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Research Report

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Performance of Interleaved Block Codes with Burst Errors

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A Gilbert-Elliott channel for symbol errors is considered to analyze the performance of interleaved error correction codes with fixed block size in the presence of burst errors. The proposed channel model for symbol errors is based on a simplified Gilbert channel for bit errors, which enables direct comparisons of the performance of block codes with different symbol sizes. The autocorrelation function between two erroneous symbols within an interleaved codeword is computed. An exact expression for the codeword-error probability is derived from the symbol-based channel model. A tight lower bound on the codeword-error probability and error floors, which are observed when the average raw bit-error rate is low, are analyzed using the bit-based channel model.

***Index Terms*—Error correction code, burst error, Markov chain, Gilbert-Elliott channel, error rate performance, error floor.**

I. Introduction

FINITE state channels that are described by a probabilistic or a deterministic function of a Markov chain are natural models for many communication and storage channels with memory in which the channel condition, characterized by a state variable, changes with time [1]–[5]. If the input symbol of the channel at time k is x_k and the output symbol of the channel at time k is y_k, the error at time k can be defined as $e_k = 0$ if $x_k = y_k$ and $e_k = 1$ if $x_k \neq y_k$. Thereby, a channel input and output symbol is usually an s-bit symbol, e.g., a bit if $s = 1$, or a byte if $s = 8$. Two-state Markov chains with a good state and a bad or burst state, proposed by Gilbert and Elliott [1], [2], have been widely used to model channels with memory that exhibit burst errors.

To evaluate the error rate performance of interleaved error correction codes (ECC) on channels with burst errors, it is necessary to express the error weight distribution at the output of the finite state channel corresponding to the probability of having m symbol errors, $0 \leq m \leq N$, in a symbol-interleaved codeword of length N as a function of channel and code parameters. The post decoding word error probability of interleaved block codes with burst errors have been analyzed for fixed symbol size and raw symbol-error rate [6]–[9].

In this paper, the error rate performance is evaluated using a Gilbert-Elliott channel for symbol errors that is based on a simplified Gilbert channel for bit errors. The proposed approach enables direct comparisons of the performance of interleaved block codes with different symbol sizes. Furthermore, it allows to evaluate the codeword-error probability as a function of the average raw bit-error rate. Post-decoding error rate curves are computed for a two-state burst error channel with a fixed average burst length. The approach to performance analysis that is used in this paper does not make use of the partial fraction technique proposed in [9], but is an extension of the method in [8]. It was noted in [10] that the approach of [8] is particularly suitable if the number of channel states is small.

In the presence of additive white Gaussian noise, the error rate performance of conventional coding schemes (e.g. Reed-Solomon (RS) codes in conjunction with algebraic decoding or convolutional codes in conjunction with maximum-likelihood decoding) exhibits only a waterfall region. However, for iterative decoding of low-density parity-check (LDPC) codes and turbo codes, the waterfall region is followed by an error floor region in which the error rate performance flattens. Error floors of iterative decoders have been extensively studied [11], but error floors of Gilbert-Elliott channels have not been analyzed. Error floors of interleaved block codes on Gilbert -Elliott channels were first observed for a simplified Gilbert channel characterizing bit errors [12]. In this paper, error floors of Gilbert-Elliott channels are analyzed and a closed-form expression for the error rate performance in the error floor region is derived.

The paper is organized as follows. In Section II, a two-state simplified Gilbert burst channel model for bit errors is described from which a two-state Gilbert-Elliott channel model for s-bit symbol errors is derived. Furthermore, the autocorrelation function between two erroneous symbols at the output of the Gilbert-Elliott channel is given. In Section III, the error weight distribution and the codeword-error probability for an N-symbol codeword, which is symbol-interleaved to depth I and has ECC capability of t symbols, is computed. In Section IV, a tight lower bound on the codeword-error probability is computed by considering all possible single burst errors within a codeword that exceed the ECC capability of the code. Moreover, it is shown that in the high signal-to-noise ratio (SNR) regime of low average raw bit-error rates the codeword-error probability is proportional to the raw bit-error rate. A closed-form expression for the error rate performance in the error floor regime is derived. Finally, Section V contains a brief summary and conclusions.

II. Channel Model

The underlying channel model for evaluating the error rate performance of block codes in the presence of burst errors is shown in Fig. 1. The simplified Gilbert channel model for bit errors is characterized by the state transition probability matrix

$$\boldsymbol{C} = \begin{bmatrix} b & 1-b \\ 1-a & a \end{bmatrix}, \tag{1}$$

where b is the probability of staying in the good state G (state 1) and a is the probability of staying in the bad state B (state

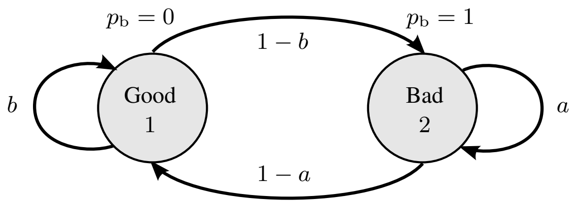

Fig. 1. Bit-based channel model.

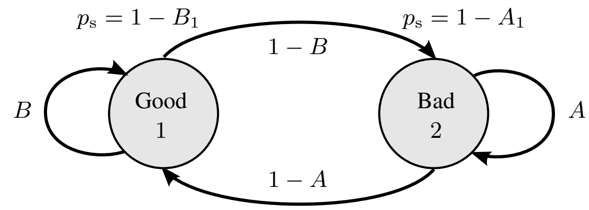

Fig. 2. Symbol-based channel model.

2). The bit-error probability at state G is 0 whereas the bit-error probability at state B is 1. The average raw bit-error rate (BER) is given by

$$p_\mathrm{b} = \frac{1-b}{(1-a)+(1-b)}. \qquad (2)$$

The stationary probability row vector $\boldsymbol{\pi} = [\pi_\mathrm{G}, \pi_\mathrm{B}]$ is defined by $\boldsymbol{\pi C} = \boldsymbol{\pi}$, where $\pi_\mathrm{G} = 1 - p_\mathrm{b}$ is the stationary probability of being at state G and $\pi_\mathrm{B} = p_\mathrm{b}$ is the stationary probability of being at state B. In a simplified Gilbert model, the distribution of occupancy times for both states G and B is geometric with means $(1-b)^{-1}$ and $(1-a)^{-1}$, respectively. In the following, a is a fixed constant for a given channel with burst errors and b changes as a function of a and p_b. Therefore, the simplified Gilbert channel can be characterized either by a and b or by a and p_b where from (2)

$$b = 1 - (1-a)\frac{p_\mathrm{b}}{1-p_\mathrm{b}} = 1 - (1-a)\sum_{i=1}^{\infty} p_\mathrm{b}^i. \qquad (3)$$

From (3), b can be tightly upper bounded by $1-(1-a)p_\mathrm{b}$ in the regime of high SNR as $p_\mathrm{b} \to 0$.

In magnetic tape storage, burst errors at the input of the ECC decoder are caused by various mechanisms that include magnetic media imperfections, head-to-media distance variations, error events at the output of the maximum-likelihood sequence detector for the equalized partial-response channel and error propagation in the modulation decoder. Measurement of the average raw BER p_b and the average length of error bursts $(1-a)^{-1}$ at the input of the ECC decoder is sufficient to characterize the simplified Gilbert channel with memory. In our channel model, the average error burst length is fixed whereas the average length of error-free intervals $(1-b)^{-1} = (1-a)^{-1}(1-p_\mathrm{b})/p_\mathrm{b}$ increases as the average raw BER decreases with increasing SNR. The simplified Gilbert channel model considered in this paper is a special case of the Gilbert channel model [1] and the Fritchman channel model with one error state [13], which are examples of renewal channels as the error-free intervals in both types of channel are independent from each other and identically distributed.

Figure 2 depicts the Gilbert-Elliott channel model for s-bit symbol errors that is obtained from the simplified Gilbert channel model in Fig. 1 by starting at a state and making s transitions, i.e., each transition in Fig. 2 corresponds to a set of s-bit error patterns. A symbol error $e_k = 1$ at time k occurs only if the s-bit error pattern is not the all-zero pattern. The Markov chain characterizing symbol errors in Fig. 2 is specified by the s-step state transition probability matrix

$$\boldsymbol{T} = \boldsymbol{C}^s = \begin{bmatrix} B & 1-B \\ 1-A & A \end{bmatrix}. \qquad (4)$$

The symbol-error probability at state G is $p_\mathrm{s} = (1 - B_1)$ where $B_1 = b^s$. The symbol-error probability at state B is $p_\mathrm{s} = (1 - A_1)$ where $A_1 = (1-a)b^{s-1}$. Clearly, the Gilbert-Elliott channel in Fig. 2 describing symbol errors is a non-renewal channel and the average symbol-error probability is

$$p_\mathrm{s} = (1-p_\mathrm{b})(1-B_1) + p_\mathrm{b}(1-A_1). \qquad (5)$$

From the correlation function between erroneous symbols at the output of a Gilbert-Elliott channel [14] (see Eq. (4.54)), the autocorrelation $R(k)$ between two symbols spaced k symbols apart in a codeword interleaved to depth I is computed to be

$$R(k) = \frac{(1-A_1-p_\mathrm{s})(p_\mathrm{s}-1+B_1)(A+B-1)^{Ik}}{p_\mathrm{s}(1-p_\mathrm{s})}. \qquad (6)$$

For $a = 15/16$, $s = 8$, an average error burst length of 16 bit and symbol interleaving depths $I = 1$ and $I = 4$, Fig. 3 illustrates the autocorrelation of adjacent symbols in a codeword $R(1)$ vs. raw BER. $R(1)$ for $I = 4$ is less than 0.1, which is about four times less than $R(1)$ for $I = 1$.

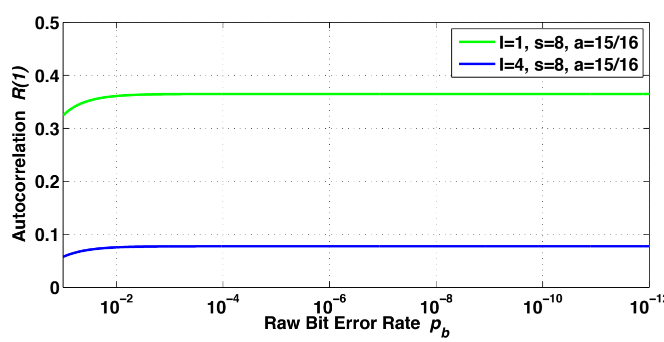

Fig. 3. Autocorrelation of adjacent symbols in a codeword vs. raw BER for interleaving depths $I = 1$ and $I = 4$.

III. Error Rate Performance

In this section, the error rate performance of block codes in the presence of burst errors is evaluated based on the underlying symbol-based channel model in Fig. 2, which was derived from the bit-based channel model in Fig. 1. In the following, we assume that the error correction code has N s-bit symbols and an error correction capability of t symbols. Furthermore, it is assumed that the symbol interleaving depth is I and the symbol errors in the interleaved codewords are described by the Gilbert-Elliott channel in Fig. 2 introduced in Section II.

The codeword-error probability p_C can be computed by

$$p_\mathrm{C} = \sum_{m=t+1}^{N} P(m,N) = 1 - \sum_{m=0}^{t} P(m,N), \qquad (7)$$

where $P(m,N)$ is the probability of having m erroneous symbols in an N-symbol codeword.

Following a similar approach as in [8], $P(m,N)$ can be expressed as the coefficient of a polynomial. Specifically,

$$P(m,N) = \left\langle \boldsymbol{\pi} \left(\boldsymbol{T}^{I-1} \boldsymbol{E}(x) \right)^{N} \mathbf{1} \right\rangle_m, \qquad (8)$$

where the m-th coefficient c_m of a polynomial $f(x) = \sum_{m=0}^{n} c_m x^m$ is denoted by $c_m = \left\langle f(x) \right\rangle_m$, $\boldsymbol{\pi} = [1-p_\mathrm{b}, p_\mathrm{b}]$ is the stationary probability row vector associated with the Markov chain in Fig. 2, $\mathbf{1}$ is a 2×1 column vector of ones, x is an indeterminate, and $\boldsymbol{E}(x)$ is a 2×2 error matrix that labels each state transition for counting purposes by the probability of making no symbol error plus the probability of making a symbol error multiplied by the counting variable x.

For the Gilbert-Elliott channel in Section II, the error matrix is given by

$$\boldsymbol{E}(x) = \begin{bmatrix} B_1 + (B - B_1)\,x & (1-B)\,x \\ A_1 + (1 - A - A_1)\,x & Ax \end{bmatrix}. \qquad (9)$$

It can readily be shown that $P(m,N)$ in (8) can be expressed more generally as

$$P(m,N) = \left\langle \boldsymbol{\pi} \left(\boldsymbol{T}^{i} \boldsymbol{E}(x) \boldsymbol{T}^{j} \right)^{N} \mathbf{1} \right\rangle_m, \qquad (10)$$

for all pairs (i,j) such that $0 \le i \le I-1$ and $j = I - 1 - i$. Therefore, the computation of the codeword-error probability using the stationary Markov chain in Fig. 2 is independent of which one of the codewords in a block of I interleaved codewords is considered.

For a typical error correction code used in magnetic tape storage with $N = 240$, $s = 8$, $t = 6$ [15], and interleaving depths $I = 1$ and $I = 4$, Fig. 4 depicts the codeword-error probability p_C as a function of the raw BER p_b and $a = 15/16$, i.e., an average error burst length of 16 bit.

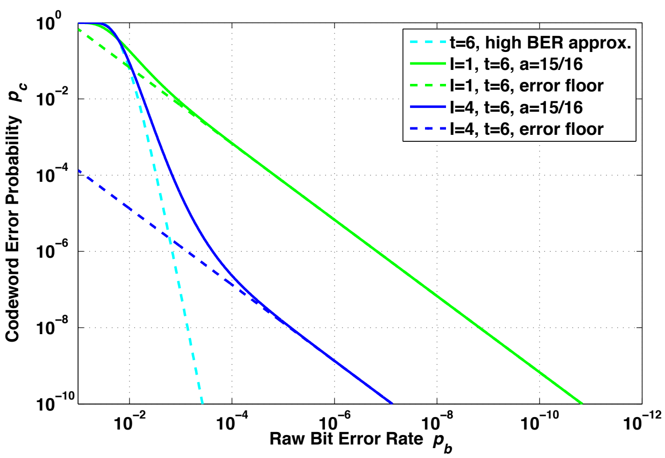

Fig. 4. Codeword error probability vs. raw BER for $N = 240$, $s = 8$, $t = 6$ and interleaving depths $I = 1$ and $I = 4$.

In the high SNR regime of low BER p_b, the error rate performance curve exhibits an error floor, which will be analyzed in the next section. The error floor for $I = 1$ is about four orders higher than for $I = 4$.

In the low SNR regime of high raw BER p_b, the codeword-error probability p_C can be approximated by assuming independent symbol errors with symbol-error probability given in (5). The validity of this assumption at high raw BER p_b was demonstrated by using measured data from magnetic tape [16]–[18]. The approximate codeword-error probability in the low SNR regime p_A is given by

$$p_\mathrm{A} = \sum_{m=t+1}^{N} \binom{N}{m} p_\mathrm{s}^{m} (1-p_\mathrm{s})^{N-m}, \qquad (11)$$

Figure 5 shows that the computed error distribution for $N = 240$, $t = 6$, $I = 4$, $s = 8$, $a = 15/16$ and $p_\mathrm{b} = 10^{-2}$ can be approximated by the binomial distribution for $N = 240$ and the symbol-error probability $p_\mathrm{s} = 1.4 \times 10^{-2}$ corresponding to average BER $p_\mathrm{b} = 10^{-2}$, thus providing an analytical justification for the high-BER approximation, which was based on the assumption of uncorrelated symbol errors.

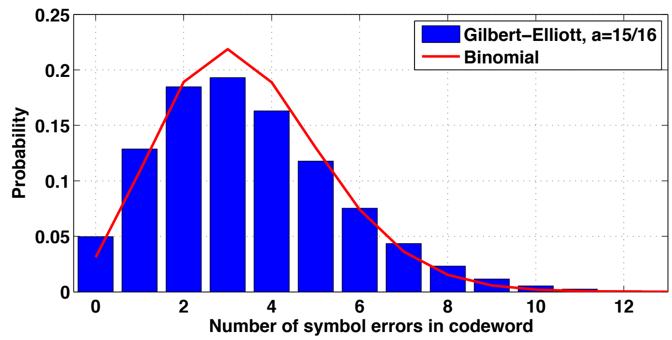

Fig. 5. Error distribution for $N = 240$, $t = 6$, $I = 4$, $s = 8$, $a = 15/16$, $p_\mathrm{b} = 10^{-2}$ and binomial distribution.

In Fig. 6, the error rate performance of two RS codes with symbol size $s = 8$, codeword length $N = 240$, interleaving depth $I = 4$ and error correction capabilities $t = 3$ and $t = 6$ are compared for a channel with $a = 15/16$, i.e., an average error burst length of 16 bit. As expected, the codeword-error probability p_C of the (240,228) RS code as a function of the raw BER p_b is lower than that of the (240,234) RS code. The error floor of the (240,234) RS code with $t = 3$ is about three orders higher than the error floor of the (240,228) RS code with $t = 6$.

In Fig. 7, the error rate performance of three RS codes with symbol sizes $s = 8$ and $s = 10$ are compared for a channel with $a = 3/4$, i.e., an average error burst length of four bit. In all three cases the encoded and interleaved block of codewords have the same size. The proposed channel model in Section II allows a direct comparison of the performance of interleaved block codes with different symbol sizes. The (768,728) RS code with 10-bit symbol size and interleaving depth one performs better than the (384,364) RS code with 10-bit symbol size and interleaving depth two, which in turn is better than the (240,228) RS code with 8-bit symbol size and interleaving depth four. The error floors, which will be

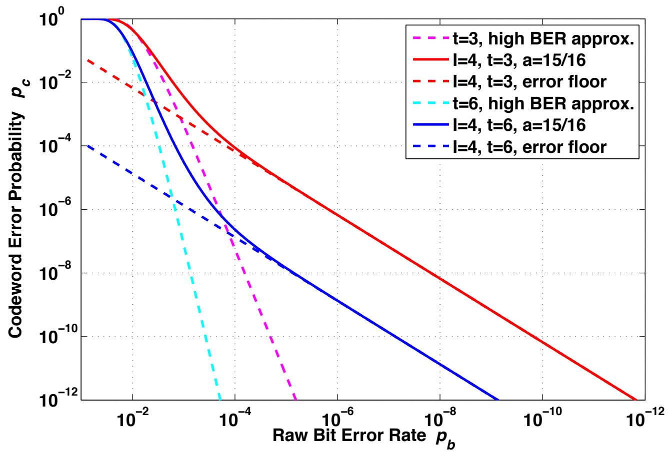

Fig. 6. Codeword error probability vs. raw BER for $N = 240$, $s = 8$, $I = 4$ and ECC capabilities $t = 3$ and $t = 6$.

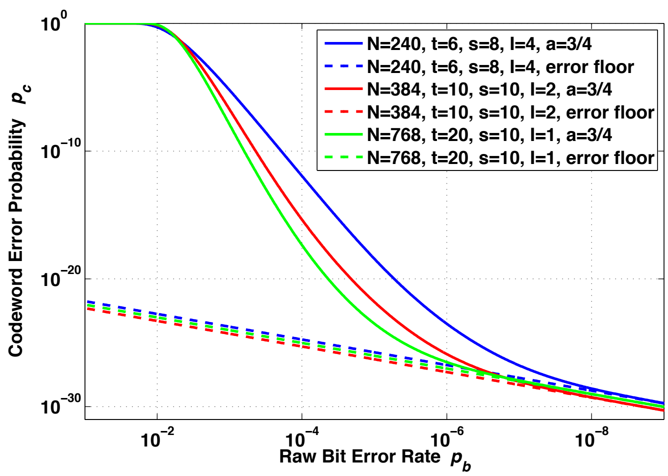

Fig. 7. Codeword error probability vs. raw BER for three RS codes with different symbol sizes s and interleaving depths I.

analyzed in the next section, occur in all three cases for a codeword-error probability $p_{\mathrm{C}} < 10^{-20}$.

IV. Lower Bound and Error Floor

In this section, a lower bound on the error rate performance and error floors of block codes in the presence of burst errors are evaluated based on the underlying bit-based channel model in Fig. 1.

A tight lower bound p_{L} on the codeword-error probability p_{C} is first derived for interleaved codewords. The probability of a codeword error within a block of I symbol-interleaved codewords is given by

$$p_{\mathrm{C}} = \pi_{\mathrm{G}} p_{\mathrm{C,G}} + \pi_{\mathrm{B}} p_{\mathrm{C,B}}, \tag{12}$$

where $\pi_{\mathrm{G}} = 1 - p_{\mathrm{b}}$ is the stationary probability of being at state G, $\pi_{\mathrm{B}} = p_{\mathrm{b}}$ is the stationary probability of being at state B, $p_{\mathrm{C,G}}$ is the probability of having at least $t+1$ symbol errors in a codeword starting from the good state G, and $p_{\mathrm{C,B}}$ is the probability of having at least $t+1$ symbol errors in a codeword starting from the bad state B.

From (3) the transition probability from the good state G to the bad state B approaches $(1-b) = (1-a)p_{\mathrm{b}}$ as $p_{\mathrm{b}} \to 0$. Since the stationary probability of being at state B is p_{b}, all dominant error events that exceed the error correction capability of the code are single error bursts, which are generated either starting from state B or from state G as $p_{\mathrm{b}} \to 0$. In the following, the index n, $1 \le n \le NIs$, indicates the first erroneous bit within the encoded and interleaved block of size NIs bit.

The probability of having at least $t+1$ symbol errors in a codeword starting from state G is then lower bounded by

$$p_{\mathrm{C,G}} > p^{L}_{\mathrm{C,G}} = \sum_{n=1}^{s} b^{n-1}(1-b)a^{Its-(n-1)} + \sum_{n=s+1}^{M} b^{n-1}(1-b)a^{f(n,s,t,I)}, \tag{13}$$

where $M = (N-t-1)Is + s$ is the total number of single burst error events of minimum length that exceed the ECC capability t and are generated by starting from state G, and

$$f(n,s,t,I) = Ist + (I-1)s - \operatorname{mod}(n-s-1, Is) \tag{14}$$

is the minimum length of a dominant single burst error generated after transitioning to state B, which depends on the index n and the code parameters s, t and I.

To generate a dominant error burst starting from state B, it is necessary to stay at state B for at least $Its+1$ transitions. Therefore, there is only one dominant single burst error of minimum length that exceeds the ECC capability of the code, and the probability of having at least $t+1$ symbol errors in a codeword starting from state B is lower bounded by

$$p_{\mathrm{C,B}} > p^{L}_{\mathrm{C,B}} = a^{Its+1}. \tag{15}$$

The lower bound p_{L} on the codeword-error probability p_{C} is then given by

$$p_{\mathrm{C}} > p_{\mathrm{L}} = \pi_{\mathrm{G}} p^{L}_{\mathrm{C,G}} + \pi_{\mathrm{B}} p^{L}_{\mathrm{C,B}}, \tag{16}$$

and p_{L} is a strict lower bound as it is the sum of probabilities of a subset of error events that exceed the ECC capability of the code. Summing up the probabilities of all dominant single burst error events starting from states G and B, we then obtain

$$p_{\mathrm{L}} = p_{\mathrm{b}} a^{Its+1} D(p_{\mathrm{b}}, a, N, s, t, I), \tag{17}$$

where

$$D(p_{\mathrm{b}}, a, N, s, t, I) = 1 + \frac{1-a}{a}\Big\{\sigma\big(b^{Is}, N-t\big)\,\sigma\big(\tfrac{b}{a}, s\big) + b^{Is}\sigma\big(b^{Is}, N-t-1\big)\,\big(\sigma\big(\tfrac{a}{b}, Is-s+1\big)-1\big)\Big\}, \tag{18}$$

and

$$\sigma(q,n) = \sum_{i=0}^{n-1} q^{i} = \frac{1-q^{n}}{1-q} \tag{19}$$

is the sum over the first n terms of a geometric series. Note that b is a function of a and p_{b} as shown in (3). The tight lower bound p_{L} depends on the raw BER p_{b}, the fixed channel parameter a and the fixed code parameters N, s, t, and I.

In the error floor regime of high SNR, there is a linear relationship between p_{b} and the error floor probability p_{F}. Specifically, $p_{\mathrm{L}} \to p_{\mathrm{F}}$ as $p_{\mathrm{b}} \to 0$, and

$$p_{\mathrm{F}} = p_{\mathrm{b}} a^{Its+1} \lim_{p_{\mathrm{b}} \to 0} D(p_{\mathrm{b}}, a, N, s, t, I). \tag{20}$$

From (3), $b \to 1$ as $p_\mathrm{b} \to 0$, and

$$p_\mathrm{F} = p_\mathrm{b} a^{Its-s+1}((N-t)(1-a^{Is}) + a^{Is}). \quad (21)$$

The proportionality factor

$$h(a, N, s, t, I) = a^{Its-s+1}((N-t)(1-a^{Is}) + a^{Is}) \quad (22)$$

in the linear relationship between p_b and the error floor probability p_F is a constant that depends on the fixed channel parameter a and the fixed code parameters N, s, t and I. At low average bit-error rates p_b and for two interleaved block codes with codeword length N, symbol size s, error correction capability t and two different interleaving depths $I_1 < I_2$, the interleaving gain γ on a channel with average error burst length $(1-a)^{-1}$ is then given by

$$\gamma = \frac{h(a, N, s, t, I_1)}{h(a, N, s, t, I_2)}. \quad (23)$$

We remark that on a $\log(p_\mathrm{F})$ vs. $\log(p_\mathrm{b})$ scale as in Figs. 4, 6 and 7, the slope of the error floor is one, i.e.,

$$\log(p_\mathrm{F}) = \log(p_\mathrm{b}) + \log(h(a, N, s, t, I)). \quad (24)$$

Finally, the high-BER approximation in Section III and the low-BER error floor derived in this Section are very close to the exact computation based on (7), (8) and (9). This allows a performance analyst to quickly determine the error rate performance at low and high BER using the closed-form expressions in (11) and (21).

V. Conclusion

A Gilbert-Elliott channel for symbol errors has been proposed for the performance analysis of interleaved block codes in the presence of burst errors. The channel model for symbol errors is based on a simplified Gilbert channel for bit errors to enable a direct comparison of the performance of block codes with different symbol sizes. The autocorrelation function between two erroneous symbols within a codeword interleaved to depth I has been given.

An exact expression for the codeword-error probability has been derived and a high-BER approximation of the codeword-error probability has been presented. A tight lower bound for the codeword-error probability has been obtained in the regime of low raw BER. It has been demonstrated that the error rate performance of interleaved codewords at the output of the simplified Gilbert channel exhibits an error floor at low raw BER. The error floor, which is proportional to the raw BER, has been analyzed and a closed-form expression has been derived.

The results of the performance evaluation presented in this paper including the analysis of error floors are valid for binary and non-binary BCH codes, RS codes, and in general for any interleaved block code with codeword length N, symbol size s, ECC capability t, and interleaving depth I.

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