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Research Report

PERFORMANCE OF SUBOPTIMAL MAXIMUM LIKELIHOOD SEQUENCE ESTIMATORS

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ABSTRACT: A broad class of sub-optimal Maximum Likelihood Sequence Estimators is presented. This class includes receivers using a simplified metric to perform the estimation, as well as approximate noise spectrum and channel function. The case of equalizing a channel to a partial response channel and using a Viterbi algorithm for the equalized channel is typical; here the sub-optimality consists in the improper handling of the noise, treated as white by the Viterbi algorithm. Additional sub-optimality can be present due to imperfect equalization.

The performance of sub-optimal receivers belonging to this broad class is analyzed. Several examples are given as a guide to performance computation.

1. INTRODUCTION

This work describes some aspects of sub-optimal Maximum Likelihood Sequence Estimators (MLSE). The sub-optimality of the MLSE is present, for example, when the channel is equalized to a partial response (PR) channel and then a Viterbi detector is used for the PR channel. The sub-optimality resides, first, on the equalization which looses information for it sub-utilizes the channel, and second on the improper use of the Viterbi algorithm assuming that the noise is white, which almost certainly is not the case *after* equalization. Additional sub-optimality may be present due to the fact that equalization may not be perfect.

In this paper we will examine the performance of sub-optimal MLSEs by calculating the probability of the receiver to choose a sequence which is different than the transmitted one. This calculation is a good approximation to the performance in case the two sequences chosen are likely to be confused by the sub-optimal receiver and the frequency of the error sequence is a good percentage of the totality of error sequences allowed by the system.

2. FRAMEWORK

The data consist of finite sequences with elements taken from a finite alphabet. Furthermore, constraints are imposed on the data sequences due to codes. The set of all possible such sequences is imbedded into an Euclidean space A of proper dimension, which we will call the "Data Space." The allowed sequences, therefore, are located in a constraint set C contained in the Data Space. The channel maps Data Space points into points of an "Observation Space" Y, as follows:

$$y = H_0 a_0 + n$$

where $a_0 \in C \subset A$ is the transmitted data sequence, H_0 is a linear map from A into Y, $n \in Y$ is an additive noise component and $y \in Y$ is the observation. The Observation Space Y will also be modeled as a Euclidean space, but of possibly very high dimension. The actual observations may be continuous processes observed over a finite time, but here we will sample these processes at a very high rate so that the energy outside the corresponding Nyquist band can be neglected. The reason for this is to avoid certain technicalities concerning the noise.

The noise is a normally distributed non-degenerate random process in the sense that any finite collection of its samples has a normal density function of full rank. This implies an invertible covariance matrix. Let R_0 be the covariance matrix of the noise n, i.e., $R_0 = Enn'$ where here $n \in Y$ is a "column" vector which is matrix-multiplied by n′, its transpose, hence a row vector, and the symbol E indicates stochastic expectation. Notice that R_0 is an operator mapping Y into Y.

Under the above conditions, it was shown in Ref. [1] that the MLSE for the input data sequence can be obtained from the observation y by the following operations:

1. Calculate the statistic $\gamma = H_0^+ y$, where $H_0^+ = (H_0' R_0^{-1} H_0)^{-1} H_0' R_0^{-1}$ is the "Zero Forcing Equalizer" followed by a sampler at clock rate, i.e., H_0^+ is the so-called Pseudo-Inverse of H_0. As usual, the prime, as in H_0', indicates the transpose of H_0.
2. Pick the point in $C \subset A$ which is closest to γ under the metric induced by the quadratic form $(., H_0' R_0^{-1} H_0 .)$, where (.,.) is the Euclidean dot product in A.

The above operations correspond to the *optimal* receiver. The invertibility and boundedness (stationary case) of $(H_0' R_0^{-1} H_0)$ were also considered in Refs. [1] and [3]. For completeness, these issues are repeated here in Appendix A.

The above framework include, for example, the case of PAM systems like the Magnetic Recording channel. In such system, the observation at instant t is given by

$$y(t) = \sum_k b_k h(t - kT) + n(t) \tag{1}$$

where $b_k \in \{1,0,-1\}$ are the components of the data sequence, h(t) is the *step response* of the magnetic channel and n(t) is the noise component, assumed additive and normally distributed. This is the NRZI formulation of the magnetic channel. In an alternative formulation, the NRZ, the observations are given by an equation similar to (1) with the modification that $b_k \in \{.5, -.5\}$ and h(t) is then the *pulse* response of the system. For finite data both formulations are equivalent and it is possible to transform one into the other by a linear transformation, as will be exemplified latter.

In the case of Magnetic Recording, in order to achieve stable solutions in the sense that $(H_0' R_0^{-1} H_0)^{-1}$ remains bounded as the amount of data goes to infinity, the channel have to be formulated in terms of the NRZI sequences. The NRZI formulation has the draw back of putting the additional constraint in the input sequence of having to alternate the signs of its non-zero components. Although such constraint neither increases the complexity of the receiver nor is difficult to incorporate in a trellis to compute the distances, the addition of a mapping from the NRZI-Data Space to a NRZ-Data Space can be incorporated *at the end* of the computation allowing the calculation to be performed in terms of NRZ data as will be exemplified later. The removal of zeros from any channel function can be done in general by a similar process.

The MLSE solution is independent of its implementation. Therefore, the performance can be computed using the above framework, regardless of the implementation. We proceed in formulating a fairly general sub-optimal receiver and computing its performance.

3. SUB-OPTIMAL RECEIVERS AND THEIR PERFORMANCE

The receiver derives the statistic γ as in (i) above, but assuming that the channel map is H (when it is, actually, H_0) and that the noise covariance is R (rather than R_0). Then, the receiver maps the data space A into a second data space B through the linear map L which is not necessarily invertible or stable (bounded in the limit as the amount of data goes to infinity). In the new data space B it uses a metric M to compute the distance from Ly to points in LC, the image of the constraint set C⊂A. The map L permit, for the example of the magnetic recording channel, to process the NRZ rather than the NRZI data, if so is desired. It also can be identified with maps that are usually employed to "simplify" the metric, such as a "whitening" processing. The arbitrary metric M permits the receiver to use simplified algorithms to reduce the size of the related trellis in the computation of distances. Figure 1 illustrates the above assumptions.

This formulation describes a broad class of receivers, including the sub-optimal receivers analyzed in Refs. [1] and [2]. In Section 4 we will give several examples of commonly used sub-optimal receivers fitting into this formulation.

We proceed in computing the probability that the above detector chooses a sequence a_1 when the actual transmitted sequence is a_0. Such error event will occur in case the statistic Ly falls closer to La_1 than to La_0 in the metric using M as kernel, that is, its probability is given by (see Fig. 2):

$$Prob\{error\} = Prob\{(L(y - a_0), ML(a_1 - a_0)) > .5(L(a_1 - a_0), ML(a_1 - a_0))\} .$$

Let's denote $e = a_1 - a_0$. Noting that $y = H^+y = H^+(H_0a_0 + n)$, it follows that

$$y - a_0 = H^+(H_0 - H)a_0 + H^+n$$

In the above expression, the left inverse property of the pseudo-inverse, i.e., $H^+Ha_0 = a_0$ was used (see expression for H^+). Hence, the error event becomes:

$$\{(LH^+n, MLe) > .5(Le, MLe) - (LH^+(H_0 - H)a_0, MLe)\} .$$

It remains to calculate the variance of the scalar (LH^+n, MLe):

$$\overline{(LH^+n,MLe)^2} = \overline{(H^{+'}L'MLe,n)(n,H^{+'}L'MLe)} = (H^{+'}L'MLe,\ R_0H^{+'}L'MLe)$$

where R_0 is the true covariance of the noise and the primes denote transposes, as stated before. Using the expression $H^+ = (H'R^{-1}H)^{-1}H'R^{-1}$, the final probability of error can be written down:

$$Pr\{error\} = Q\left\{ \frac{\frac{1}{2}(e,\ L'MLe) - \left((H'R^{-1}H)^{-1}H'R^{-1}(H_0 - H)a_0,\ L'MLe\right)}{\sqrt{(e,\ L'ML(H'R^{-1}H)^{-1}H'R^{-1}R_0R^{-1}H(H'R^{-1}H)^{-1}L'MLe)}} \right\} \tag{2}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left[-\frac{\zeta^2}{2} \right] d\zeta\ .$$

Observe that the probability of error in not only a function of the error sequence, but also depends on the actual sequence a_0 transmitted. This is the same feature observed in the performance of the suboptimal receivers considered in Ref. [2].

Expression (2) seems formidable, but we will see that it simplifies in most applications and its usefulness will be demonstrated by the following examples.

4. EXAMPLES

4.1 A Simplification

Let us first simplify expression (2) for the case when L is invertible. In this case, for the following examples, we will select the metric M so that $L'ML = H'R^{-1}H$, that is,

$$M = L'^{-1}H'R^{-1}HL^{-1}\,. \tag{3}$$

The reason for this is not evident at this point, but such selection leads to the optimal receiver, as will be seen. Denote $HL^{-1} = \hat{H}$. $\hat{H}$ is the channel function in case the input sequences are elements of the second data space B. For consistence, call $La_0 = \hat{a}_0$ and $Le = \hat{e}$, the sequence and the error sequence in the B data space. Under these assumptions, (2) simplifies to:

$$Pr\{error\} = Q\left\{ \frac{\frac{1}{2}\left(\hat{e}, \hat{H}'R^{-1}\hat{H}\hat{e}\right) - \left(\hat{H}'R^{-1}\left(\hat{H}_0 - \hat{H}\right)\hat{a}_0, \hat{e}\right)}{\sqrt{\left(\hat{e}, \hat{H}'R^{-1}R_0R^{-1}\hat{H}\hat{e}\right)}} \right\}. \tag{4}$$

4.2 An Optimal NRZI Channel

We start with some optimal receivers in order to become familiar with the approach taken in this paper and to clarify some possible misconceptions on the use of equalization to simplify the metric (Section 4.4). The system considered here is the PAM system described by Equation (1). Consider a minimum bandwidth channel (channel function within the Nyquist bandwidth corresponding to the sampling rate; non-minimum bandwidth channels are discussed in Appendix B) having channel function

$$h_0(D) = 1 + .9D + .2D^2\,.$$

Here, in addition to the regular matrix representation for the channel, we also use the channel representation in terms of the D-transform. This representation is equivalent to the matrix representation in case of stationary minimum bandwidth channels. Stationarity is invoked to avoid end effects in finite sequences. The noise is white, with in band variance σ_0^2. The inputs are NRZI sequences, i.e., the elements are taken from the set {1,0,-1} with the constraint that the non-zero entries have to alternate signs.

The receiver is an optimal MLSE, i.e., it assumes $H = H_0$, $R = R_0 = \sigma_0^2 I$, $L = I$ and the metric M as in (3). Therefore, (4) reduces further to:

$$Pr\{error\} = Q\left\{ \frac{1}{2\sigma_0} \sqrt{(e, H_0' H_0 e)} \right\} = Q\left\{ \frac{1}{2\sigma_0} \|H_0 e\| \right\}$$

The worst error sequence occurs when a "1" is shifted to an adjacent location, i.e., $e = ...0\ 1\ \text{-}1\ 0...$ (error sequences here have elements from the set {2,1,0,-1,-2}). Then, $\|H_0 e\|$ can be computed as follows:

H_0:	1	.9	.2	
$-H_0 D$:		-1	-.9	-.2
$H_0 e$ "=" $h_0(D)(1-D)$:	1	-.1	-.7	-.2

As a result, $\|H_0 e\|^2 = (1)^2 + (-.1)^2 + (-.7)^2 + (-.2)^2 = 1.54$ yielding:

$$Pr\{error\} = Q\{\sqrt{1.54}/2\sigma_0\} = Q\{1.241/2\sigma_0\}.$$

4.3 An optimal NRZ Channel

Here we recompute the performance of the above channel, but using the NRZ description. For that purpose, the transformation that brings the NRZI sequences into the NRZ is $L = 1/1 - D$. Consequently, the new channel function, as described in 4.1, is:

$$\hat{H}_0 \text{ "=" } (1 - D)h_0(D) = 1 - .1D - .7D^2 - .2D^3 .$$

The error sequences, in this case, have entries from the set {1,0,-1}. The worst case is $\hat{e} = ...0\ 0\ 0\ 1\ 0\ 0\ 0...$, i.e., a single error, yielding exactly the same performance as calculated in Section 4.2.

The great advantage of using the NRZ description is that, in absence of code constraints, the sequences, hence the error sequences are unconstrained. If codes are used, then the error sequences have to be compatible with the code constraints.

4.4 Optimal Channel With Perfect Equalization

An equalizer is used with the channel described in 4.2 to change its function to $1 + D$, a PR-1 function, i.e., the equalizer transfer function is

$$f(D) = (1 + D)/(1 + .9D + .2D^2) .$$

At the output of the equalizer, the noise has covariance

$$R_0(D) = \sigma_0^2 f(D)f(D^{-1}) .$$

The optimal detector for the equalized channel is the one that uses $h_0(D) = 1 + D$ and $R_0(D)$ as above. The kernel for its metric is, then,

$$M(D) = h_0(D^{-1})R_0^{-1}(D)h_0(D)$$

$$= \sigma_0^{-2}(1 + D)(1 + .9D + .2D^2)(1 + .9D^{-1} + .2D^{-2})(1 + D^{-1})/(1 + D)(1 + D^{-1})$$

$$= \sigma_0^{-2}(1 + .9D + .2D^2)(1 + .9D^{-1} + .2D^{-2})$$

which is *exactly* the same as before equalization! In spite of the reduction of interfering bits, there is no savings in the number of states in the trellis! The performance of this receiver follows from (4):

$$Pr\{error\} = Q\{.5(e,Me)/\sqrt{(e,Me)}\} = Q\{1.241/2\sigma_0\} .$$

The above estimate uses the worst case error sequence, $e = ...0\ 0\ 1\ \text{-}1\ 0\ 0...$

4.5 Sub-Optimal Channel with Perfect Equalization

We now consider the above equalized channel, but we use a standard Viterbi detector, which assumes white noise at the output of the channel. Since the noise is assumed white, the metric in this case is $M(D) = \sigma^{-2}(1 + D)(1 + D^{-1})$, hence the number of trellis states is reduced from 4 to 2. The value of σ^2 is the value of the noise variance at the output of the equalizer ($\sigma^2 = 1.19\sigma_0^2$. This value, however, will not be needed). The Viterbi algorithm assumes $h(D) = h_0(D) = 1 + D$ and $R(D) = \sigma^2 I$. Hence, the performance can be computed from (4):

$$Pr\{error\} = Q\{.5(e,Me)/\sqrt{(e, H'R^{-1}R_0R^{-1}He)}\} .$$

Let's compute Pr{error} for the same error event, $e(D) = 1 - D$. The numerator of the argument of the Q function is easily computed as the coefficient of D^0 in the expression

$$e(D)e(D^{-1})h_0(D)h_0(D^{-1})/\sigma^2 = \frac{1}{\sigma^2}(1 + D)(1 + D^{-1})(1 - D)(1 - D^{-1})$$

or, equivalently, as the sum of the squares of the coefficients of the expansion

$$(1 + D)(1 - D)/\sigma^2 = \frac{1}{\sigma^2}(1 - D^2)$$

i.e.,

$$(e,Me) = 2/\sigma^2 .$$

This quadratic is the standard (distance/sigma)^2 traditionally used in the performance evaluation of the Viterbi algorithm. It only holds for the *optimal* receiver, i.e., when the channel is actually $1 + D$ and the noise is actually white.

The quadratic in the denominator is calculated here as the coefficient of D^0 in the power series expansion of

$$e(D)h(D)R_0(D)h(D^{-1})e(D^{-1})/\sigma^4$$

$$= \frac{\sigma_0^2}{\sigma^4}(1 - D)(1 + D)^2(1 + D^{-1})^2(1 - D^{-1})/(1 + .9D + .2D^2)(1 + .9D^{-1} + .2D^{-2})$$

yielding

$$(e,H'R^{-1}R_0R^{-1}He) = 2.75\frac{\sigma_0^2}{\sigma^4} .$$

For the above evaluation the IBM symbolic language SCRATCHPAD was used. The computation can also be done in the frequency domain by putting $D = e^{-j\omega T}$, carrying out the operations and integrating. This technique will be used subsequently (see Appendix C for details).

As a result,

$$Pr\{error\} = Q\{2/2\sigma_0\sqrt{2.75}\} = Q\{1.21/2\sigma_0\}\,.$$

Note: It must be emphasized here that the choice of the error sequence e was arbitrary and, due to the complex argument of the Q function in (4), there is no systematic approach (other than exhaustive search) to find the worst error event. The same observation also apply to the following example.

4.6 Sub-Optimal Receiver with Misequalized Channel

In this example, an attempt to execute the sub-optimal receiver described in 4.5 is made. However, the equalizer's transfer function turns out to be:

$$g(D) = .1D^{-1} + 1 + D + .1D^2/1 + .9D + .2D^2$$

rather than f(D) as in Section 4.4. Hence, its output to a step is

$$h_0(D) = .1D^{-1} + 1 + D + .1D^2\,.$$

The noise after equalization is, then,

$$R_0(D) = \sigma_0^2 g(D) g(D^{-1})\,.$$

A receiver that assumes R_0 and $h_0(D)$ will be optimal and perform as in 4.2 or 4.4. Here, the receiver proceeds as in 4.5 assuming

$$h(D) = 1 + D$$

$$R(D) = \sigma^2 I = 1.19\sigma_0^2 I .$$

The performance of such sub-optimal receiver is given by (4):

$$Pr\{error\} = Q\left\{ \frac{\frac{1}{2}\left(e, H'R^{-1}He\right) - \left(H'R^{-1}(H_0 - H)a_0, e\right)}{\sqrt{\left(e, H'R^{-1}R_0R^{-1}He\right)}} \right\} .$$

We proceed computing the various dot products in the above expression. The quadratic $(e,H'R^{-1}He)$ was already computed, yielding, for $e(D) = 1 - D$,

$$(e,H'R^{-1}He) = 2/\sigma^2 .$$

The last term in the numerator is a dot product which depends on a_0 as well as on the error sequence e. First note that

$$(H'R^{-1}(H_0 - H)a_0,e) = (a_0,(H_0 - H)'R^{-1}He) .$$

Then compute:

$$h_0(D) - h(D) = (.1D^{-1} + 1 + D + .1D^2) - (1 + D) = .1D^{-1} + .1D^2$$

$$h(D)e(D) = (1 + D)(1 - D) = 1 - D^2 .$$

Hence,

$$(h_0(D^{-1}) - h(D^{-1}))h(D)e(D)/R(D) = (.1D + .1D^{-2})(1 - D^2)/\sigma^2$$

$$= (.1D^{-2} - .1 + .1D - .1D^3)/\sigma^2 .$$

Since we are looking for the worst situation, i.e., a sequence a_0 yielding the largest possible dot product with the sequence

... 0 0 0 .1 0 -.1 .1 0 -.1 0 0 0 ... ,

as described above, we choose

$$a_0 = \cdots \text{x x x 1 0 -1 1 0 -1 x x x} \ldots ,$$

a valid sequence if no further code constraints is imposed in the input sequences. The x's in the above sequence indicate don't care entries, since they are multiplied by zeroes. Hence, $a_0 = D^{-2} - 1 + D - D^3 +$ don't care terms, yielding

$$(a_0, (H_0 - H)' R^{-1} He) = .4/\sigma^2 .$$

The numerator of the argument of the Q-function is then,

$$(1 - .4)/\sigma^2 = .6/\sigma^2 .$$

The denominator is computed as in 4.5, with the new $R_0(D)$, yielding:

$$\sqrt{(e, H'R^{-1}R_0R^{-1}He)} = \frac{\sigma_0}{\sigma^2}\sqrt{1.79} .$$

The final result for the performance is, then,

$$Pr\{error\} = Q\{1.2/2\sigma_0\sqrt{1.79}\} = Q\{.897/2\sigma_0\} .$$

APPENDIX A

The Invertibility of $H'R^{-1}H \equiv M_0$

Let $a \neq 0$ be an arbitrary non-null sequence in A. M_0 is invertible if, and only if

$$(Ha, R^{-1}Ha) = (a, H'R^{-1}Ha) = (a, M_0 a) > 0 .$$

Here R is assumed positive definite, hence the above quadratic is always non-negative. Expanding the above quadratic (see also Appendices B and C),

$$(a, M_0 a) = \sum_k \sum_\ell a_k a_\ell m_{k-\ell} = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} M_0(\omega) \,|\, a_0 + a_1 z + \cdots + a_n z^n \,|^2 d\omega$$

where $z = e^{-j\omega T}$, $n+1$ is the dimension of A,

$$m_k = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} M_0(\omega) e^{j\omega k T} d\omega$$

$$M_0(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{\left| H\left(\omega - \frac{2\pi m}{T}\right) \right|^2}{R\left(\omega - \frac{2\pi m}{T}\right)} \geq 0 \qquad \omega \in \left(-\frac{\pi}{T}, \frac{\pi}{T}\right).$$

Here, H(ω) is the Fourier Transform of h(t) and R(ω) is the average power spectrum of the noise.

Then, a sufficient condition for $(a, M_0 a) \neq 0$ for arbitrary $a \neq 0$ is $M_0(\omega) \neq 0$ in some set S of non-zero measure inside $\left(-\frac{\pi}{T}, \frac{\pi}{T}\right)$. This is because

$$\sum_{l=0}^{n} a_l z^l$$

is a polynomial in z, hence can have at most n roots in S, i.e., cannot vanish identically there.

Although this condition is sufficient for the existence of the inverse of M_0, this inverse may become "ill conditioned" when the dimension of A is large, i.e., $n \to \infty$ may cause $(H'R^{-1}H)^{-1}$ to become "unbounded." A necessary and sufficient condition for a bounded inverse is that $1/M_0(\omega)$ be integrable, $\omega \in \left(-\frac{\pi}{T}, \frac{\pi}{T}\right)$. To see the sufficiency, note that

$$\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{d\omega}{M_0(\omega)} < \infty$$

imply that $M_0(\omega)$ can be zero only in a set of measure zero, hence, the previous sufficient condition for the existence of the inverse for finite sequences is satisfied. For any n, consider the problem

$$\min_{a \in A}(Ha, R^{-1}Ha) \quad \text{subject to } a_0 \triangleq (a, e_0) = 1$$

where e_0 is a sequence of all zeros except for a "1" at position "0". The uniformly boundedness of M_0^{-1} is equivalent to the fact that the above minimum is uniformly bounded away from zero as $n \to \infty$. The minimum value of $(Ha, R^{-1}Ha)$ is obviously decreasing with n.

Since we already established the existence of M_0^{-1} for every n, the above minimization yields

$$\min_{a \in A}(Ha, R^{-1}Ha) = \left(e_0, M_0^{-1}e_0\right)^{-1}.$$

Hence, $\left(e_0, M_0^{-1}e_0\right)$ increases with n. It's limit, as $n \to \infty$, is

$$\lim_{n \to \infty}\left(e_0, M_0^{-1}e_0\right) = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{d\omega}{M_0(\omega)} < \infty .$$

This is so because the Fourier transform of the rows of M_0^{-1}, as $n \to \infty$, converges to $1/M_0(\omega)$ as can be easily verified.

The integrability of $1/M_0(\omega)$ is also necessary, since the existence of a bounded M_0^{-1} implies that

$$\frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{d\omega}{M_0(\omega)} = \left(e_0, M_0^{-1}e_0\right) < \infty.$$

APPENDIX B

Non-Minimum Bandwidth Systems

The expressions for the computation of the probability of error involve inner products of sequences in the data space Λ. They can be readily computed in the frequency domain (see Appendix C) from the polynomial representation of the sequences and Toeplitz matrices involved in the inner products. However several maps from Λ to Λ appearing in the expressions are resultant of mapping first Λ into Y then Y back into Λ. The maps from Y to Λ can be identified by the transpose operation. We now will show how the entries of these total maps from Λ to Λ, matrices, can be expressed in terms of the Fourier Transforms of the channel function and noise spectrum in the stationary, time invariant case, and then as D-polynomials. This is accomplished by the use of the Poisson formula (the transpose operations involve sampling). In the examples of section 4 this was not necessary since the systems were assumed to be of minimum bandwidth, that is, the spectra involved where confined within the Nyquist band and hence the channel function could be identified with its samples at clock rate.

We proceed by giving, as an example, the expression for the entries of the matrix $H'R^{-1}H$, as well as the Poisson formula, allowing us to represent the operations involved in terms of D-polynomials as in the case of minimum bandwidth systems. Other matrices are treated in a similar way.

Let a and $b \in A$, H mapping Λ into Y as in (1) and R mapping Y into Y. We will compute $(Hb, R^{-1}Ha) = (b, H'R^{-1}Ha)$, the first inner product being taken with time functions in Y and the second with sequences in Λ.

Let's compute first $g = R^{-1}Ha$:

$$g(t) = (R^{-1}Ha)(t) = \sum_k a_k \int_{-\infty}^{\infty} h(\zeta - kT)R^{-1}(t-\zeta)d\zeta\ .$$

Now, let's calculate the inner product of g(t) with $Hb = \sum_{k'} b_{k'}h(t - k'T)$ and using the transposition to calculate $(b, H'R^{-1}Ha)$:

$$(Hb,g) \triangleq \int g(t)\sum_{k'} b_{k'}h(t-k'T)dt = \sum_{k,k'} b_{k'}a_k \iint_{-\infty}^{\infty} h(t-k'T)R^{-1}(t-\zeta)h(\zeta - kT)dtd\zeta\ .$$

Hence, the Toeplitz matrix $M = H'R^{-1}H$ has entries

$$m_{k,k'} = \iint_{-\infty}^{\infty} h(t-k'T)R^{-1}(t-\zeta)h(\zeta-kT)dtd\zeta\ .$$

The integral above can be computed in the frequency domain, yielding:

$$m_{k,k'} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 R^{-1}(\omega)e^{-j\omega T(k'-k)}d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\sum_{\ell=-\infty}^{\infty} \frac{\left|H\left(\omega - \frac{2\pi\ell}{T}\right)\right|^2}{R\left(\omega - \frac{2\pi\ell}{T}\right)} \right] e^{-j\omega T(k'-k)}d\omega\ .$$

Therefore, the operator $H'R^{-1}H = M$ can be represented as a D-polynomial using Poisson's formula:

$$M(\omega) = \sum_k m_k e^{-j\omega kT} = \frac{1}{T} \sum_{\ell=-\infty}^{\infty} \frac{\left| H\left(\omega - \frac{2\pi\ell}{T}\right)\right|^2}{R\left(\omega - \frac{2\pi\ell}{T}\right)}$$

and

$$M(D) = \sum_k m_k D^k .$$

Poisson's formula can be easily verified by multiplying both sides of the above expression by $e^{j\omega k'T}$ and integrating from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

APPENDIX C

Computing Inner Products in the Frequency Domain

The computation of inner products of sequences using their frequency representation was exemplified in Appendix A. Here we will describe the computation in more detail.

Let a, b, and c be sequences in A and let M be a Toeplitz matrix mapping A into itself. Let the kk' entry of M be $m_{k-k'}$. Define $\Lambda(\omega)$ as

$$A(\omega) = \sum_k a_k e^{-j\omega kT}.$$

Notice that $\Lambda(\omega)$ is obtained from the D-Polynomial representation of the sequence a by putting $D = e^{-j\omega T}$. The components of the original sequence a can be obtained from $A(\omega)$ by the Inverse Fourier Transform as follows:

$$a_k = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} A(\omega)e^{j\omega kT}d\omega .$$

Let now $b = Ma$. Its transform $B(\omega)$ is $M(\omega)A(\omega)$ where $M(\omega)$ is defined in a similar way as $\Lambda(\omega)$ taking into account the Toeplitz nature of M. Indeed,

$$\frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} M(\omega)A(\omega)e^{j\omega kT}d\omega = \frac{T}{2\pi}\sum_{k',\,k''} a_{k'}m_{k''}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} d\omega e^{j\omega T(k-k'-k'')}$$

$$= \sum_{k'} m_{k-k'}a_{k'} = b_k$$

Similarly, (b,Ma) can be computed as follows:

$$(b,Ma) = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} B^*(\omega)M(\omega)A(\omega)d\omega = \sum_{k,\,k'} a_k m_{k-k'} b_{k'} .$$

As an example,

$$(b,H'R^{-1}Ha) = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} d\omega \left[\frac{1}{T}\sum_{\ell=-\infty}^{\infty} \frac{\left| H\left(\omega - \ell\frac{2\pi}{T}\right)\right|^2}{R\left(\omega - \ell\frac{2\pi}{T}\right)} \right] \cdot \left[\sum_k a_k e^{-j\omega kT}\right] \cdot \left[\sum_{k'} b_{k'} e^{j\omega k'T}\right] .$$

REFERENCE

1. L. C. Barbosa, "Maximum Likelihood Sequence Estimators: A Geometric View," to appear, IEEE Trans on Information Theory. Also available as IBM Research Report RJ5273 (54313) 8/20/86.
2. J. B. Anderson, T. Aulin, and C. E. Sundberg, Chapters 3 and 8, Plenum Press, New York and London, 1986.
3. D. Messerschmitt, "A Geometric Theory of Intersymbol Interference," The Bell System Technical Journal, 52(9), November 1973.

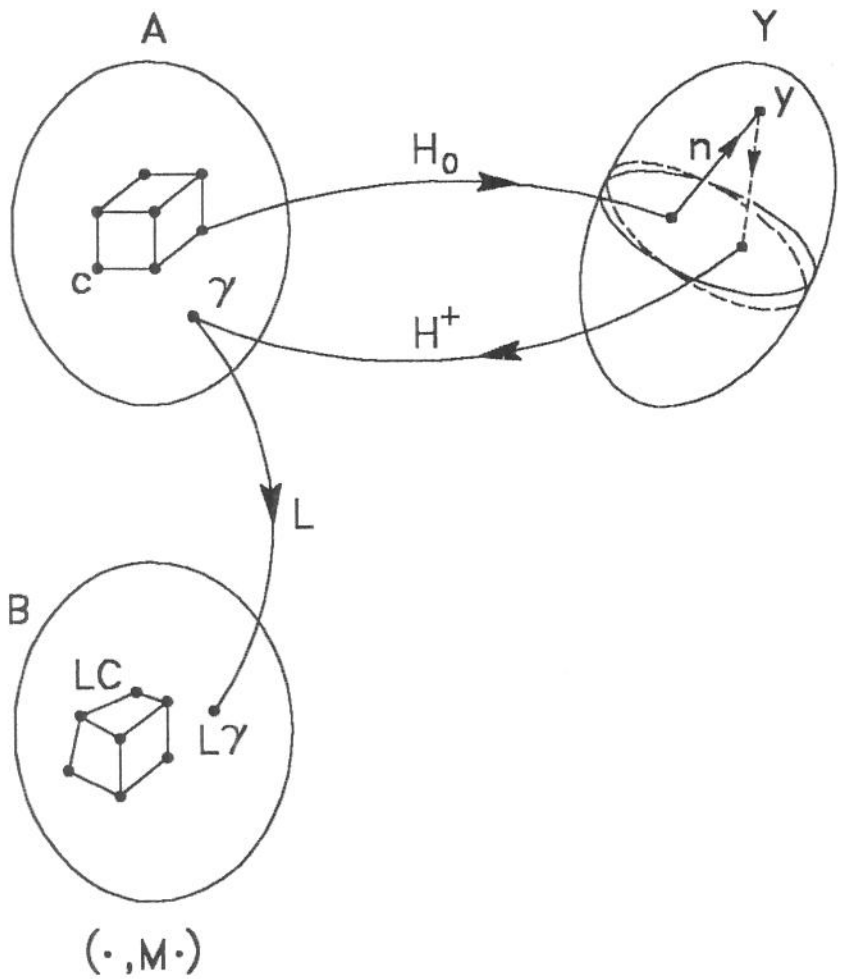

Figure 1. The maps and metric of the sub-optimal receiver: H^+, L and M.

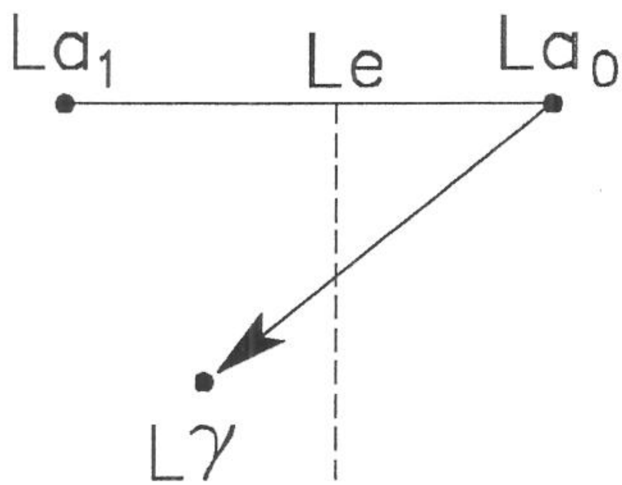

Figure 2. Condition for error occurrence: a_0 is sent and $L\gamma$ falls closer to La_1 than to La_0.