}_2$, a contradiction. Therefore $\mathcal{I}_2 = \phi$.

Similarly, by the definition of $\mathcal{I}_1$, $N_{11} + N_{12} \geq M_{11} + M_{12}$. Since $M_{11} \geq N_{11}$, we have $N_{12} \geq M_{12}$. And, since $N_{22} \geq M_{22}$, this implies $N_{12} + N_{22} \geq M_{12} + M_{22}$, a contradiction. There $\mathcal{T}_2 = \phi$ also.
Step 4: $\eta = m$.
Proof: Suppose $\eta \neq m$. Since $\overline{\eta} \geq \overline{m}$, there exists $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$ such that $\eta_i^\alpha > m_i^\alpha$. By Step 3, $\overline{\eta}^\alpha \geq \overline{m}^\alpha$. This, along with (16) and (17) and (AVII), implies

$$\mu_i^\alpha(\eta_i^\alpha, \overline{\eta}_i) \geq \mu_i^\alpha(m_i^\alpha, \overline{m}_i)$$

But, by Step 2, $\overline{\eta}_i \geq \overline{m}_i$ for all $i \in \mathcal{I}$. Assumption (AVI) on μ_i^α then yields $\eta_i^\alpha \leq m_i^\alpha$, a contradiction. □

Assumption (AVI) does not hold in general in our canonical case $\gamma_i^\alpha(m) = m_i^\alpha/\overline{m}_i$. But it does hold if we restrict to profiles m such that $m_i^\alpha/\overline{m}_i \leq 1/2$ for all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$. It is easy to place conditions on the exogenous parameters of our model in order to ensure that any Nash Equilibirum profile m will automatically satisfy the restriction (e.g. for any company there is another with the same characteristics, or several with close characteristics). In the presence of such conditions, Theorem 3 establishes uniqueness of NE in the canonical case.

5.2 The Anonymous Case

NE are unique in the canonical case in other settings as well, which are not covered by Theorem 3. We explore one such in Theorem 4 below. (As was said, there may well be a general theorem which yields Theorems 2, 3 and 4 as corollaries.)

Consider the following generalization of the model of Section 4.4. Suppose that the cost is a differentiable and convex function of total expenditure: $C^\alpha(m^\alpha) \equiv C^\alpha(\overline{m}^\alpha)$ for all $\alpha \in \mathcal{T}$. Further suppose

AVIII For all $\alpha \in \mathcal{T}$ and $i \in \mathcal{I}$

$$\lambda_i^\alpha(r_i^\alpha, \overline{m}_i) = \frac{\lambda^\alpha(r_i^\alpha)}{f_i(\overline{m}_i)}$$

(when $\overline{m}_i > 0$), where f_i is strictly increasing and λ^α is strictly decreasing. (Recall $r_i^\alpha \equiv m_i^\alpha / \overline{m}_i$.)

Note that, in our canonical case, $\lambda_i^\alpha(r_i^\alpha, \overline{m}_i) = (1 - r_i^\alpha)/\overline{m}_i$ and so (AVIII) is satisfied. The related examples in footnote 2 also satisfy (AVIII).

Theorem 4 *There exists a unique Nash Equilibrium in the model of Section 4.4, with convex costs as above, if assumption (AVIII) holds.*

Proof: Let m be an NE. As argued in the proof of Theorem 3, $\overline{m}_i > 0$ for all $i \in \mathcal{I}$, so that the derivatives $\lambda_i^\alpha(r_i^\alpha, \overline{m}_i)$ are well defined.

Step 1: $r_i^\alpha = r_j^\alpha \equiv r^\alpha$ for all $i \in \mathcal{I}, j \in \mathcal{I}$ and $\alpha \in \mathcal{T}$.

Proof: Suppose $r_i^\alpha > r_j^\alpha$ for some α, i, j. Since $\sum_{\beta \in \mathcal{T}} r_i^\beta = 1 = \sum_{\beta \in \mathcal{T}} r_j^\beta$, there exists β such that $r_i^\beta < r_j^\beta$. Since $r_i^\alpha > 0$, the first-order conditions for α at i and j are (where $\xi^\alpha(\overline{m}^\alpha) = (d/d\overline{m}^\alpha)C^\alpha(\overline{m}^\alpha)$)

$$v_i \frac{\lambda^\alpha(r_i^\alpha)}{f_i(\overline{m}_i)} = \xi^\alpha(\overline{m}^\alpha)$$

and

$$v_j \frac{\lambda^\alpha(r_j^\alpha)}{f_j(\overline{m}_j)} \leq \xi^\alpha(\overline{m}^\alpha)$$

which gives

$$\frac{v_i \lambda^\alpha(r_i^\alpha)}{f_i(\overline{m}_i)} \geq \frac{v_j \lambda^\alpha(r_j^\alpha)}{f_j(\overline{m}_j)} \tag{18}$$

Similarly

$$\frac{v_i \lambda^\beta(r_i^\beta)}{f_i(\overline{m}_i)} \leq \frac{v_j \lambda^\beta(r_j^\beta)}{f_j(\overline{m}_j)} \tag{19}$$

From (18) and (19) we obtain

$$\frac{\lambda^\alpha(r_i^\alpha)}{\lambda^\alpha(r_j^\alpha)} \geq \frac{\lambda^\beta(r_i^\beta)}{\lambda^\beta(r_j^\beta)} \tag{20}$$

But, by (AVIII), LHS of (20) < 1 and RHS of (20) > 1, a contradiction. □

Let η be another NE and define $s_i^\alpha \equiv \eta_i^\alpha / \overline{\eta}_i$. As shown in Step 1, $s_i^\alpha = s_j^\alpha \equiv s^\alpha$ for all $\alpha \in \mathcal{T}$ and $i, j \in \mathcal{I}$.

Step 2: $r^\alpha = s^\alpha$ for all $\alpha \in \mathcal{T}$.

Proof: Suppose not. W.l.o.g. let $\overline{m} \geq \overline{\eta}$. Clearly there exists $i \in \mathcal{I}$ such that $\overline{m}_i \geq \overline{\eta}_i$ and (since $\sum_{\beta \in \mathcal{T}} r^\beta = 1 = \sum_{\beta \in \mathcal{T}} s^\beta$) there exists $\alpha \in \mathcal{T}$ such that $r^\alpha > s^\alpha \geq 0$. The first order conditions give the following:

$$\frac{v_i \lambda^\alpha(r^\alpha)}{f_i(\overline{m}_i)} = \xi^\alpha(r^\alpha \overline{m}) \tag{21}$$

$$\frac{v_i \lambda^\alpha(s^\alpha)}{f_i(\overline{\eta}_i)} \leq \xi^\alpha(s^\alpha \overline{\eta}) \tag{22}$$

By the convexity of the cost function and the fact that $r^\alpha > s^\alpha$ and $\overline{m} \geq \overline{\eta}$, RHS of (21) $\geq$ RHS of (22), so

$$\frac{\lambda^\alpha(r^\alpha)}{f_i(\overline{m}_i)} \geq \frac{\lambda^\alpha(s^\alpha)}{f_i(\overline{\eta}_i)}$$

But, since $r^\alpha > s^\alpha$ and $\overline{m}_i \geq \overline{\eta}_i$, (AVIII) implies $\lambda^\alpha(r^\alpha) < \lambda^\alpha(s^\alpha)$ and $f_i(\overline{m}_i) \geq f_i(\overline{\eta}_i)$, contradicting the last displayed inequality. □

Step 3: $\overline{m} = \overline{\eta}$.
Proof: Suppose w.l.o.g. $\overline{m} > \overline{\eta}$. Then there exists i such that $\overline{m}_i > \overline{\eta}_i$. Also there clearly exists α such that $r^\alpha > 0$. Now, consider the first order conditions (21) and (22). Since $r^\alpha = s^\alpha$ by Step 2, RHS of (21) $\geq$ RHS of (22), which implies $f_i(\overline{\eta}_i) \geq f_i(\overline{m}_i)$. Thus $\overline{\eta}_i \geq \overline{m}_i$ since f_i is strictly monotonic by (AVIII). This is a contradiction. □

Steps 1, 2 and 3 imply that $m = \eta$, proving Theorem 4. □

5.3 Structure of Nash Equilibrium

Consider the model of Section 5.2, but with assumption AVIII replaced by

AIX $\lambda_i^\alpha = \lambda_i^\beta \equiv \lambda_i$ for all α, β and i.

Also assume that λ_i satisfies condition (i) of Section 4.1.

Even though we cannot prove uniqueness of NE, we can establish an interesting property that is common to all of them. Recall that $\xi^\alpha(\overline{m}^\alpha) = (d/d\overline{m}^\alpha)C^\alpha(\overline{m}^\alpha)$.

Theorem 5 *Consider the model stated above. Let m be any NE. Then*

$$m_i^\alpha > 0 \Rightarrow m_i^\beta > 0$$

for all β, α such that $\xi^\beta(0) \leq \xi^\alpha(0)$.

Proof: Suppose, to the contrary, that $m_i^\alpha > 0$ and $m_i^\beta = 0$ for some β such that $\xi^\beta(0) \leq \xi^\alpha(0)$. The first order conditions for α and β become

$$v_i \lambda_i(r_i^\alpha, \overline{m}_i) = \xi^\alpha(\overline{m}^\alpha)$$

$$v_i \lambda_i(0, \overline{m}_i) \leq \xi^\beta(0)$$

where $r_i^\alpha \equiv m_i^\alpha/\overline{m}_i > 0$. Since $\xi^\alpha(\overline{m}^\alpha) > \xi^\alpha(0) \geq \xi^\beta(0)$, we have $\lambda_i(r_i^\alpha, \overline{m}_i) > \lambda_i(0, \overline{m}_i)$ contradicting condition (i) on λ_i in Section 4.1. □

Corollary Let m be any NE and denote $\mathcal{T}(i) = \{\alpha \in \mathcal{T} : m_i^\alpha > 0\}$. Then, for any two customers i and j, either $\mathcal{T}(i) \subset \mathcal{T}(j)$ or $\mathcal{T}(j) \subset \mathcal{T}(i)$. (Equivalently, denoting $\mathcal{I}(\alpha) = \{i \in \mathcal{I} : m_i^\alpha > 0\}$, there is an ordering $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ of the firms such that $\mathcal{I}(\alpha_1) \subset \mathcal{I}(\alpha_2) \subset \ldots \subset \mathcal{I}(\alpha_n)$.)

6 Acknowledgements

We are grateful to Sachindra Joshi and Chandan Deep Nath for introducing us to the social network literature (in particular to the references [1, 4]), and for stimulating initial discussions.

7 Appendix

Lemma 1 *For any strategy profile m, the system of equations (1) has a unique solution $p(m)$.*

Proof: Fix m and firm $\alpha \in \mathcal{T}$. Define the map $G^{\alpha,m} : [0,1]^{\mathcal{I}} \to [0,1]^{\mathcal{I}}$ by

$$G_i^{\alpha,m}(p^\alpha) = (1-\theta_i^\alpha)\gamma_i^\alpha(m) + \theta_i^\alpha F_i^\alpha(p^\alpha)$$

for all $i \in \mathcal{I}$. Then $G^{\alpha,m}$ is a contraction. Indeed,

$$||G^{\alpha,m}(p^\alpha) - G^{\alpha,m}(q^\alpha)|| \leq \lambda ||p^\alpha - q^\alpha||$$

where $\lambda = \max_{i \in \mathcal{I}} \theta_i^\alpha < 1$. Thus there exists (see, e.g., [2]) a unique $p^\alpha(m)$ such that

$$G^{\alpha,m}(p^\alpha(m)) = p^\alpha(m),$$

showing that $p(m) = (p^\alpha(m))_{\alpha \in \mathcal{T}}$ is the unique solution to (1). (Indeed $p^\alpha(m) = \lim_{k\to\infty}(G^{\alpha,m})^k(q^\alpha)$ for any $q^\alpha \in [0,1]^{\mathcal{I}}$.) □

Proof of Theorem 1

Step 1: Since C^α is strictly monotonic and convex, it is easy to see that $C^\alpha(x) \to \infty$ as $||x|| \to \infty$. Thus there exists a scalar b such that $C^\alpha(m^\alpha) > U^\alpha(1,1,\ldots,1) - U^\alpha(0,0,\ldots,0)$ whenever $||m^\alpha|| \geq b$. Define $S^\alpha = \{m^\alpha \in R_+^{\mathcal{I}} : ||m^\alpha|| \leq b\}$. Clearly, no firm α would spend more than b, for it then could be better off spending zero on every customer. W.l.o.g. we may confine α's strategies to the compact convex set S^α.

Step 2: For any $\alpha \in \mathcal{T}$ and $j \in \mathcal{I}$, if[17] $m >> 0$ then $p_i^\alpha(m^\alpha, m^{-\alpha})$ is concave and monotonic in m^α for every fixed $m^{-\alpha}$.

Proof of Step 2: For any positive integer k, customer $i \in \mathcal{I}$, denote

$$H_i^{\alpha,k}(m^\alpha, m^{-\alpha}) \equiv (G^{\alpha,m})^k(0)$$

where $m \equiv (m^\alpha, m^{-\alpha})$ and $G^{\alpha,m}$ is as in the proof of Lemma 1.

For brevity, say that a real-valued function $h(x,y)$ defined on a vector space "has property (*) with respect to x" if it is concave and monotonic in x for every fixed choice of y.

Now $H_i^{\alpha,1}(m^\alpha, m^{-\alpha})$ has property (*) w.r.t. m^α, since it equals $(1-\theta_i^\alpha)\gamma_i^\alpha(m_i^\alpha, m_i^{-\alpha}) + \theta_i^\alpha F_i^\alpha(0)$ and since γ_i^α has the property (*) w.r.t. m_i^α by assumption AIV.

Make the inductive assumption that $H_i^{\alpha,k}(m^\alpha, m^{-\alpha})$ has the property (*) w.r.t. m^α. Consider

$$\begin{aligned} H_i^{\alpha,k+1}(m^\alpha, m^{-\alpha}) &= G_i^{\alpha,m}((H_j^{\alpha,j}(m^\alpha, m^{-\alpha}))_{j\in\mathcal{I}}) \\ &= (1-\theta_i^\alpha)\gamma_i^\alpha(m_i^\alpha, m_i^{-\alpha}) + \theta_i^\alpha F_i^\alpha((H_j^{\alpha,k}(m^\alpha, m^{-\alpha}))_{j\in\mathcal{I}}) \end{aligned}$$

The function F_i^α has the property (*) w.r.t. all its variables by assumption (AIII). The functions $H_j^{\alpha,k}$ (for all j) also have property (*) w.r.t. m^α by the inductive assumption. It is easy to check that their composition as above will have the property (*) w.r.t. m^α.

Since $H_i^{\alpha,k+1}$ is the sum of two functions which have the property (*) w.r.t. m^α, it too has property (*) w.r.t. m^α.

[17] i.e., each component of m is strictly positive.

But $p_i^\alpha(m^\alpha, m^{-\alpha}) = \lim_{k\to\infty} H_i^{\alpha,k}(m^\alpha, m^{-\alpha})$ as shown in the proof of Lemma 1. Thus p_i^α inherits (from $H_i^{\alpha,k}$) the property (*) w.r.t. m^α by an obvious limiting argument.

Step 3: For $\epsilon > 0$, define the game Γ^ϵ by truncating the strategy sets to $S^{\epsilon,\alpha} = S^\alpha \cap \{m^\alpha \in R_+^{\mathcal{I}} : m_j^\alpha \geq \epsilon \ \forall j \in \mathcal{I}\}$. Then Γ^ϵ has an NE.

Proof of Step 3: Obviously $p_i^\alpha(m^\alpha, m^{-\alpha})$ is continuous in $m \equiv (m^\alpha, m^{-\alpha})$ and (by Step 2,) concave in m^α. Moreover U^α is continuous, concave and monotonic in all its variables by assumption AII. It follows that $\Pi^\alpha(m) = U^\alpha(p^\alpha(m))$ is continuous in m and concave in m^α. The existence of NE now follows from the standard Nash argument [3].

Step 4: Let $m(\epsilon)$ be an NE of Γ^ϵ and select a subsequence $\epsilon_n \to 0$ so that $m(\epsilon_n) \to m$ as $n \to \infty$. Then m is an NE of $\mathcal{T}$.

Proof of Step 4: We need only verify that m is a point of continuity of the payoff functions. To this end, it suffices to show that if there exists $\tilde{\alpha} \in \mathcal{T}$ and $j \in \mathcal{I}$ such that $\gamma_j^{\tilde{\alpha}}$ is not continuous at zero, then $m_j \neq 0$. Suppose, to the contrary, $m_j = 0$ i.e., $m^\beta(\epsilon_n) \to 0$ for all $\beta \in \mathcal{T}$. By assumption AV, there exist distinct firms α, α' such that AV(α, j) and AV(α', j) both hold and α, α' both value customer j. By going to a subsequence if necessary, assume $m_j^\alpha(\epsilon_n) \leq m_j^{\alpha'}(\epsilon_n)$ for all n. Choose $\delta_n \to 0$ such that $\delta_n / m^\alpha(\epsilon_n) \to \infty$ as $n \to \infty$, for all $\alpha \in \mathcal{T} \backslash \{\alpha\}$ (e.g. take $\delta_n = \max\{\sqrt{m^\alpha(\epsilon_n)} : \alpha \in F\backslash\{\alpha\}\}$). Let α spend δ_n more on j. The incremental cost of this to α is at most $C_+^\alpha \delta_n$ where C_+^α is the maximum of the right hand derivative of C^α evaluated at (see Step 1) the point $(b, b, \ldots, b)$. We will show that α's gain in benefit is strictly more for small enough δ_n. Let $p^\alpha(-), p^\alpha(+) \in R_+^{\mathcal{I}}$ be the probabilities achieved before and after α's unilateral deviation to the extra expenditure δ_n. As shown in Step 2, $p^\alpha(+) \geq p^\alpha(-)$ component-wise. But, since U^α is weakly-monotonic, the gain in benefit is at least $\kappa[p_j^\alpha(+) - p_j^\alpha(-)]$ where $\kappa = \min\{\partial U^\alpha(p^\alpha)/\partial p_j^\alpha : p^\alpha \in [0,1]^{\mathcal{I}}\} > 0$ with $\partial U^\alpha / \partial p_j^\alpha$ denoting the right-hand derivative of the concave function U^α. Now

$$
\begin{aligned}
p_j^\alpha(+) - p_j^\alpha(-) &= (1-\theta_j^\alpha)\gamma_j^\alpha(+) - \theta_j^\alpha F_j^\alpha(p^\alpha(+)) - (1-\theta_j^\alpha)\gamma_j^\alpha(-) + \theta_j^\alpha F_j^\alpha(p^\alpha(-)) \\
&\geq (1-\theta_j^\alpha)(\gamma_j^\alpha(+) - \gamma_j^\alpha(-) \\
&\geq (1-\theta_j^\alpha)\mu\delta_n
\end{aligned}
$$

where, by assumption AV, μ can be chosen arbitrarily large for small enough δ_n; in particular $\mu > C_+^\alpha / (\kappa(1-\theta_j^\alpha))$. But then the gain in benefit is at least $\kappa(1-\theta_j^\alpha)\mu\delta_n$ which exceeds $C_+^\alpha \delta_n$ for small enough δ_n. This shows that α can benefit from a unilateral deviation at $m(\epsilon_n)$, for small enough ϵ_n, contradiction that $m(\epsilon_n)$ is an NE of Γ^{ϵ_n}. We conclude that $m_j \neq 0$ as was to be shown. □

References

[1] Pedro Domingos and Matt Richardson. Mining the network value of customers. In *Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 57–66, San Francisco, USA, August 2001.

[2] Vasile I. Istratescu. *Fixed Point Theory, An Introduction.* D. Reidel, Holland, 1981.

[3] John Nash. Equilibrium points in n-person games. *PNAS*, 36:48–49, 1950.

[4] Matthew Richardson and Pedro Domingos. Mining knowledge-sharing sites for viral marketing. In *Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 61–70, Edmonton, Alberta, Canada, July 2002.