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Progressively Refining the Orientation and Speed of a Gravitational Wave or Plane Wave from Time of Arrival Detector Measurements

Dilip Krishnaswamy
IBM Research Division
IBM India Research Lab
Bangalore-560045, India.
Email: dilikris@in.ibm.com , dilip@ieee.org
Web: http://researcher.watson.ibm.com/researcher/view.php?person=in-dilikris

Abstract

A method to progressively refine the orientation and speed of a gravitational wave or plane wave is described based on a coordinate system that is oriented along the detectors that successively detected the wave in time. A set of normalized wave propagation constraints are determined for the successive wave traversal events through the detectors based on the time of arrival at the detectors, to estimate the orientation and speed of the wave based on the number of detectors available for measurement. When multiple detectors are available, additional normalized wave propagation constraint equations can be used or different coordinate systems can be constructed to refine the estimates for the wave orientation and the wave speed. It is hoped that the presented approach with the normalized wave propagation constraints and choice of coordinate axes will be useful for efficient computations to determine wave orientation and speed for practitioners in this area. The described technique is applicable to any plane wave propagating through detectors successively in time.

1. Introduction

Gravitational waves [1][2][3][4][5] can be detected by gravitational wave detectors [6][7] on the surface of the earth. Based on the location of the detectors, and the orientation of the wave in space, each detector may detect the wave at a different time. Source localization [1][8][9]10][11] of gravitational waves has been studied in the past with multiple detectors. The relative timing of the signals arriving at the detectors provides the most useful information for source localization. Additional properties of the wave such as the signal amplitude, spin, and precession effects, or differences in the detectors such as antenna pattern asymmetry can also be utilized [1][8][9][10][11] to accomplish source localization. At times, only some detectors may be active to detect a wave. This paper focuses on source localization (determining the orientation of the wave) using a coordinate system aligned along these detectors that were involved in the detection of the wave, and to provide for a correction of the wave speed if desired.

The approach presented in this paper for source localization and speed estimation utilizes the knowledge regarding the wave propagation in time successively through the detectors, to progressively refine the knowledge related to the orientation of the wave, and to then estimate the speed of the wave, based on the available measurements from the detectors. The work in [9] was the earliest effort to present a solution to the problem, where the focus was primarily on how best to utilize time-of-arrival difference information from 3 detectors, and augment the information with additional information such as antenna pattern asymmetry, to determine the orientation of the wave. The work in [8] provides good intuition regarding the problem and focuses on a null-stream based technique to discriminate gravitational wave bursts from noise glitches. Timing accuracy and uncertainty related to the detector measurements are addressed in [10][11]. In [10], it is suggested that a precise form for localization with more than 3 detectors has not been calculated, which we hope to address in this paper, purely from the standpoint of the time of arrival information across multiple detectors. With a good number of detectors getting installed around the world [1], techniques for efficient computation and estimation across detectors can be expected to be useful. We assume, in this paper, that a plane wave propagation through detectors or a gravitational wave burst propagation through detectors has been observed and isolated from any noise glitches. The paper then focuses purely on utilizing the time-of-arrival information at the detectors to determine wave orientation and speed. For this objective, a set of normalized wave propagation constraints are determined for the wave traversal events through the detectors. The coordinate system is oriented along the detectors that successively detected the wave in time. The difference in time arrival at the detectors and the relative location of the detectors are used to study the orientation and speed of the wave. With two detectors, an angle of arrival can be determined relative to the vector associated with the path between the two detectors (chosen as the +z direction in this paper), assuming that the speed of the wave is known. Given the relative time information for three detectors, the +x direction is chosen in the plane of the detectors based on the direction of propagation of the wave through the detectors in sequence. The orientation of the wave relative the plane of the three detectors is determined. An ambiguity exists (in this paper, this translates to a determination of wave propagation relative to the +y or –y direction) related to this orientation which can be resolved [8][9] based on variations in signal amplitudes and the antenna pattern asymmetry in the detectors. Alternatively, a fourth detector may be used to determine the exact orientation of the wave (as suggested in [1] and [8]) to resolve the ambiguity. With four detectors, the speed of the wave can be validated as well. If the initial estimate on the speed of the wave is incorrect, then the wave speed can be scaled to refine the values of both the speed and the orientation of the wave in space. When multiple measurements from more than 4 detectors are available, different coordinate systems may be constructed using different sets of detectors, or additional constraint equations can be used, to refine the estimates for the orientation and speed of the wave. When multiple measurements from a given set of detectors that form a coordinate system are available, then additional wave propagation constraint equations can be obtained for each new set of measurements. When measurements from multiple detectors are available, or when multiple measurements are available for the detectors, then a pseudo-inverse can be computed to obtain a least-squares estimate for the orientation and the speed based on the wave propagation constraints arising from each of the detectors. The method described in the paper is general and it can be applied to any plane wave or plane wave train that traverses a sequence of detectors successively in time. It is hoped that the presented approach with the normalized wave propagation constraints and choice of coordinate axes will be useful for efficiently determining wave orientation and speed for practitioners in this area.

2. Determination with 2 detectors (informal geometrical interpretation)

Let us assume that a gravitational wave is detected by two gravitational wave detectors D1 and D2 on the surface of the earth, where the wave first reaches detector D1 and then reaches detector D2. Let the detectors D1 and D2 subtend an angle α at the center of the earth. Then the straight line distance, d, between them (through the earth) is given by the following equation, where R is the radius of the earth d = 2 R sin (α / 2) (assuming that the earth is a perfect sphere). In practice, the exact distance d between the detectors in space needs to be determined. The wave traverses the distance d from detector D1 to detector D2 as it propagates, arriving at the detectors at different times relative to the detectors. Let us assume that the detectors detect the same signal at a time interval Δt relative to each other, and that the speed of the gravitational wave is v. Let the wave arrive at an angle θ relative to the direct path between the two detectors. Then the additional distance traveled by the wave front to reach detector D2 after reaching detector D1 is given by v Δt. Then, cos θ = v Δt / d. Therefore, the angle of arrival θ relative to the vector $\overrightarrow{D_1D_2}$ is given by θ = cos^{-1} (v Δt / d).

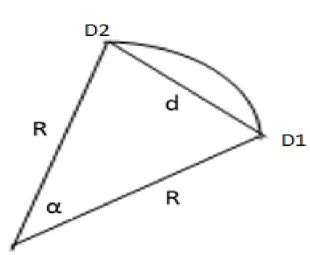

Figure 1: 2 Detectors separated by distance d

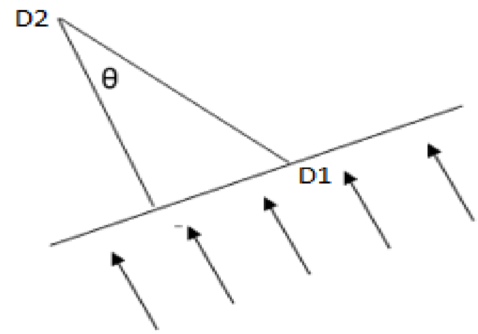

Figure 2: Incident Gravitational Wavefront

For example, if d = 3000km, Δt = 7ms, and v = c (speed of light in vacuum) = 3 x 10^8 m/s, then θ = cos^{-1} (0.7) = 45.57 degrees. If v = (1/0.7) c, then θ = cos^{-1} (1) = 0 degrees. If v = 0.9 c, then θ = cos^{-1} (0.63) = 50.94 degrees. It can be seen that as the speed varies, the orientation of the wave relative to the direct path between the detectors can vary as well. Intuitively, this resolves the orientation of the source to be lying on the surface of a cone that is centered around the direct path between the detectors.

3. Progressive Wave Orientation and Speed determination with Multiple Detectors

Let us assume that a gravitational wave (or any plane wave for that matter) passes through *n* detectors at locations D_1, D_2, D_3 ,.... D_l, D_{l+1},.... D_n in sequence in time. We will interchangeably use the same representation for the location of the detector and the index associated with the detector. Let us assume that the wave propagates with speed v. Let the signal (such as a peak associated with a gravitational wave burst) be received at detector D_2 *at* a time $\Delta t_{1,2}$ relative to detector D_1. The signal is subsequently detected at detector D_3 at a time $\Delta t_{2,3}$ relative to detector D_2. In general, the signal is detected at detector D_{l+1} at a time $\Delta t_{l,l+1}$ relative to the detector D_l.

3.1 Choosing a coordinate system oriented along the detectors

Without loss of generality, let us assume an x-y-z coordinate system, where the unit vector z along the z-axis is oriented along the vector $\overrightarrow{D_1D_2}$, and that the z-x plane is formed by the plane consisting of the points D_1, D_2, and D_3,corresponding to the first three detectors that the wave passed through in sequence.

Let us assume that the distance between detectors D_l and D_{l+1} is given by $d_{l,l+1}$ where *l = 1,2, 3, …(n-1)*.

Let $\hat{d}_{l,l+1}$ denote the unit vector in the direction $\overrightarrow{D_lD_{l+1}}$.

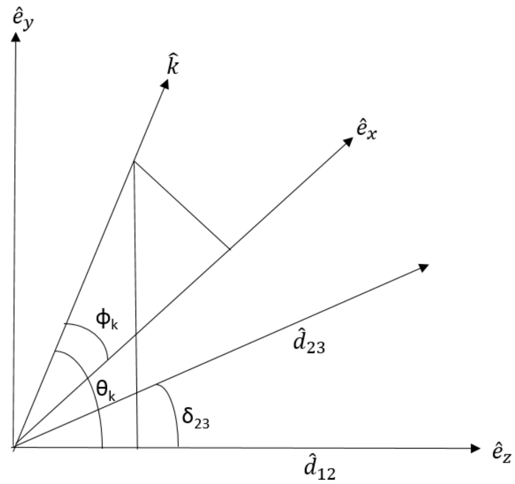

Figure 3: A coordinate system aligned along the detectors

Let $\hat{d}_{l,l+1}$ have unit vector components ($e_{l,l+1,x}$, $e_{l,l+1,y}$, $e_{l,l+1,z}$) in the x, y, and z directions. Then,

$\overrightarrow{D_l D_{l+1}} = d_{l,l+1} (e_{l,l+1,x}\ \hat{e}_x + e_{l,l+1,y}\ \hat{e}_y + e_{l,l+1,z}\ \hat{e}_z)$ .... (1)

Let us also define the wave propagation ratios

$$\eta_{l,l+1} = \frac{v\,\Delta t_{l,l+1}}{d_{l,l+1}} \quad \text{..(2)}$$

Then $\hat{d}_{1,2} \equiv (0,0,1)$. Therefore, $\overrightarrow{D_1 D_2}$ = d_{12} $\hat{e}_z$ .....(3)

Let the vector $\overrightarrow{D_2 D_3}$ be oriented at a non-zero angle δ_{23} relative to the vector $\overrightarrow{D_1 D_2}$ in the z-x plane, where the $\hat{e}_x$ direction is chosen orthogonal to $\hat{e}_z$ (in the plane consisting of D_1, D_2, and D_3) such that $0 < \delta_{23} < \pi$. The unit vector $\hat{e}_y$ is chosen in the direction of the vector product $\hat{e}_z$ x $\hat{e}_x$.

Then $\hat{d}_{2,3} \equiv$ (sin (δ_{23}), 0, cos (δ_{23}))

$\Rightarrow$ $\overrightarrow{D_2 D_3}$ = d_{23} (cos (δ_{23}) $\hat{e}_z$ + sin (δ_{23}) $\hat{e}_x$) ....(4)

Let the direction of propagation of the gravitational wave be given by the unit vector $\hat{k}$, where

$$\hat{k} = \cos(\theta_k)\ \hat{e}_z + \sin(\theta_k)\cos(\varphi_k)\ \hat{e}_x + \sin(\theta_k)\sin(\varphi_k)\,\hat{e}_y \ \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$= \cos(\theta_k)\ \hat{e}_z + \sin(\theta_k)\,\hat{k}_{xy}$$

where $\hat{k}_{xy} = \cos(\varphi_k)\ \hat{e}_x + \sin(\varphi_k)\,\hat{e}_y$

Here $\hat{k}_{xy}$ is a unit vector component in the x-y plane, θ_k is the angle made by the vector $\hat{k}$ with the z-axis, and, φ_k is the angle made the vector $\hat{k}_{xy}$ with x-axis in the xy-plane. Based on the chosen directions for the axes, and the fact that the wave propagation was intercepted by the detectors D_1, D_2 and D_3 in sequence (in time), we have $\cos(\theta_k) \geq 0$, and $\cos(\varphi_k) \geq 0$.

3.2 Wave propagation constraints for successive traversal through the detectors

Since the amount of time taken by the wave to reach detector D_{l+1} from detector D_l is based on the component of the vector $\overrightarrow{D_l D_{l+1}}$ in the direction of the propagation vector $\hat{k}$, we have the set of propagation constraints through the detectors given by, $\hat{k}\,.\,\overrightarrow{D_l D_{l+1}}$ = v $\Delta t_{l,l+1}$ or alternatively,

$\hat{k}\,.\,\hat{d}_{l,l+1} = \eta_{l,l+1}$, $\forall l \in \{1, 2, 3, \ldots, (n\text{-}1)\}$ ....(6)

where $\eta_{l,l+1} = \frac{v\,\Delta t_{l,l+1}}{d_{l,l+1}}$

It should be noted that the above normalized wave propagation constraints can be applied to any pair of detectors D_i and D_j, where the wave reaches detector D_j at a later time $\Delta t_{i,j}$ relative to detector D_i. This results in the wave propagation constraint $\hat{k}\,.\,\overrightarrow{D_i D_j}$ = v $\Delta t_{i,j}$, or alternatively,

$\hat{k}\,.\,\hat{d}_{i,j} = \eta_{i,j}$, where $\eta_{i,j} = \frac{v\,\Delta t_{i,j}}{d_{i,j}}$ . ...(7)

However, it should be noted that any such constraint for an arbitrary pair of detectors would merely be a linear combination of the constraints in equation (6), so that such constraints for any other arbitrary pair of detectors do not provide any additional information, once the constraints in equation (6) are specified.

3.3 Progressively Refining the Orientation of the wave

Let us assume that v = v_0, where v_0 can be set equal to c (the speed of light in vacuum) if desired.

Now let us successively apply these constraints in the chosen coordinate system described above.

$\hat{k}\,.\,\overrightarrow{D_1 D_2}$ = v $\Delta t_{1,2}$ or alternatively,

$\hat{k}\,.\,\hat{d}_{1,2} = \eta_{1,2}$ ....(8)

where $\eta_{1,2} = \frac{v\,\Delta t_{1,2}}{d_{1,2}}$

$\Rightarrow \cos(\vartheta_k) = \eta_{1,2}$

Let $\beta = cos^{-1}(\eta_{1,2})$ such that $\beta \in [\,0, \frac{\pi}{2}\,]$

Then, $\vartheta_k = \pm\beta$...(9)

Thus, with two detectors, we estimate an angle of arrival ϑ_k relative to the vector $\overrightarrow{D_1 D_2}$.

Also, $\hat{k} \,.\, \overrightarrow{D_2 D_3} = \text{v}\,\Delta t_{23}$, or alternatively,

$\hat{k} \,.\, \hat{d}_{2,3} = \eta_{2,3}$ ....(10)

where $\eta_{2,3} = \frac{v\,\Delta t_{2,,3}}{d_{2,3}}$

$$\cos(\vartheta_k)\,\cos(\delta_{23}) + \cos(\varphi_k)\sin(\vartheta_k)\,\sin(\delta_{23}) = \eta_{2,3}$$

$$\Rightarrow\ \sin(\vartheta_k)\,\cos(\varphi_k) = \frac{\eta_{2,3} - \cos(\vartheta_k)\,\cos(\delta_{23})}{\sin(\delta_{23})}$$

Since $\cos(\varphi_k) \geq 0$, the sign of ϑ_k is chosen based on the sign of the RHS. Thus,

If $\frac{\eta_{2,3} - \cos(\vartheta_k)\cos(\delta_{23})}{\sin(\delta_{23})} \geq 0$, then $\vartheta_k = +\beta$,

else $\vartheta_k = -\beta$ ....(11)

Now, $\cos(\varphi_k)$ may be determined, using the equation

$$\cos(\varphi_k) = \frac{\eta_{2,3} - \cos(\vartheta_k)\cos(\delta_{23})}{\sin(\delta_{23})\sin(\vartheta_k)}$$

Let $\gamma = cos^{-1}\left(\frac{\frac{v\,\Delta t_{23}}{d_{23}} - \cos(\vartheta_x)\cos(\delta_{23})}{\sin(\vartheta_x)\sin(\delta_{23})}\right)$

where $\gamma \in [\,0, \frac{\pi}{2}]$

Then, $\varphi_k = \pm\gamma$ ....(12)

Intuitively, with ϑ_k determined as specified in equation (11), the remaining choice of $\varphi_k = \pm\gamma$ effectively reduces to that of determining whether the wave is propagating in the +y or the –y direction in the chosen coordinate system. As noted in [8][9], this ambiguity can be resolved for gravitational waves by accounting for variations in the signal amplitudes, and the antenna pattern asymmetry in the data. However, let us continue the determination using relative time arrivals at detectors only, assuming that an additional fourth detector is available, so that the treatment is applicable to any plane wave traversing detectors in time as well. Let us use the arrival time $\Delta t_{3,4}$ at the fourth detector D_4 relative to the detector D_3. Let the vector $\overrightarrow{D_3 D_4}$ be given by

$$\overrightarrow{D_3 D_4} = d_{3,4}\,(e_{3,4,x}\,\hat{e}_x + e_{3,4,y}\,\hat{e}_y + e_{3,4,z}\,\hat{e}_z)$$

where $e_{3,4,x}$, $e_{3,4,y}$, and $e_{3,4,z}$ are components of the unit vector $\hat{d}_{3,4}$ in the direction $\overrightarrow{D_3 D_4}$.

Then $\hat{k} \,.\, \overrightarrow{D_3 D_4} = \text{v}\,\Delta t_{34}$ or,

alternatively $\hat{k} \,.\, \hat{d}_{3,4} = \eta_{3,4}$, where $\eta_{3,4} = \frac{v\,\Delta t_{3,4}}{d_{3,4}}$

$$\Rightarrow \cos(\vartheta_k)\ e_{3,4,z} + \sin(\vartheta_k)\cos(\varphi_k)\ e_{3,4,x} + \sin(\vartheta_k)\sin(\varphi_k)\ e_{3,4,y} = \eta_{3,4}$$

$$\Rightarrow \sin(\varphi_k) = \frac{\eta_{3,4} - \cos(\vartheta_k)\ e_{3,4,z} - \sin(\vartheta_k)\cos(\varphi_k)\ e_{3,4,x}}{\sin(\vartheta_k)\, e_{3,4,y}}$$

Since all the quantities on the RHS are known, the sign of φ_k can be determined and chosen based on the sign of the RHS. Thus,

If $\left(\frac{\eta_{3,4} - \cos(\vartheta_k)\, e_{3,4,z} - \sin(\vartheta_k)\cos(\varphi_k)\, e_{3,4,x}}{\sin(\vartheta_k) e_{3,4,y}}\right) \geq 0$,

then $\varphi_k = +\gamma$, else $\varphi_k = -\gamma$ ....(13)

Now, with the knowledge of both ϑ_k and φ_k, from equations (11) and (13), the direction of propagation (equation (5)) of the wave $\hat{k}$ (its orientation in space) is known relative to the chosen coordinate system.

If the direction of propagation in any alternate coordinate system is desired, then the appropriate coordinate transformation may be performed.

3.4 Progressively refining the estimate of the wave speed and the orientation

Now, we can validate whether the speed of the wave that was assumed earlier to determine both ϑ_k and φ_k is correct or not. Since $\hat{k} \,.\, \hat{d}_{3,4} = \eta_{3,4} = \frac{v\,\Delta t_{3,4}}{d_{3,4}}$, then based on the known value of $\hat{d}_{3,4}$ and based on the current estimated value of $\hat{k}$, one can determine v, using

$$\text{v} = (\hat{k} \,.\, \hat{d}_{3,4}) \,.\, \frac{d_{3,4}}{\Delta t_{3,4}} \text{ ....(14)}$$

Equation (14) can be used to validate whether the initial setting of $\text{v} = \text{v}_0 = c$ (say) is indeed valid. If this setting is not valid, then one would have to scale v to refine the set of constraints in equation (6), until equations are valid, or the until the mean squared error difference between the RHS and the LHS is minimized for the system of equations (6), to obtain the best possible estimate for the orientation and wave speed. Since the wave speed is fixed in value, there is only one solution that minimizes the mean squared error, and the system of equations can be expected to converge. In general, if $\hat{k} \equiv (k_x, k_y, k_z)^T$ and $\hat{d}_{l,l+1} \equiv (e_{l,l+1,x}\,,\ e_{l,l+1,y}\,,\ e_{l,l+1,z})^T$ $\forall l \in \{1, 2, 3\}$, then the set of equations (6) for the 4 detectors for the chosen coordinate system reduces to the set of equations,

$e_{1,2,z}\,k_z = \eta_{1,2}$, $e_{2,3,y}\,k_y + e_{2,3,z}\,k_z = \eta_{2,3}$, and $e_{3,4,x}\,k_x + e_{3,4,y}\,k_y + e_{3,4,z}\,k_z = \eta_{3,4}$, ...(15)

Alternatively, $\hat{k} = U^{-1}\ N$ ....(16)

where U = $\begin{pmatrix} e_{3,4,x} & e_{3,4,y} & e_{3,4,z} \\ 0 & e_{2,3,y} & e_{2,3,z} \\ 0 & 0 & e_{1,2,z} \end{pmatrix}$ and N = $\begin{pmatrix} \eta_{3,4} \\ \eta_{2,3} \\ \eta_{1,2} \end{pmatrix}$

If additional detectors are available, one can continue to use the general formula

$\hat{n} \,.\, \overrightarrow{D_l D_{l+1}}$ = v $\Delta t_{i\,(i+1)}$, or alternatively,

$$\hat{k} \,.\, \hat{d}_{l,l+1} = \eta_{l,l+1} \text{ .... (17)}$$

to derive additional wave propagation constraints through the detectors that can be used to further validate or check or improve the accuracy of the results (please see section 3.7).

3.5 Correcting the wave speed if required

Since $\hat{k}$ is a unit vector, we require that $\| \hat{k} \| = 1$. Therefore, if $v = v_0 = c$ (say) for an initial setting, and if an estimate for the propagation vector $\hat{k}_{est}$ is obtained using equation (16) for $v = v_0$ where $\| \hat{k}_{est} \| \neq 1$, then v can be scaled using $v = \frac{v_0}{\|\hat{k}_{est}\|}$, and correspondingly, $\hat{k}$ can be scaled using $\hat{k} = \frac{\hat{k}_{est}}{\|\hat{k}_{est}\|}$ to satisfy $\| \hat{k} \| = 1$. The corresponding values of the angular coordinates (such as in equation (5)) can be computed once the values for each of the components $k_x, k_y,$ and k_z, are known.

3.6 Coordinate systems based on alternate sets of detectors

If the information from the first 4 detectors related to the relative time arrivals for the measured signal does not have any uncertainty, then no additional measurements or additional coordinate systems with additional detectors are required to refine the estimates on the wave orientation and speed. However, if the wave was detected by more than 4 detectors, and additional corroboration or reduction in uncertainty related to the estimates of the wave speed and orientation are required, then new coordinate systems can be constructed and the estimates obtained relative to each coordinate system. For example, a coordinate system based on detectors D_2, $D_3, D_4,$ and D_5, can be constructed starting with the unit vector $\hat{e}_z$ chosen parallel to the vector $\overrightarrow{D_2 D_3}$. If there are *n* detectors where *n > 4*, then there exist *(n-3)* sets of detectors that detected the wave successively in time, so that *(n-3)* such coordinate systems can be constructed to obtain estimates for the orientation and the speed of the wave in each of these coordinate systems. In general, one can choose any 4 detectors, to create a coordinate system, to estimate the orientation and speed of the wave in such a coordinate system (for example, when 7 detectors are available, a coordinate system based on detectors D_2, D_4, D_5 and D_7, can be created to obtain an estimate for the orientation and the speed). When an arbitrary set of detectors are selected to create a coordinate system, then the general equation (7) is utilized to obtain the wave propagation constraints for wave traversal through the chosen detectors (it should be noted that constraint equations across non-successive detector pairs are merely linear combinations of constraints from successive detectors pairs). Additionally, when different sets of measurements are available (such as two bursts separated in time propagating through the detectors) for the same set of detectors forming a coordinate system, then alternate estimates can be obtained for the orientation and speed of the wave. Processing related to each of the coordinate systems can be processed in parallel to estimate the results across the coordinate systems faster. When estimates from each of these coordinate systems are available, it would be desirable to transform coordinates from all coordinate systems to a common coordinate system, to then compare the estimates for the orientation and speed of the wave across the coordinate systems. To select a common coordinate system, one could utilize a coordinate system for which the angular deviation (ϑ_k) of the wave from the unit vector $\hat{e}_z$ across the systems is a minimum, so that estimates of ($\vartheta_k, \varphi_k, v$) for the wave could be refined around this common coordinate system. Among the detector coordinate subsets that are chosen, if the presence of a detector in a subset causes significant deviation from results predicted from other coordinate system detector subsets, then one could eliminate such a detector from further analysis, and from the detector sequence. If such an anomalous detector is found, one could check if there might have been a time synchronization or other error that needs to get fixed so that the estimates for ($\vartheta_k, \varphi_k, v$) get closer to the alternate available estimates from other coordinate systems.

3.7 Progressive refinement using additional detectors or measurements

In general, in any coordinate system, one can consider, wave propagation vector $\hat{k} \equiv (k_x, k_y, k_z)^T$, $\hat{d}_{l,l+1} \equiv (e_{l,l+1,x}, e_{l,l+1,y}, e_{l,l+1,z})^T$, and $\eta_{l,l+1} = \frac{v\,\Delta t_{l,l+1}}{d_{l,l+1}}$. Let the *(n-1) x 3* matrix of pairwise detector unit vectors, $E \equiv (\hat{d}_{n-1,n}, \hat{d}_{n-2,n-1}, \ldots.., \hat{d}_{3,4}, \hat{d}_{2,3}, \hat{d}_{1,2})^T$, and the *(n-1) x 1* vector of wave propagation ratios $\mathrm{H} \equiv (\eta_{n-1,n}, \eta_{n-2,n-1}, \ldots\ldots\ldots, \eta_{3,4}, \eta_{2,3}, \eta_{1,2})^T$, then one can write the matrix equation

$$E \,.\, \hat{k} = \mathrm{H} \text{ ....(18)}$$

so that $\hat{k} = (E^T E)^{-1} E^T \mathrm{H}$ ....(19)

Equation (18) may be refined by scaling the wave speed v. If $v = v_0 = c$ (say) for an initial setting, and an estimate $\hat{k}_{est}$ is obtained with $v = v_0$ where $\| \hat{k} \| \neq 1$, then v and the terms in H, can be scaled using $v = \frac{v_0}{\|\hat{k}_{est}\|}$ to scale $\hat{k}$ correspondingly using $\hat{k} = \frac{\hat{k}_{est}}{\|\hat{k}_{est}\|}$ to satisfy $\| \hat{k} \| = 1$. When utilizing constraint equations from different coordinate systems for different sets of detectors, then it is necessary to transform coordinates into a common coordinate system, to then compute the pseudo-inverse to obtain a least squares estimate for $\hat{k}$. If different measurements are available for each detector for different measured instances of a gravitational wave train (or any plane wave train for that matter) as it passes through the detectors, then the each of these instances produces corresponding constraints (of the form in equation (17)) that can be utilized (using equations (18) and (19)) to refine and estimate the orientation and speed of the wave.

Summary

The paper has described a technique to utilize a coordinate system aligned with the gravitational wave detectors that detected a gravitational wave in sequence in time, to progressively refine estimates related to the orientation and speed of the gravitational wave. A set of normalized wave propagation constraints relative to the detectors and the wave propagation vector are determined. With two detectors, the z-axis is oriented along the vector corresponding to the directional path between the two detectors, and an angle of arrival relative to this vector can be determined if the speed of the wave is known. With three detectors, the x axis is chosen based on the direction of wave propagation through the detectors in sequence, in the plane formed by the 3 detectors, and then the orientation of the detected wave relative to the z-x plane is determined. Four detectors can be used to determine the exact orientation of wave propagation in space (by resolving the wave orientation in the +y or –y direction), and the speed of the wave can be validated as well. The computations may be refined if the initial estimate of the wave speed did not satisfy the propagation constraints through the detectors. When more than 4 detectors are available, then additional coordinate systems can be used to obtain alternate estimates for the orientation and speed. Any additional detectors or available additional measurements on the detectors can provide additional propagation constraints through the detectors to further validate or improve the accuracy of the results based on the measurements. The technique described in the paper is generic enough to be applicable to any plane wave or plane wave train that traverses a sequence of detectors in time.

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