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A Conditional Logic Approach for Strengthening Mixed 0-1 Linear Programs

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Abstract

We study a conditional logic approach for tightening the continuous relaxation of a mixed 0-1 linear program. The procedure first constructs quadratic inequalities by computing select pairwise products of the problem constraints, and then surrogates modified versions of these inequalities to produce valid linear restrictions. There are two key ingredients that work cooperatively to yield improved representations. First, conditional logic is employed to adjust the coefficients on the quadratic restrictions while maintaining their validity. Second, the quadratic inequalities are strategically computed so that nonnegative surrogates having coefficients of zero on all the quadratic terms are available. The approach can be viewed as a unifying framework for published single and multiple coefficient increasing and decreasing methods that exploit such structure as special ordered sets, plant location and capacity expansion constraints, variable upper bounds, and logical implication restrictions that arise from preprocessing and probing techniques. It also generalizes the process of sequential lifting, which takes a valid linear inequality in a given variable space and lifts it to a higher-dimensional space. We give illustrative examples, and discuss

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various extensions, including the use of more complex conditional logic constructs that produce linear restrictions by computing and surrogating polynomial expressions, and the application to general integer programs.

Key words: Integer programming, coefficient adjustment, cutting planes, continuous relaxation.

1 Introduction

Our ability to solve a mixed 0-1 linear program is dramatically affected by the problem formulation. Since state-of-the-art solution methodologies typically rely on linear programming (continuous) relaxations, reformulating the problem so that the convex hull of feasible solutions is closely approximated by the continuous relaxation has substantial benefits. This well-known fact has motivated much interest in the development of methods for generating tight polyhedral outer-approximations. A variety of methods are prevalent throughout the literature, including preprocessing, probing, constraint disaggregation, automatic reformulation, and cutting plane generation. In this paper, we are concerned with coefficient adjustment via conditional logic.

In a recent publication, Sherali, Adams, and Driscoll [20] introduced a conditional logic reasoning to tighten linear inequalities for mixed 0-1 linear programs. Their basic idea was to compute pairwise products of constraints, and to "condition" the coefficients of one such constraint based on the values of the other. The concept was introduced in the context of a reformulation-linearization technique (RLT) of Sherali and Adams [17, 18, 19]. While the approach was shown to encompass Balas and Zemel's [1] sequential lifting process for the full-dimensional knapsack polytope and to motivate the Miller-Tucker-Zemlin [21] inequalities for the traveling salesman problem, the contribution is predominantly theoretical, establishing no definitive strategies for generating valid linear inequalities.

Independently, various authors have suggested coefficient adjustment strategies for devising tight continuous relaxations of pure 0-1 linear programs. Given a valid linear inequality, the intent of these strategies is to exploit problem information to alter the inequality's coefficients in such a manner that all feasible binary solutions are preserved, but so that fractional solutions lying outside the convex hull of feasible binary solutions are eliminated. Typically, these adjustments either increase or decrease the coefficients, with the associated techniques referred to as coefficient increasing or decreasing (reduction) methods, respectively.

Investigations into coefficient adjustment strategies have focused on both relaxation strength and computational tractability. The early computational studies of Crowder, Johnson, and Padberg [2] on real-world, large-scale, 0-1 linear programs demonstrated that decreasing a single coefficient of a linear inequality was extremely efficient in strengthening certain inequalities formulated with "Big M" type coefficients. The computational utility of both single-coefficient increasing and single-coefficient decreasing methods for more general inequalities was shown by Johnson, Kostreva, and Suhl [12]. They studied how a linear inequality in binary variables can be tightened based on variable fixing, and explained how special ordered sets (SOS) could lead to further improvements. Hoffman and Padberg [11] also considered tightening methods for inequalities when SOS constraints are present. They incorporated SOS-tightened inequalities, as well as inequalities strengthened using plant location constraints (in a manner similar to Dietrich and Escudero [3]), as part of a preprocessing scheme in a successful branch-and-cut algorithm. A more extensive study of coefficient increasing and coefficient decreasing methods for knapsack inequalities was conducted in a sequence of papers (see [3, 4, 5, 6, 7, 8, 9, 10] among others), where special structures such as clique constraints, cover restrictions, and variable upper bounding were effectively exploited. The central theory to this body of research, along with a good overview of the general approach and promising computational results, can be found in Dietrich, Escudero and Chance [7]. In one of the latest of these articles, Escudero, Martello, and Toth [9] show how additional gains can be achieved by incorporating probing analysis and new advances in algorithms for variants of the classical knapsack problem [15]. Their computational results are encouraging. Most recently, Lougee-Heimer [13] proposed a clarification of [12] and an extension to the SOS-based methods in [11, 12]. She showed that the results in [12] can be viewed as adjusting the coefficient on a single binary variable, while exploiting the SOS restrictions. A key observation used here is that the slack variable on an SOS inequality is itself binary-valued.

Interestingly, it turns out that the conditional logic reasoning of [20] can be expanded to form a general framework that motivates a rich family of valid linear inequalities, including the inequalities obtainable via all the above-cited coefficient adjustment methods. This paper presents the essential ingredients of the framework, both motivating the concept of conditional logic and prescribing the steps required to generate the inequalities. Our study extends the initial efforts of [20] by strategically computing quadratic inequalities in such a manner that they can be readily surrogated to yield valid linear inequalities. We also discuss how cubic and higher-degreed inequalities can be utilized in the same spirit to form linear inequalities which are even stronger than their quadratic-based counterparts, and we extend our logic to accommodate general integer programs.

Due to the nature of the conditional logic approach, there exists a great deal of latitude relative to the types and strengths of cuts that can be generated, as well as to the computational effort required to compute these cuts. In Section 2 we present the general conditional logic framework to create strengthened quadratic inequalities, and then describe our surrogate operation to obtain linear restrictions. In Section 3 we present specific numeric examples that demonstrate the utility of these cuts, and show how various published works are in fact special cases of this general approach. We focus on both the theoretical derivation and the computational requirements. Finally, in Section 4 we explain how the conditional logic arguments generalize to employ higher-ordered inequalities, which in turn lead to progressively tighter representations. We also discuss extensions to integer programs.

2 Derivation of the Conditional-Logic Cut

This paper is concerned with constructing tight polyhedral outer-approximations of the set

$$S \equiv \{\boldsymbol{x} \in \mathcal{R}^n \mid \boldsymbol{x} \in X, x_j \text{ binary } \forall\, j = 1, \ldots, p\}, \tag{1}$$

where $p \leq n$ and where X is a nonempty polyhedral set in the nonnegative variables $\boldsymbol{x}$ defined in terms of the $m \times n$ matrix A and the m-component column vector $\boldsymbol{b}$ as

$$X = \{\boldsymbol{x} \in \mathcal{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}. \tag{2}$$

For notational convenience we assume without loss of generality that the inequalities $x_j \leq 1\ \forall\ j = 1, \ldots, p$ are implied by the restrictions in X, with X then giving the standard continuous relaxation of S. We refer to the inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$, less any explicit inequalities of the form $x_j \leq 1$, as the *structural* constraints of S. We adopt the notation that $conv(S)$ represents the convex hull of the set S, so that $conv(S) \subseteq X$ by definition. The idea behind using conditional logic to tighten the continuous relaxation of S provided by X is based on the generation of "conditional" nonlinear inequalities in $\boldsymbol{x}$ that are valid for S but are not valid for X. The logic is conditioned on the relative values that the slacks of the constraints defining X can realize. By computing special surrogates of these nonlinear inequalities, we are able to obtain linear restrictions that tighten the continuous relaxation provided by X. Throughout this paper, we will be applying conditional logic on a linear inequality of the form $\boldsymbol{\alpha x} \leq \beta$ implied by X to modify $\boldsymbol{\alpha}$ and β so that an improved polyhedral representation of S is achieved.

To demonstrate the concept of conditional logic proposed in [20], consider any two inequalities within X, say $\boldsymbol{\alpha x} \leq \beta$ and $\boldsymbol{\delta x} \leq \gamma$. Then clearly the following quadratic inequality holds true.

$$(\gamma - \boldsymbol{\delta x})(\beta - \boldsymbol{\alpha x}) \geq 0 \;\forall\; \boldsymbol{x} \in S. \tag{3}$$

In fact, this inequality holds true for all $\boldsymbol{x} \in X$ and, as pointed out in [20], will remain true for all $\boldsymbol{x} \in S$ if we perform standard logical preprocessing steps on the inequality $\boldsymbol{\delta x} \leq \gamma$, the inequalities in X, and the x_j binary $\forall\; j = 1, \ldots, p$ restrictions to tighten $\boldsymbol{\alpha x} \leq \beta$ to $\boldsymbol{\alpha}'\boldsymbol{x} \leq \beta'$.

Building on this conditional logic thought process, we introduce our approach. Let us temporarily restrict attention to the set of all $\boldsymbol{x} \in T$ for which $\boldsymbol{\delta x} < \gamma$, where $T \equiv S$ or any relaxation so that $S \subseteq T$. We compute

$$\beta^* = \min\{\beta,\; \max\{\boldsymbol{\alpha x} |\; \boldsymbol{x} \in T, \boldsymbol{\delta x} < \gamma\}\}, \tag{4}$$

and consequently have the "conditional logic" statement that

$$\text{if } (\boldsymbol{\delta x} < \gamma \text{ and } \boldsymbol{x} \in S) \text{ then } \boldsymbol{\alpha x} \leq \beta^*. \tag{5}$$

We assume throughout this paper that the optimization problem in (4) is feasible for any selected set T with $S \subseteq T$. If such is not the case for some chosen set T, then the inequality $\boldsymbol{\delta x} \leq \gamma$ can be expressed as an equality in X without changing S, and an alternate inequality chosen to construct (3). As a result, the quadratic inequality

$$(\gamma - \boldsymbol{\delta x})(\beta^* - \boldsymbol{\alpha x}) \geq 0 \;\forall\; \boldsymbol{x} \in S \tag{6}$$

holds true. Observe that, unlike inequality (3), if $\beta^* < \beta$ then we can have that (6) does not hold for all $\boldsymbol{x} \in X$, and thus (6) can serve to strengthen the approximation of $conv(S)$ provided by X. This strengthening can occur even if $T = X$ in (4), as will be discussed in Section 3.1.

To accomplish our goal of obtaining improved *polyhedral* representations of $conv(S)$, however, we must generate valid *linear* inequalities. Toward this end, we proposed the following strategy. Suppose we once again have an inequality $\boldsymbol{\alpha x} \leq \beta$ in X, but now also have R additional inequalities, distinct from $\boldsymbol{\alpha x} \leq \beta$, of the form $\boldsymbol{\delta x} \leq \gamma$ in X, say

$$\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r \ \forall\, r = 1, \ldots, R. \tag{7}$$

Then we can compute

$$\beta^r = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{x} \in T_r, \boldsymbol{\delta}_r \boldsymbol{x} < \gamma_r\}\} \,\forall\, r = 1, \ldots, R \tag{8}$$

where

$$T_r \equiv S \text{ or any relaxation so that } S \subseteq T_r \tag{9}$$

to obtain

$$(\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta^r - \boldsymbol{\alpha x}) \geq 0 \,\forall\, r = 1, \ldots, R \tag{10}$$

which by construction are valid for all $\boldsymbol{x} \in S$, but not necessarily for all $\boldsymbol{x} \in X$. Of course, if $\boldsymbol{\alpha x} \leq \beta$ is an inequality implied by T_r, then the computing of β^r in (8) simplifies to $\beta^r = \max\{\boldsymbol{\alpha x} |\ \boldsymbol{x} \in T_r, \boldsymbol{\delta}_r \boldsymbol{x} < \gamma_r\}$, with (4) analogously simplifying. Regardless of the specific form of T_r, however, (8) ensures that

$$\beta^r \leq \beta \,\forall\, r = 1, \ldots, R. \tag{11}$$

Further suppose that associated with the inequalities $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ are nonnegative multipliers λ_r, $r = 1, \ldots, R$, such that

$$\sum_{r=1}^{R} \lambda_r (\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x}) \leq \kappa \,\forall\, \boldsymbol{x} \in S, \tag{12}$$

where κ is a positive scalar. Then by surrogating inequalities (10) using these multipliers λ_r, we obtain the following *linear* inequality which is valid for all $\boldsymbol{x} \in S$.

$$(\beta - \boldsymbol{\alpha x}) + \sum_{r=1}^{R} \frac{\lambda_r}{\kappa} (\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta^r - \beta) \geq 0 \tag{13}$$

It is precisely this inequality that will promote superior polyhedral representations.

To clarify the derivation of (13), observe that

$$0 \leq \sum_{r=1}^{R} \lambda_r (\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta^r - \boldsymbol{\alpha x})$$

$$= \sum_{r=1}^{R} \lambda_r(\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta - \boldsymbol{\alpha x}) + \sum_{r=1}^{R} \lambda_r(\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta^r - \beta)$$

$$\leq \kappa(\beta - \boldsymbol{\alpha x}) + \sum_{r=1}^{R} \lambda_r(\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x})(\beta^r - \beta) \text{ for each } \boldsymbol{x} \in S, \tag{14}$$

where the first inequality is due to (10) and the nonnegativity of λ_r for $r = 1, \ldots, R$; the equality is purely algebraic; and the second inequality follows from (12) and the nonnegativity of $\beta - \boldsymbol{\alpha x}$ for all $\boldsymbol{x} \in S$. Inequality (13) is then obtained by dividing (14) by the positive quantity κ.

As opposed to simply appending inequality (13) to the set of restrictions defining X, we purport to replace the motivating inequality $\boldsymbol{\alpha x} \leq \beta$ with (13), thereby maintaining the same number of restrictions. Let X' denote the set obtained from X by replacing the inequality $\boldsymbol{\alpha x} \leq \beta$ with (13). Recall that $\kappa > 0$, and for each $r = 1, \ldots, R$, that $\lambda_r \geq 0$, $\beta^r - \beta \leq 0$ by (11), and $\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x} \geq 0 \ \forall \ \boldsymbol{x} \in X$, so it follows that every point $\boldsymbol{x}$ satisfying $\boldsymbol{x} \in X'$ must also satisfy $\boldsymbol{x} \in X$. The formal relationships between the sets X, X', and $conv(S)$ are summarized in the Theorem and Corollary below.

Theorem 1 *$conv(S) \subseteq X' \subseteq X$.*

Proof. $conv(S) \subseteq X'$. Given any $\hat{\boldsymbol{x}} \in conv(S)$, $\hat{\boldsymbol{x}}$ satisfies (13) by construction and $\hat{\boldsymbol{x}} \in X$ by definition. Thus, $\hat{\boldsymbol{x}} \in X'$.

$X' \subseteq X$. It suffices to show that $\boldsymbol{\alpha x} \leq \beta$ can be expressed as a nonnegative linear combination of the inequalities defining X'. Surrogate these constraints as follows. For each $r = 1, \ldots, R$, add to inequality (13) the nonnegative multiple $(\lambda_r/\kappa)(\beta - \beta^r)$ of the inequality $\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x} \geq 0$ to obtain $\beta - \boldsymbol{\alpha x} \geq 0$ as desired. ■

Corollary 1 *Given any $\hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{x}}_j$ binary $\forall \ j = 1, \ldots, p$, then $\hat{\boldsymbol{x}} \in X$ if and only if $\hat{\boldsymbol{x}} \in X'$.*

Proof. By definition of the sets S and X, any $\hat{\boldsymbol{x}} \in X$ satisfying $\hat{\boldsymbol{x}}_j$ binary $\forall \ j = 1, \ldots, p$ must also have $\hat{\boldsymbol{x}} \in S$, and consequently $\hat{\boldsymbol{x}} \in conv(S)$. The result then follows directly from the Theorem. ■

Theorem 1 and Corollary 1 combine to show that the sets X and X' define the same regions when the restrictions x_j binary for $j = 1, \ldots, p$ are enforced, and that the polyhedral outer-approximations of $conv(S)$

obtained by relaxing these binary restrictions satisfy $X' \subseteq X$. They do not explicity address when (13) provides sufficient strength beyond that of $\boldsymbol{\alpha x} \leq \beta$ to ensure that $X' \subset X$. Theorem 2 below addresses this concern in the following manner. Suppose that we define the set $X_{\boldsymbol{\alpha}}$ as

$$X_{\boldsymbol{\alpha}} \equiv \{\boldsymbol{x} \in \mathcal{R}^n |\, A_{\boldsymbol{\alpha}}\boldsymbol{x} \leq \boldsymbol{b}_{\boldsymbol{\alpha}}, \boldsymbol{x} \geq \boldsymbol{0}\} \tag{15}$$

where the system of linear inequalities $A_{\boldsymbol{\alpha}}\boldsymbol{x} \leq \boldsymbol{b}_{\boldsymbol{\alpha}}$ is precisely the system $A\boldsymbol{x} \leq \boldsymbol{b}$ with the restriction $\boldsymbol{\alpha x} \leq \beta$ removed. In essence, the set $X_{\boldsymbol{\alpha}}$ is the relaxation of the set X obtained when the restriction $\boldsymbol{\alpha x} \leq \beta$ is not enforced. Theorem 2 establishes that, provided the inequalities $\boldsymbol{\alpha x} \leq \beta$ and (13) are distinct (in the sense that for at least one $r \in \{1, \ldots, R\}$, $\lambda_r(\beta^r - \beta) < 0$), then $X' \subset X_{\boldsymbol{\alpha}}$ if and only if $X' \subset X$. This theorem is in the same spirit as a result due to Escudero and Muñoz [10], who provide a related argument for a special case of inequality (13).

Before presenting Theorem 2, for notational convenience we let $\boldsymbol{\alpha}'\boldsymbol{x} \leq \beta'$ denote inequality (13), and $s_{\boldsymbol{x}}$ and $s'_{\boldsymbol{x}}$ denote the slack variables on the inequalities $\boldsymbol{\alpha x} \leq \beta$ and $\boldsymbol{\alpha}'\boldsymbol{x} \leq \beta'$ respectively so that

$$s_{\boldsymbol{x}} = \beta - \boldsymbol{\alpha x} \text{ and } s'_{\boldsymbol{x}} = \beta' - \boldsymbol{\alpha}'\boldsymbol{x}. \tag{16}$$

Then we have that

$$s'_{\boldsymbol{x}} - s_{\boldsymbol{x}} = \sum_{r=1}^{R} \frac{\lambda_r}{\kappa}(\gamma_r - \boldsymbol{\delta}_r\boldsymbol{x})(\beta^r - \beta) \leq 0 \;\forall\; \boldsymbol{x} \in X_{\boldsymbol{\alpha}} \tag{17}$$

where the equality follows from (13) and (16), and where the inequality is due to (11), the definition of $X_{\boldsymbol{\alpha}}$ in (15), the nonnegativity of $\lambda_r \;\forall\; r = 1, \ldots, R$, and the positivity of κ.

Theorem 2 *Given that the inequalities $\boldsymbol{\alpha x} \leq \beta$ and (13) are distinct, then $X' \subset X_{\boldsymbol{\alpha}}$ if and only if $X' \subset X$.*

Proof. Since $X \subseteq X_{\boldsymbol{\alpha}}$, if $X' \subset X$ then $X' \subset X_{\boldsymbol{\alpha}}$. Now, suppose $X' \subset X_{\boldsymbol{\alpha}}$ so that there exists an $\tilde{\boldsymbol{x}} \in X_{\boldsymbol{\alpha}}$ with $s'_{\tilde{\boldsymbol{x}}} < 0$. If $s_{\tilde{\boldsymbol{x}}} \geq 0$ then $X' \subset X$ and the result follows, so we consider the case where $s_{\tilde{\boldsymbol{x}}} < 0$. ($s'_{\tilde{\boldsymbol{x}}} \leq s_{\tilde{\boldsymbol{x}}}$ by (17) since $\tilde{\boldsymbol{x}} \in X_{\boldsymbol{\alpha}}$). By the assumption that the inequalities $\boldsymbol{\alpha x} \leq \beta$ and (13) are distinct, there exists a $q \in \{1, \ldots, R\}$ such that $\lambda_q(\beta^q - \beta) < 0$. Let $\hat{\boldsymbol{x}}$ be an optimal solution to the inner minimization problem in (8) when $r = q$ and $T_r = S$, and observe from (17) that $s'_{\hat{\boldsymbol{x}}} - s_{\hat{\boldsymbol{x}}} < 0$ since $S \subseteq X_{\boldsymbol{\alpha}}$. In addition, $s'_{\hat{\boldsymbol{x}}} \geq 0$ since $\hat{\boldsymbol{x}} \in S$. Summarizing, we have $s'_{\tilde{\boldsymbol{x}}} \leq s_{\tilde{\boldsymbol{x}}} < 0 \leq s'_{\hat{\boldsymbol{x}}} < s_{\hat{\boldsymbol{x}}}$ so that

$$0 \leq \frac{s'_{\hat{\boldsymbol{x}}}}{s'_{\hat{\boldsymbol{x}}} - s'_{\tilde{\boldsymbol{x}}}} < \frac{s_{\hat{\boldsymbol{x}}}}{s_{\hat{\boldsymbol{x}}} - s_{\tilde{\boldsymbol{x}}}}. \tag{18}$$

Initialize at the point $\hat{x}$ and proceed in the direction $(\tilde{x} - \hat{x})$ by computing $\overline{x} = \hat{x} + \theta(\tilde{x} - \hat{x})$ for $\theta \geq 0$ to obtain

$$s_{\overline{x}} = s_{\hat{x}} + \theta(s_{\tilde{x}} - s_{\hat{x}}) \text{ and } s'_{\overline{x}} = s'_{\hat{x}} + \theta(s'_{\tilde{x}} - s'_{\hat{x}}). \tag{19}$$

Observe that (19) defines $s_{\overline{x}}$ and $s'_{\overline{x}}$ as linear functions of θ (having negative slopes) with $s_{\overline{x}} = s_{\hat{x}} > 0$ and $s'_{\overline{x}} = s'_{\hat{x}} \geq 0$ at $\theta = 0$, with $s_{\overline{x}} = 0$ at $\theta = s_{\hat{x}}/(s_{\hat{x}} - s_{\tilde{x}})$, and with $s'_{\overline{x}} = 0$ at $\theta = s'_{\hat{x}}/(s'_{\hat{x}} - s'_{\tilde{x}})$. Consequently, in light of (18), for any $\theta \in (s'_{\hat{x}}/(s'_{\hat{x}} - s'_{\tilde{x}}), s_{\hat{x}}/(s_{\hat{x}} - s_{\tilde{x}})]$, we have $s'_{\overline{x}} < 0$ and $s_{\overline{x}} \geq 0$ so that each such point $\overline{x}$ has $\overline{x} \in X$ but $\overline{x} \notin X'$. Thus $X' \subset X$ and the proof is complete. ■

We make three final remarks in this section. First, though inequalities of (7) were used to compute (13), equality restrictions may also be employed. However, any such equation $\boldsymbol{\delta}_r \boldsymbol{x} = \gamma_r$ used to replace the strict inequality in (8) would yield $\beta^r < \beta$ only if $\boldsymbol{\alpha x} \leq \beta$ were redundant to the set T_r in the presence of this equality restriction. Second, an interesting interpretation of Theorem 2 is that, provided the computed inequality (13) is both not redundant to the set $X_{\boldsymbol{\alpha}}$ (so that $X' \subset X_{\boldsymbol{\alpha}}$) and distinct from the restriction $\boldsymbol{\alpha x} \leq \beta$, then it serves to tighten the approximation of the set $conv(S)$ afforded by X. If, however, either of these two conditions fail, then (13) provides no additional strength to the set X beyond that of $\boldsymbol{\alpha x} \leq \beta$. Third, since by definition the inequalities $\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x} \geq 0$ for $r = 1, \ldots, R$ serve to partially define the set X', the proof of Theorem 1 suggests that the scalar κ used to construct (13) should be chosen as small as possible while still satisfying (12). Such a minimal selection of κ will lend maximal strength to the representation X'.

3 Examples and Illustrative Relationships with the Literature

In this section we demonstrate the utility and general applicability of the conditional logic approach by providing specific examples that illustrate how inequality (13) can strengthen the continuous relaxations of different sets S. In the process, we show how this strengthening encompasses published results. As indicated by (8), (9), and (13), the versatility of conditional logic when applied to an inequality $\boldsymbol{\alpha x} \leq \beta$ is primarily due to the selection of the inequalities $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$, $r = 1 \ldots, R$, and the sets T_r for their combined role in determining the values β^r. The scalars λ_r and κ are then defined in terms of these selections to satisfy (12). By varying our choice of these inequalities and sets, we will be able to motivate cuts of differing strengths and computational requirements.

Our discussion in this section is organized as follows. Section 3.1 provides a simple numerical example and some generally applicable statements to illustrate the approach. The remaining sections then focus on the selections of $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$ and T_r to demonstrate how special structures can be exploited, leading to generalizations of published coefficient adjustment methods. Section 3.2 examines instances where $R \leq 2$ and where $\boldsymbol{\delta}_1\boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2\boldsymbol{x} \leq \gamma_2$ correspond to the simple lower and upper bounds on a single binary variable x_k. These limitations lead to conditional logic inequalities which can be obtained from $\boldsymbol{\alpha x} \leq \beta$ by adjusting a single α_k coefficient, and possibly the scalar β. Both general sets T_1 and T_2, and sets having exploitable structures such as plant location constraints and various types of variable upper bounding restrictions are considered. Section 3.3 studies cases where R may exceed 2 and where the inequalities $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$, $r = 1, \ldots, R$, can include a single structural constraint of X, in addition to simple lower and upper bounding restrictions. The resulting conditional logic inequalities lead to the adjustment of multiple α_k coefficients. It turns out that published methods can be viewed as special cases of this multiple-coefficient adjustment where the structural constraint is either a clique or cover. We examine general and specially-structured sets T_r. Section 3.4 considers inequalities $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$, $r = 1, \ldots, R$, of special form so that more than one structural constraint in X can be employed. An illustrative example is presented where $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$ consist of variable upper-bounding restrictions.

3.1 Numerical example and general remarks

Consider the simple numeric example below where the set S has $n = p = 2$ binary variables, and where the set X in $\mathcal{R}^2$ contains a single structural inequality.

Example 1. Let $S \equiv \{(x_1, x_2) \in X |\, x_1, x_2 \text{ binary}\}$ with $X \equiv \{(x_1, x_2) \in \mathcal{R}^2 |\, 5x_1 + 2x_2 \leq 6, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$. Further let $\boldsymbol{\alpha x} \leq \beta$ be given by $5x_1 + 2x_2 \leq 6$ and let $R = 2$ with $\boldsymbol{\delta}_1\boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2\boldsymbol{x} \leq \gamma_2$ given by $x_1 \leq 1$ and $-x_1 \leq 0$. Then $\lambda_1 = \lambda_2 = \kappa = 1$ will satisfy (12) since $1(1 - x_1) + 1(x_1) \leq 1 \;\forall\; \boldsymbol{x} \in S$. Let $T_1 = T_2 = S$, and compute β^1 and β^2 as prescribed in (8) to obtain

$$\beta^1 = \min\{6,\; \max\{5x_1 + 2x_2 |\; 5x_1 + 2x_2 \leq 6, x_1, x_2 \text{ binary}, x_1 < 1\}\}$$

and

$$\beta^2 = \min\{6,\; \max\{5x_1 + 2x_2 |\; 5x_1 + 2x_2 \leq 6, x_1, x_2 \text{ binary}, -x_1 < 0\}\}.$$

We must have that $x_1 = 0$ in the optimization problem of β^1 and $x_1 = 1$ in the optimization problem of β^2. Then $\beta^1 = 2$ and $\beta^2 = 5$ so that inequality (13) becomes $2x_1 + 2x_2 \leq 2$ which, together with the $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$ restrictions in X, gives $conv(S)$.

There exists a great deal of flexibility in the derivation of conditional logic cuts. For instance, in Example 1 we could have instead let $R = 1$ with $\boldsymbol{\delta}_1\boldsymbol{x} \leq \gamma_1$ given by either $x_1 \leq 1$ or $-x_1 \leq 0$, $\lambda_1 = \kappa = 1$, and $T_1 = S$, and produced the restrictions $x_1 + 2x_2 \leq 2$ or $6x_1 + 2x_2 \leq 6$, respectively. Alternately in Example 1, we could have chosen less restrictive sets for T_1 and T_2 such as $T_1 = T_2 = \{\boldsymbol{x} \in X |\; x_1 \text{ binary}\}$. In this case, since x_1 binary together with either $x_1 < 1$ or $-x_1 < 0$ fixes the value of x_1, computing β^1 and β^2 simplifies to solving linear programs over the single variable x_2. This results in $\beta^1 = 2$, $\beta^2 = 6$, and the cut $x_1 + 2x_2 \leq 2$. This scenario notably demonstrates that even when linear programs define β^1 and β^2, a strengthened inequality can still result.

Yet another option is to apply conditional logic *sequentially* as opposed to *simultaneously*, to further tighten the inequality $\boldsymbol{\alpha x} \leq \beta$. To illustrate, reconsider the set S and the inequality $\boldsymbol{\alpha x} \leq \beta$ given in Example 1. If we select $R = 2$ with $\boldsymbol{\delta}_1\boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2\boldsymbol{x} \leq \gamma_2$ given by $x_1 \leq 1$ and $x_2 \leq 1$ respectively, and let $T_1 = \{(x_1, x_2) |\; 5x_1 + 2x_2 \leq 6, x_1 \text{ binary}, 0 \leq x_2 \leq 1\}$ and $T_2 = \{(x_1, x_2) |\; 5x_1 + 2x_2 \leq 6, 0 \leq x_1 \leq 1, x_2 \text{ binary}\}$, then $\lambda_1 = \lambda_2 = 1$ and $\kappa = 2$ will satisfy (12) since $(1 - x_1) + (1 - x_2) \leq 2 \;\forall\; \boldsymbol{x} \in S$. Here, $\beta^1 = 2$ and $\beta^2 = 5$ so that (13) becomes $3x_1 + (3/2)x_2 \leq 7/2$. In essence, we have simultaneously applied conditional logic via the constraints $x_1 \leq 1$ and $x_2 \leq 1$ on the given inequality $\boldsymbol{\alpha x} \leq \beta$. Suppose, on the other hand, that we sequentially apply conditional logic using the same inequalities $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2 \boldsymbol{x} \leq \gamma_2$ and the same sets T_1 and T_2 in the following manner. First, let $R = 1$ with $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ given by $x_1 \leq 1$ and T_1 as defined above. Using $\lambda = \kappa = 1$, we have $\beta^1 = 2$ so that (13) becomes $x_1 + 2x_2 \leq 2$. Next, redefine $\boldsymbol{\alpha x} \leq \beta$ as $x_1 + 2x_2 \leq 2$, let $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ be given by $x_2 \leq 1$, let T_1 be as given by T_2, and let $\lambda = \kappa = 1$. Then $\beta^1 = 1$, with the new cut (13) becoming $x_1 + x_2 \leq 1$. Thus, by sequentially applying conditional logic, the cut $5x_1 + 2x_2 \leq 6$ is effectively strengthened to $x_1 + x_2 \leq 1$, an improvement over the cut $3x_1 + (3/2)x_2 \leq 7/2$ obtained using the simultaneous approach. For this example, the additional strength afforded by sequential logic promotes the same strengthened inequality as the simultaneous approach of Example 1, but allows us to compute β^1 and β^2 as linear programs, as opposed to the integer programs. Of course, in applying sequential logic on larger and more complex problems, the inequalities $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ and the sets T_r can be strategically defined at each iteration so as to take maximum advantage of the previous-computed tightened inequalities.

3.2 Single-coefficient adjustment

The simple example of Section 3.1 motivates a slightly more involved application of conditional logic. In this section we will once again have $R = 2$ and invoke inequalities of the form $1 - x_j \geq 0$ and $x_j \geq 0$ to represent the restrictions $\gamma_1 - \boldsymbol{\delta}_1 \boldsymbol{x} \geq 0$ and $\gamma_2 - \boldsymbol{\delta}_2 \boldsymbol{x} \geq 0$. However, the set X will now be permitted to contain more than a single structural inequality. As we will see, by varying the sets T_1 and T_2, this generalization will permit us to subsume a variety of coefficient adjustment methods. Consider the example below.

Example 2. Let the sets S and X be generally defined as in (1) and (2) respectively, with $\boldsymbol{\alpha x} \leq \beta$ implied by X. Let $R = 2$ with $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2 \boldsymbol{x} \leq \gamma_2$ given as $x_k \leq 1$ and $-x_k \leq 0$ respectively for some $k \in \{1, \ldots, p\}$. Define T_1 and T_2 to be any relaxations of S that enforce x_k binary, and compute β^1 and β^2 as prescribed in (8). Since in the presence of the restriction x_k binary, the strict inequality $x_k < 1$ implies $x_k = 0$ and $-x_k < 0$ implies $x_k = 1$, we have

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha x} \mid \boldsymbol{x} \in T_1, x_k = 0\}\}$$

and

$$\beta^2 = \min\{\beta,\ \max\{\boldsymbol{\alpha x} \mid \boldsymbol{x} \in T_2, x_k = 1\}\}.$$

Letting $\lambda_1 = \lambda_2 = \kappa = 1$ so that $(1 - x_k) + (x_k) = 1\ \forall\ \boldsymbol{x} \in S$ and (12) is satisfied, we obtain the following form of the conditional logic cut (13).

$$\boldsymbol{\alpha x} + (\beta^1 - \beta^2)x_k \leq \beta^1 \tag{20}$$

Inequality (20) is obtained from $\boldsymbol{\alpha x} \leq \beta$ by adjusting the coefficient of a single variable x_k and by potentially decreasing the righthand-side (henceforth rhs) from β to the value β^1. To show the relationship between (20) and existing coefficient adjustment methods, we point out here that possibly weakened versions of (20) in which the coefficient adjustment on x_k is ensured to be either nonnegative or nonpositive can be obtained by letting $R = 1$ with $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ given by either $-x_k \leq 0$ or $x_k \leq 1$. For these two cases, we can define T_1 to be any relaxation of S with x_k binary and $\lambda_1 = \kappa = 1$ to obtain the two inequalities

$$\boldsymbol{\alpha x} + (\beta - \beta^2)x_k \leq \beta \tag{21}$$

and

$$\boldsymbol{\alpha x} + (\beta^1 - \beta)x_k \leq \beta^1, \tag{22}$$

respectively. Here, the notation β^1 and β^2 is used to reflect the identical values found in (20), and clearly indicate that (21) increases the single coefficient α_k on x_k in $\boldsymbol{\alpha x} \leq \beta$ by the nonnegative value $\beta - \beta^2$ while (22) decreases both the single coefficient α_k on x_k and the rhs of $\boldsymbol{\alpha x} \leq \beta$ by the nonnegative value $\beta - \beta^1$. Observe that Example 1 is a special case of Example 2 in that if we let $S, X, \lambda_1, \lambda_2$, and κ be as defined in

Example 1, we let $\boldsymbol{\alpha x} \leq \beta$ be given by $5x_1 + 2x_2 \leq 6$, and we let $T_1 = T_2 = S$ with $k = 1$, then (20) is precisely the inequality $2x_1 + 2x_2 \leq 2$. Moreover, (21) is $6x_1 + 2x_2 \leq 6$ and (22) is $x_1 + 2x_2 \leq 2$, exactly as computed in the discussion immediately following Example 1. Here, the point $(x_1, x_2) = (\frac{4}{5}, \frac{3}{5})$ is permitted by the inequalities (21) and (22), but is not feasible to the more general coefficient adjustment inequality (20).

As we will show throughout this section, a variety of coefficient adjustment methods for pure 0-1 linear programs can be cast within the framework of inequality (20). A recent paper by Dietrich, Escudero, and Chance [7] conducts a detailed study of coefficient increasing and coefficient decreasing methods, proposes new techniques that unify earlier works, and provides promising computational experience. These authors essentially consider special cases of (21) and (22) where the set S has $p = n$, and where $\boldsymbol{\alpha x} \leq \beta$ is a *knapsack* constraint, that is $0 < \alpha_j \leq \beta \; \forall \; j = 1, \ldots, n$. As noted in [7], while the choice of S for T_1 and T_2 promotes attractive values of β^1 and β^2, an optimization problem over $n-1$ binary variables must be solved to determine each value. Consequently, they exploit special problem structure in determining sets T_1 and T_2 that give rise to more computationally tractable problems. Escudero, Martello, and Toth [8, 9] continue the research in this vein. As pointed out in [7, 9], and in a different context in [20], the process of sequential lifting, which takes a valid linear inequality in a given space and "lifts" it to a higher-dimensional space, can be explained in terms of coefficient adjustment; this lifting turns out to be implementable as a special case of (21).

We address different special cases of (20) in Sections 3.2.1 through 3.2.7 below, continually referring back to Example 2 and varying only the sets T_1 and T_2. In certain of these examples, we show how special structures of S can be exploited. As all cited works deal with pure 0-1 linear programs, we restrict attention in these sections to those instances of S having $p = n$.

3.2.1 $T_1 = T_2 = \{\boldsymbol{x} \in \mathcal{R}^n | \; \boldsymbol{x}$ binary$\}$

Consider Example 2, and let T_1 and T_2 be the relaxations of S obtained by removing all structural constraints so that $T_1 = T_2 = \{\boldsymbol{x} \in \mathcal{R}^n | \; \boldsymbol{x} \text{ binary}\}$. With this definition, the values of β^1 and β^2 are readily calculated as

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha x}|\ \boldsymbol{x} \text{ binary}, x_k = 0\}\} = \min\{\beta,\ \sum_{j\neq k} \max\{0, \alpha_j\}\}$$

and

$$\beta^2 = \min\{\beta,\ \max\{\boldsymbol{\alpha x}|\ \boldsymbol{x} \text{ binary}, x_k = 1\}\} = \min\{\beta,\ \alpha_k + \sum_{j\neq k} \max\{0, \alpha_j\}\}.$$

If $\beta^2 < \beta$, then (21) simplifies to

$$\sum_{j\neq k} \alpha_j x_j + (\beta - \sum_{j\neq k} \max\{0, \alpha_j\})x_k \leq \beta, \tag{23}$$

while if $\beta^1 < \beta$, then (22) simplifies to

$$\sum_{j\neq k} \alpha_j x_j + (\alpha_k - (\beta - \sum_{j\neq k} \max\{0, \alpha_j\}))x_k \leq \sum_{j\neq k} \max\{0, \alpha_j\}. \tag{24}$$

The coefficient adjustment method proposed by Crowder, Johnson, and Padberg [2] to strengthen an inequality of the form $\boldsymbol{\alpha x} \leq \beta$ having $\alpha_j \leq 0\ \forall\ j$ and $\beta \leq 0$, by modifying some coefficient $\alpha_k < \beta$, is a special case of (23). In this instance, $\max\{0, \alpha_j\} = 0\ \forall\ j$, and (23) simplifies to $\sum_{j\neq k} \alpha_j x_j + \beta x_k \leq \beta$. More generally, Johnson, Kostreva, and Suhl [12] posed (23) for indices k with $\alpha_k < 0$ and (24) for indices k with $\alpha_k > 0$. We see from our derivation that these methods apply regardless of the sign of α_k, however it is clear that (23) for $\alpha_k \geq 0$ and (24) for $\alpha_k \leq 0$ would each necessarily be implied by $0 \leq \boldsymbol{x} \leq 1$. As noted in the following section, inequality (24) also arises as a special case of a cut posed by Dietrich, Escudero, and Chance [7], with inequality (23) a natural consequence.

3.2.2 T_1 and T_2 each contain a single structural inequality, together with $\boldsymbol{x}$ binary

By assigning T_1 and T_2 tighter approximations to S than $\{\boldsymbol{x} \in \mathcal{R}^n|\ \boldsymbol{x} \text{ binary}\}$ we can obtain stronger inequalities than those derived in Section 3.2.1. Suppose in Example 2 we let T_1 and T_2 be the set of binary points satisfying a single structural constraint of S, say $\boldsymbol{a}_1\boldsymbol{x} \leq b_1$ and $\boldsymbol{a}_2\boldsymbol{x} \leq b_2$, respectively. Then the resulting optimization problems for β^1 and β^2 are as follows.

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha x}|\ \boldsymbol{a}_1\boldsymbol{x} \leq b_1, \boldsymbol{x} \text{ binary}, x_k = 0\}$$

and

$$\beta^2 = \min\{\beta,\ \max\{\boldsymbol{\alpha x}|\ \boldsymbol{a}_2\boldsymbol{x} \leq b_2, \boldsymbol{x} \text{ binary}, x_k = 1\}.$$

Dietrich, Escudero, and Chance [7] developed the single-coefficient decreasing cut (22) with β^1 as given. Their idea naturally generalizes to the single-coefficient increasing inequality (21) with the above value β^2. As calculating β^1 requires solving a 0-1 knapsack problem, these authors suggest bounding procedures, including the trivially-solvable continuous relaxation and the relaxation obtained by ignoring the $\boldsymbol{a}_1\boldsymbol{x} \leq b_1$ restriction. This latter relaxation yields inequality (24) of Section 3.2.2 whenever $\beta^1 < \beta$. They also considered the *subset-sum problem* (SSP) special case where $\boldsymbol{a}_1\boldsymbol{x} \leq b_1$ is given by $\boldsymbol{\alpha x} \leq \beta$, as specialized solution and bounding procedures exist for this problem class (see [14]). As all the bounds they consider can be obtained by suitably redefining the sets T_1 or T_2, all the cuts they derive are special cases of (22) and (21).

3.2.3 T_1 and/or T_2 exploits plant location constraints

The relaxations T_1 and T_2 discussed in Sections 3.2.1 and 3.2.2 make no use of special structure in the constraint set of S. By exploiting available structure in S, sets T_1 and T_2 may be defined so that reduced computational effort is required to obtain β^1 and β^2. One such structure occurs when S enforces a plant location constraint of the form $\sum_{j \in C} x_j \leq h x_k$, where C is a nonempty subset of the binary variable indices not including k, and h is an integer satisfying $1 \leq h \leq |C|$. Let T_1 be the relaxation of S obtained by replacing $\boldsymbol{Ax} \leq \boldsymbol{b}$ in X with the single plant location constraint and $\boldsymbol{\alpha x} \leq \beta$. Since $\boldsymbol{\alpha x} \leq \beta$ is implied by T_1, the calculation of β^1 given in (8) simplifies to

$$\beta^1 = \max\,\{\boldsymbol{\alpha x} |\ \sum_{j \in C} x_j \leq h x_k, \boldsymbol{\alpha x} \leq \beta, \boldsymbol{x} \text{ binary}, x_k = 0\}.$$

Hoffman and Padberg [11] developed the instance of (22) with this value of β^1. Observe that the plant location constraint, together with $x_k = 0$ and $\boldsymbol{x}$ binary, enforce $x_j = 0\ \forall\ j \in C \cup \{k\}$, decreasing the problem size by $|C| + 1$ binary variables, and creating a subset-sum problem as discussed in Section 3.2.2. In practice, Hoffman and Padberg do not solve this optimization problem, but instead suggest approximating the optimal solution with an upperbound. Their bound is equivalent to the value β^1 defined by letting T_1 be the (weaker) relaxation of S constructed by replacing $\boldsymbol{Ax} \leq \boldsymbol{b}$ with only the plant location constraint, so that $\beta^1 = \min\{\beta,\ \sum_{j \notin C \cup \{k\}} \max\{0, \alpha_j\}\}$. Here, since $\boldsymbol{\alpha x} \leq \beta$ is not enforced in the set T_1, the minimization operation is once again needed and the problem of calculating β^1 reduces to the form discussed in Section

3.2.1.

Of course, identical arguments can be developed for computing β^2 if T_2 is given by $T_2 = \{\boldsymbol{x} | \sum_{j \in C} x_j \leq h(1-x_k),\ \boldsymbol{\alpha x} \leq \beta,\ \boldsymbol{x}$ binary$\}$ for some index k and set C as defined above, since in this case $x_k = 1$ implies $x_j = 0\ \forall\ j \in C$. If such structure arises in both sets T_1 and T_2, then the strengthened inequality (20) may be superior to either (22) or (21). In any case, such plant location restrictions serve to reduce the size of the associated optimization problems regardless of the other inequalities in the sets T_1 and T_2. Furthermore, since the essential ingredient in the argument for computing β^1 is that $x_k = 0$ implies $x_j = 0\ \forall\ j \in C$, the plant location restriction $\sum_{j \in C} x_j \leq hx_k$ can be generalized to $\sum_{j \in C} a_j x_j \leq hx_k$ when $a_j > 0\ \forall\ j \in C$. Likewise, $\sum_{j \in C} x_j \leq h(1-x_k)$ can be generalized to $\sum_{j \in C} a_j x_j \leq h(1-x_k)$.

3.2.4 T_1 and/or T_2 exploits variable upper bounds

Besides plant location constraints, there are other useful structures of S that may permit us to define sets T_1 and T_2 so that β^1 and β^2 can be computed with reduced effort. For instance, suppose that the index set of binary variables can be partitioned into a set of upper bounding variables U and a set of selection variables C, where the selection variables are grouped into (not necessarily disjoint) sets C_u of C for $u \in U$ such that S enforces the variable upper-bounding (VUB) constraints

$$x_j \leq x_u\ \forall\ j \in C_u, u \in U.$$

Then, by defining T_1 to be the relaxation of S obtained by replacing $\boldsymbol{Ax} \leq \boldsymbol{b}$ in X with $\boldsymbol{\alpha x} \leq \beta$ and these VUB constraints, the value β^1 specified in (8) can be computed as

$$\beta^1 = \max\{\boldsymbol{\alpha x} |\ x_j \leq x_u,\ \forall\ j \in C_u, u \in U, \boldsymbol{\alpha x} \leq \beta, \boldsymbol{x} \text{ binary}, x_k = 0\},$$

since $\boldsymbol{\alpha x} \leq \beta$ is enforced by T_1. If k is chosen to be in U so that x_k is an upper-bounding variable, then fixing x_k to 0 implies that $x_j = 0\ \forall\ j \in C_k$, and the size of the optimization problem to compute β^1 is reduced by $|C_k| + 1$ binary variables. A coefficient adjustment strategy posed by Dietrich, Escudero, and Chance [7] is precisely (22) with this value of β^1, where $\boldsymbol{\alpha x} \leq \beta$ is a knapsack constraint. Observing that β^1 is still likely to be computationally challenging to compute, these authors suggest an upper bound. Essentially, they decompose the above maximization problem into $|U| - 1$ subproblems, one corresponding to each $u \in U - \{k\}$, where each subproblem contains only the associated variables x_j having $j \in C_u \cup \{u\}$. In our notation, this bound is $\beta^1 = \min\{\beta,\ \sum_{u \in U-\{k\}} \theta_u\}$, where

$$\theta_u = \max\{\boldsymbol{\alpha x} |\ x_j \leq x_u\ \forall\ j \in C_u, \boldsymbol{\alpha x} \leq \beta, \boldsymbol{x} \text{ binary}, x_j = 0\ \forall\ j \notin C_u \cup \{u\}\} \text{ for each } u \in U - \{k\}.$$

Of course, when computing θ_u for each $u \in U - \{k\}$ we can also fix $x_j = 0\ \forall\ j \in C_u \cap C_k$. See [3, 7] for details.

3.2.5 T_1 exploits sets of capacity expansion constraints

Motivated by manufacturing problems involving resource allocation and capacity expansion questions, Dietrich, Escudero, and Chance [7] sought to strengthen a knapsack inequality $\boldsymbol{\alpha x} \leq \beta$ using restrictions of the form

$$\sum_{j \in C_u} a_{uj} x_j \leq d_u + a_u x_u\ \forall\ u \in U$$

where for each $u \in U$, the integer scalars a_{uj}, d_u, and a_u satisfy $a_{uj} > 0\ \forall\ j \in C_u$, $d_u < \sum_{j \in C_u} a_{uj}$, and $a_u > 0$. Referred to as *capacity expansion constraints*, these inequalities generalize both the plant location constraints considered in Section 3.2.3 and the VUB restrictions discussed in Section 3.2.4. The coefficient adjustment technique proposed by these authors turns out to be the special case of (22) where $k \in U$ and T_1 is defined to be the relaxation of S obtained by replacing $A\boldsymbol{x} \leq \boldsymbol{b}$ in X with the capacity expansion constraints and the inequality $\boldsymbol{\alpha x} \leq \beta$, so that

$$\beta^1 = \max\{\boldsymbol{\alpha x} |\ \sum_{j \in C_u} a_{uj} x_j \leq d_u + a_u x_u\ \forall\ u \in U, \boldsymbol{\alpha x} \leq \beta, \boldsymbol{x} \text{ binary}, x_k = 0\}.$$

Since the optimization problem to compute β^1 is likely difficult to solve, even when the sets C_u are disjoint or when $a_{uj} = 1\ \forall\ j \in C_u, u \in U$, Dietrich, Escudero, and Chance employ a simpler alternative. Based on a similar decomposition approach to the one found in Section 3.2.4, they use the upper bound $\beta^1 = \min\{\beta,\ \sum_{u \in U} \phi_u\}$, where

$$\phi_u = \max\{\boldsymbol{\alpha x} |\ \sum_{j \in C_u} a_{uj} x_j \leq d_u + a_u x_u, \boldsymbol{\alpha x} \leq \beta, \boldsymbol{x} \text{ binary}, x_j = 0\ \forall\ j \notin C_u \cup \{u\}, x_k = 0\} \text{ for each } u \in U.$$

3.2.6 T_1 and/or T_2 exploit logical implications

Sections 3.2.3 through 3.2.5 attempt to minimize the computational effort required to compute β^1 and/or β^2 by reducing the problem size. The central idea of these three sections is to selectively incorporate constraints within the sets T_1 and T_2 so that the fixing of the binary variable x_k to the value 0 or 1 respectively would dictate binary values for other variables. In Section 3.2.4, for example, we showed how to exploit inequalities of the form $x_j \le x_u$ when the variable x_u is fixed to the value 0. It turns out that there exists an entire family of *logical implication inequalities* that can be analogously exploited.

Suppose, either given as part of the set S, or through some probing and/or preprocessing steps [16], that we are able to obtain logical implication inequalities that restrict the values of certain binary variables in terms of other variables as follows. For a given binary variable x_k, let us define the sets I_1 and I_2 so that $x_k = 0$ implies $x_i = 0 \ \forall \ i \in I_1$ and implies $x_i = 1 \ \forall \ i \in I_2$. Similarly, define the sets I_3 and I_4 so that $x_k = 1$ implies $x_i = 0 \ \forall \ i \in I_3$ and implies $x_i = 1 \ \forall \ i \in I_4$. Mathematically, we have

$$x_i \le x_k \ \forall \ i \in I_1 \text{ and } x_i + x_k \ge 1 \ \forall \ i \in I_2 \tag{25}$$

and

$$x_i + x_k \le 1 \ \forall \ i \in I_3 \text{ and } x_i \ge x_k \ \forall \ i \in I_4. \tag{26}$$

Incorporating these two pairs of restrictions into T_1 and T_2 respectively, simplifies the computing of β^1 and β^2 by fixing the variables in the associated sets $I_1 \cup I_2$ and $I_3 \cup I_4$. Since the reduction in problem size is independent of all other constraints present within T_1 and T_2, any of the methods previously discussed can potentially be improved by the inclusion of such logical implication constraints. For instance, suppose T_1 is defined to be the set of binary points satisfying a single structural constraint of X, say $\boldsymbol{a}\boldsymbol{x} \le b$, together with any logical implication restrictions expressed in terms of the sets I_1 and I_2. Then β^1 can be computed as

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha}\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} \le b, x_i \le x_k \ \forall \ i \in I_1, x_i + x_k \ge 1 \ \forall \ i \in I_2, \boldsymbol{x} \text{ binary}, x_k = 0\},$$

which is equivalent to

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha}\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} \le b, x_i = 0 \ \forall \ i \in I_1, x_i = 1 \ \forall \ i \in I_2, \boldsymbol{x} \text{ binary}, x_k = 0\}.$$

Similarly, if T_2 is defined to be the set of binary points satisfying $\boldsymbol{ax} \leq b$ together with any logical implication restrictions expressed in terms of the sets I_3 and I_4, then β^2 can be computed as

$$\beta^2 = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{ax} \leq b, x_i = 0\ \forall\ i \in I_3, x_i = 1\ \forall\ i \in I_4, \boldsymbol{x} \text{ binary}, x_k = 1\}.$$

The instances of (21) and (22) with these values of β^1 and β^2 are, in fact, coefficient adjustment methods presented by Escudero, Martello, and Toth [8, 9]. Observe that these optimization problems for β^1 and β^2 are of the form discussed in Section 3.2.2, but contain $|I_1| + |I_2|$ and $|I_3| + |I_4|$, respectively fewer variables.

3.2.7 T_1 and/or T_2 exploits cover inequalities

In addition to exploiting variable fixing schemes to reduce the computational effort of calculating β^1 and β^2, we can also construct the sets T_1 and T_2 so that specialized exact and heuristic algorithms can be employed. One such structure occurs when T_1 and T_2 are defined as in the previous section but must additionally enforce a *cover* constraint. A cover constraint is an inequality of the form $\sum_{j \in C} x_j \leq h$ where C is an index set of binary variables and h is an integer satisfying $1 \leq h \leq |C| - 1$. Then β^1 and β^2 are computed as

$$\beta^1 = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{ax} \leq b, x_i \leq x_k\ \forall\ i \in I_1, x_i + x_k \geq 1\ \forall\ i \in I_2, \sum_{j \in C} x_j \leq h, \boldsymbol{x} \text{ binary}, x_k = 0\}$$

and

$$\beta^2 = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{ax} \leq b, x_i + x_k \leq 1\ \forall\ i \in I_3, x_i \geq x_k\ \forall\ i \in I_4, \sum_{j \in C} x_j \leq h, \boldsymbol{x} \text{ binary}, x_k = 1\}.$$

The above maximization problems are 0-1 knapsack problems with cover side-constraints, special instances of a two-constraint knapsack problem for which specialized techniques are available [15]. Escudero, Martello, and Toth [9] propose instances of (21) and (22) having the above values of β^1 and β^2. These authors also consider the subset sum special case where $\boldsymbol{\alpha x} \leq \beta$ is precisely $\boldsymbol{ax} \leq b$, as well as the clique side-constraint special case where either $b = 1$. For both cases they report promising computational results.

3.3 Multi-coefficient adjustment

Section 3.2 details various ways in which the conditional logic inequality (13) can be used to tighten $\boldsymbol{\alpha x} \leq \beta$ by adjusting a single coefficient α_k and possibly the scalar β. Permitting only a single coefficient to be adjusted is rather restrictive. A far richer family of linear inequalities can potentially be generated by broadening our approach to address multiple coefficients. Two imposed limitations on the construction of (13) directly lead to single-coefficient adjustment: namely, R cannot exceed 2, and the inequalities $\gamma_1 - \boldsymbol{\delta}_1\boldsymbol{x} \geq 0$ and $\gamma_2 - \boldsymbol{\delta}_2\boldsymbol{x} \geq 0$ must be of the form $1 - x_k \geq 0$ and $x_k \geq 0$ respectively. We now relax these restrictions to accommodate a single structural inequality $\sum_{j \in C} a_j x_j \leq a_0$ in X, where C represents the index set of variables x_j having nonzero coefficients a_j, and where a_0 is a scalar. We also let $R = |C| + 1$. It turns out that the resulting conditional logic inequalities subsume various published tightening methods which depend on either covering structures, clique structures, or special ordered sets. As with Sections 3.2.1 through 3.2.7, and motivated by published results, we consider only pure $0-1$ linear programs so that $p = n$ in S. However, the techniques readily apply to mixed $0 - 1$ linear programs. The general methodology is illustrated in the following example.

> **Example 3.** Let the sets S and X be generally defined as in (1) and (2) respectively, with $\boldsymbol{\alpha x} \leq \beta$ implied by X. Consider any inequality $\sum_{j \in C} a_j x_j \leq a_0$ in X, where C represents the index set of binary variables x_j having nonzero coefficients a_j, and where a_0 is a scalar. For notational convenience, assume without loss of generality that the indices are labeled in such a manner that $C = \{1, \ldots, |C|\}$. Partition C into the sets C^+ and C^-, where $C^+ = \{j \in C | \; a_j > 0\}$ and $C^- = \{j \in C | \; a_j < 0\}$. Let $R = |C| + 1$, so that for each $r = 1, \ldots, R-1$, r is either in C^+ or C^-. Define $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ to be $-x_r \leq 0$ if $r \in C^+$, $x_r \leq 1$ if $r \in C^-$, and $\boldsymbol{\delta}_R \boldsymbol{x} \leq \gamma_R$ to be $\sum_{j \in C} a_j x_j \leq a_0$. For each $r = 1, \ldots, R-1$, let T_r be any relaxation of S enforcing x_r binary, and let T_R be any relaxation of S. Compute $\beta^r \; \forall r$ as defined in (8). Since in the presence of binary restrictions $x_r < 1$ $(-x_r < 0)$ implies $x_r = 0$ $(x_r = 1)$, we have

Example (continued).

$$\beta^r = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{x} \in T_r, x_r = 1\}\}\ \forall\ r \in C^+,$$

$$\beta^r = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{x} \in T_r, x_r = 0\}\}\ \forall\ r \in C^-,$$

and

$$\beta^R = \min\{\beta,\ \max\{\boldsymbol{\alpha x} |\ \boldsymbol{x} \in T_R, \sum_{j \in C} a_j x_j < a_0\}\}.$$

(The value β^R can be made computationally plausible by incorporating problem structure to accommodate the strict inequality $\sum_{j \in C} a_j x_j < a_0$. For instance, given that the scalars a_0 and $a_j\ \forall\ j \in C$ are integers, one option is to instead use $\sum_{j \in C} a_j x_j \leq a_0 - 1$.) Choose $\lambda_r = a_r\ \forall\ r \in C^+$, $\lambda_r = -a_r\ \forall\ r \in C^-$, $\lambda_R = 1$, and $\kappa = a_0 - \sum_{r \in C^-} a_r$ so that $\sum_{r \in C^+} a_r x_r + \sum_{r \in C^-} -a_r(1 - x_r) + (a_0 - \sum_{r \in C} a_r x_r) \leq a_0 - \sum_{r \in C^-} a_r\ \forall\ \boldsymbol{x} \in S$, and (12) is satisfied. (Recall, that κ must be strictly positive. We can assume without loss of generality that $\kappa > 0$, since if $\kappa < 0$, then no binary solution to $\sum_{r \in C} a_r x_r \leq a_0$ exists so that $S = \emptyset$ while if $\kappa = 0$, then we have that $x_r = 1\ \forall\ r \in C^-$ and $x_r = 0\ \forall\ r \in C^+$ at all binary solutions to $\sum_{r \in C} a_r x_r \leq a_0$, and hence at all solutions to S.) Using these given values, the conditional logic cut (13) becomes

$$\boldsymbol{\alpha x} + \sum_{r \in C} \frac{a_r}{\kappa}(\beta^R - \beta^r)x_r \leq \beta + \frac{a_0}{\kappa}(\beta^R - \beta) + \sum_{r \in C^-} \frac{a_r}{\kappa}(\beta - \beta^r). \tag{27}$$

The multi-coefficient adjustment inequality (27) is obtained from $\boldsymbol{\alpha x} \leq \beta$ by changing the coefficients α_j having $j \in C$, and the rhs β. A variety of published coefficient adjustment methods are special cases of (27) where $C^- = \emptyset$. To facilitate this comparison, we note here that when $C^- = \emptyset$ then $\kappa = a_0$, and (27) simplifies to

$$\boldsymbol{\alpha x} + \sum_{r \in C} \frac{a_r}{a_0}(\beta^R - \beta^r)x_r \leq \beta^R. \tag{28}$$

In the same manner that the single-coefficient adjustment inequality (20) was relaxed to obtain (21) and (22), we can also compute two modified versions of (28) for which all the adjustments are either nonnegative or nonpositive. To illustrate, suppose in Example 3 we let $C^- = \emptyset$ and $R = |C|$. Otherwise preserving the values of $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$, T_r, and λ_r for all $r = 1, \ldots, |C|$, as well as κ, we produce the inequality

$$\boldsymbol{\alpha x} + \sum_{r \in C} \frac{a_r}{a_0}(\beta - \beta^r)x_r \leq \beta. \tag{29}$$

Alternately, suppose in Example 3 we once again have $C^- = \emptyset$, but now $R = 1$. Selecting $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ to be $\sum_{j \in C} a_j x_j \leq a_0$, T_1 to be any relaxation of S, $\lambda_1 = 1$, and $\kappa = a_0$ we obtain (30), where β^R is used to denote β^1 for consistency with (27).

$$\boldsymbol{\alpha x} + \sum_{r \in C} \frac{a_r}{a_0}(\beta^R - \beta)x_r \leq \beta^R. \tag{30}$$

Since a_0 and a_r $\forall\, r \in C$ are positive, (29) increases each coefficient α_r indexed by $r \in C$ in $\boldsymbol{\alpha x} \leq \beta$ by the nonnegative amount $(a_r/a_0)(\beta - \beta^r)$, while (30) decreases the rhs of $\boldsymbol{\alpha x} \leq \beta$ by the nonnegative quantity $(\beta - \beta^R)$ and decreases every coefficient α_r, $r \in C$, by the nonnegative quantity $(a_r/a_0)(\beta - \beta^R)$.

The single-coefficient adjustment of Section 3.2 is a special case of the multi-coefficient strategy, so that all related works can be characterized in terms of (27). Suppose that the inequality $\sum_{j \in C} a_j x_j \leq a_0$ has $|C| = 1$ with $a_1 = 1$ and $a_0 = 1$ so that it can be considered without loss of generality to be $x_1 \leq 1$. Then $C^+ = \{1\}$, $C^- = \emptyset$, $R = 2$, $\kappa = 1$, and $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ and $\boldsymbol{\delta}_2 \boldsymbol{x} \leq \gamma_2$ are given by $-x_1 \leq 0$ and $x_1 \leq 1$ respectively. For this special case, inequality (27) is precisely (20) and the single-coefficient relaxations (21) and (22) are (29) and (30), respectively.

Various published cover-based and SOS-based coefficient adjustment strategies for pure 0-1 problems can be explained in terms of (29) and (30), including the two "ideal" reformulation methods of Dietrich, Escudero, and Chance [7]. Both of these methods strengthen an inequality $\boldsymbol{\alpha x} \leq \beta$ having $\boldsymbol{\alpha} \geq \mathbf{0}$ by exploiting a cover inequality which, as defined in Section 3.2.7, is the special instance of $\sum_{j \in C} a_j x_j \leq a_0$ where $a_j = 1$ $\forall\, j \in C$ and a_0 is an integer having $1 \leq a_0 \leq |C| - 1$. Using $p = n$ and $T_r = S$ $\forall\, r$, one such method (29) for $C \subseteq \{1, \ldots, p\}$ while the other yields (30) where C is an *implied* cover of the knapsack inequality $\boldsymbol{\alpha x} \leq \beta$. In our derivations, we do not require that $\boldsymbol{\alpha} \geq \mathbf{0}$ (so that $C^- = \emptyset$), nor do we rely on any special covering structures in $\sum_{j \in C} a_j x_j \leq a_0$.

Just as with the single-coefficient adjustment of Section 3.2, the relaxations T_r for the multi-coefficient method can be strategically defined to reduce the computational effort in calculating β^r. In fact, all of the relaxation arguments discussed in Sections 3.2.1 through 3.2.7 can be analogously applied in determining

(27). For example, we can define the sets T_r to be $T_r = \{\boldsymbol{x} \in \mathcal{R}^n | \ \boldsymbol{x} \text{ binary}\}$ when $p = n$ as in Section 3.2.1, or we can define T_r to include a single structural constraint in addition to the binary restrictions as in Section 3.2.2. For the instances where $\sum_{j \in C} a_j x_j \leq a_0$ is a cover, Dietrich, Escudero, and Chance [7] employ this latter approach in computing (30), so that calculating β^R requires solving a two-constraint 0-1 linear program, and for (29) when each α_j is binary, so that the optimization problems to calculate $\beta^r, r = 1, \ldots, |C|$ become 0-1 knapsacks whose non-zero objective function coefficients are all 1. An efficient procedure for this class of problems is presented in [6]. Naturally, the same variable fixing schemes developed for single-coefficient adjustment can be used to extend these cover-based cuts. Escudero, Martello, and Toth [8, 9] show how the logical implication constraints in Section 3.2.6 can be included to further enhance the cover-based methods of [7].

An interesting instance of Example 3 is the special case where $\sum_{j \in C} a_j x_j \leq a_0$ is a *clique* inequality. This type of cover inequality is notable for it leads to the fixing of all the variables indexed by C in the calculation of β^r for every r. More precisely, if $\sum_{j \in C} a_j x_j \leq a_0$ is of the form $\sum_{j \in C} x_j \leq 1$ then

$$\beta^r = \min\{\beta, \ \max\{\boldsymbol{\alpha x} | \ \boldsymbol{x} \in T_r, x_j = 0 \ \forall \ j \in C - \{r\}, x_r = 1\}\} \ \forall \ r = 1, \ldots, R-1$$

and

$$\beta^R = \min\{\beta, \ \max\{\boldsymbol{\alpha x} | \ \boldsymbol{x} \in T_R, x_j = 0 \ \forall \ j \in C\}\}.$$

Dietrich, Escudero, and Chance [7] employ this latest value of β^R, where T_R includes a single structural constraint, to compute (30). Escudero, Martello, and Toth more fully exploit cliques in [8, 9], developing instances of both (29) and (30) where the sets T_r include variable upper bounding restrictions as in Section 3.2.4, and a form of capacity expansion constraints as in Section 3.2.5.

Clique structure is not only useful when arising in the inequality $\sum_{j \in C} a_j x_j \leq a_0$, but it can also be advantageously used in defining the relaxations T_r. Sets of clique inequalities, known as *special ordered set* (SOS) constraints, are restrictions of the form $\sum_{j \in C_u} x_j \leq 1$ for $u = 1, \ldots, U$, where the sets C_u partition the index set of binary variables. Such a partition will always exist, as the trivial partition has $C_u = \{u\}$ for each $u = 1, \ldots, p$. Johnson, Kostreva, and Suhl [12] consider the scenario where X in (2) is given by $X = \{\boldsymbol{x} \in \mathcal{R}^n | \ \boldsymbol{\alpha x} \leq \beta, \sum_{j \in C_u} x_j \leq 1 \ \forall \ u = 1, \ldots, U, \boldsymbol{x} \geq \boldsymbol{0}\}$, and suggest two tightening methods for the inequality $\boldsymbol{\alpha x} \leq \beta$. Hoffman and Padberg [11] essentially study this same system with the additional requirement that $\boldsymbol{\alpha} \geq \mathbf{0}$, and propose an alternate strengthening method. A clarification of [12] and a generalization of the techniques in [11, 12] are given in Lougee-Heimer [13]. These results directly derive from our conditional logic arguments in the following manner. Consider any $q \in \{1, \ldots, U\}$ and assume without loss of generality that $C_q = \{1, \ldots, |C_q|\}$. Then the key element in the method of Lougee-Heimer is precisely (27) with $\kappa = 1, a_j = 1 \; \forall \; j \in C \equiv C_q$, and $C^- = \emptyset$, so that the inequality takes the form

$$\boldsymbol{\alpha x} + \sum_{r \in C_q} (\beta^R - \beta^r) x_r \leq \beta^R, \tag{31}$$

where $R = |C_q| + 1$ and the sets T_r are defined to be all the binary points satisfying the SOS-constraints. This leads to the following simple statements of β^r.

$$\beta^r = \min\{\beta, \; \alpha_r + \psi\} \; \forall \; k = 1, \ldots, |C_q| \text{ and } \beta^R = \min\{\beta, \; \psi\}$$

where

$$\psi = \sum_{u \neq q} \max\{0, \; \max\{\alpha_j \,|\, j \in C_u\}\}.$$

The adjustments of the coefficients indexed by C_q in (31) do not alter the value of $\max\{0, \; \max\{\alpha_j \,|\, j \in C_u\}\}$ for any u so as noted in [13], the remaining coefficients indexed by C_u, $u \neq q$, can be simultaneously adjusted as in (31). The resulting simultaneous SOS-adjustment inequality does not rely on the nonnegativity of $\boldsymbol{\alpha}$ and, although SOS-constraints are a type of cover inequalities, it is potentially tighter than the cover-based methods of Dietrich, Escudero, and Chance [7] which follow from the relaxations (29) and (30).

Interestingly, the approach of Lougee-Heimer [13] gives rise to a novel interpretation of our SOS-tightened restriction (31). Suppose that for each $u = 1, \ldots, U$ we define x_{n+u} to be the binary slack variable on the associated inequality $\sum_{j \in C_u} x_j \leq 1$ so that we have $\sum_{j \in C'_u} x_j = 1$, where $C'_u = \{C_u \cup \{n+u\}\}$. The inequality $\boldsymbol{\alpha x} \leq \beta$ can then be viewed as $\sum_{j=1}^{n'} \alpha_j x_j \leq \beta$ where $n' = n + U$ and $\alpha_j = 0 \; \forall \; j = n+1, \ldots, n'$. Now, given any $q \in \{1, \ldots, U\}$ where we once again assume without loss of generality that $C_q = \{1, \ldots, |C_q|\}$, we can compute β^r for each $r \in C'_q$ as $\beta^r = \min\{\beta, \alpha_r + \psi'\}$ where $\psi' = \sum_{u \neq q} \max\{\alpha_j \,|\, j \in C'_u\}$ and obtain (29) as $\boldsymbol{\alpha x} + \sum_{r \in C'_q} (\beta - \beta^r) x_r \leq \beta$. Since by definition $x_{n+q} = 1 - \sum_{j \in C_q} x_j$ and $\beta^{|C'(q)|} = \beta^R$ from Example 3, we obtain the inequality given by (31). In essence, for any given $q \in \{1, \ldots, U\}$, the conditional logic argument applied to the nonnegativity restriction on the slack variable $-x_{n+q} \leq 0$ leads to the same tightened inequality as when applied to the inequality $\sum_{k \in C_q} x_k \leq 1$.

3.4 Exploiting structure within $\delta_r \boldsymbol{x} \leq \gamma_r$

In our discussion of Section 3.3, we showed how a single structural inequality $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ can lead to strengthened representations of S. Special structure in any family of $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ inequalities can also be exploited to derive tightened representations of S. As noted in (12), any collection of inequalities $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$, $r = 1, \ldots, R$, can be potentially utilized, provided there exists a positive scalar κ and nonnegative multipliers λ_r so that $\sum_{r=1}^{R} \lambda_r (\gamma_r - \boldsymbol{\delta}_r \boldsymbol{x}) \leq \kappa \; \forall \; \boldsymbol{x} \in S$. These relatively mild conditions permit considerable latitude. The following example shows how one type of structure, variable upper bounding, can be exploited.

> **Example 4.** Consider a set S defined as in (1) and (2) where $X = \{\boldsymbol{x} \in \mathcal{R}^n | \; A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{\alpha x} \leq \beta, 0 \leq x_k \leq x_{k+1}$ for $k = 1, \ldots, K, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{1}\}$. Let $R = K + 2, \lambda_r = 1 \; \forall \; r = 1, \ldots, R$, and define $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ as $x_{r-1} \leq x_r$ for $r = 1, \ldots R$, where for notational convenience we let $x_0 = 0$ and $x_R = 1$ in these $\boldsymbol{\delta}_r \boldsymbol{x} \leq \gamma_r$ inequalities. Further let $\kappa = 1$ so that (12) is satisfied, and for each $r = 1, \ldots R$, let the set T_r be any relaxation of S so that x_r and x_{r-1} are binary. Then (8) gives us that
>
> $$\beta^r = \min\{\beta, \; \max\{\boldsymbol{\alpha x} | \; \boldsymbol{x} \in T_r, x_{r-1} = 0, x_r = 1\}\} \; \forall \; r = 1, \ldots R,$$
>
> and we obtain the conditional logic cut (13) as
>
> $$\boldsymbol{\alpha x} + \sum_{r=1}^{R-1} (\beta^{r+1} - \beta^r) x_r \leq \beta^R.$$

To illustrate Example 4 with specific numerical values, suppose that $S = \{\boldsymbol{x} \in \mathcal{R}^5 | \; \boldsymbol{x} \in X, x_j \text{ binary } \forall \; j = 1, \ldots, 5\}$ where $X = \{\boldsymbol{x} \in \mathcal{R}^5 | \; 2x_1 - x_2 + x_3 - x_4 + x_5 \leq 2, 0 \leq x_1 \leq x_2 \leq 1, 0 \leq x_j \leq 1 \text{ for } j = 3, 4, 5\}$. Since $0 \leq x_1 \leq x_2 \leq 1$, we have $K = 1$, $R = 3$, and $\lambda_1 = \lambda_2 = \lambda_3 = \kappa = 1$. Let $\boldsymbol{\alpha x} \leq \beta$ be given by $2x_1 - x_2 + x_3 - x_4 + x_5 \leq 2$, and set $T_k = S \; \forall \; k = 1, 2, 3$. Then

$$\beta^1 = \min\{2, \; \max\{2x_1 - x_2 + x_3 - x_4 + x_5 | \; \boldsymbol{x} \in T_1, x_1 = 1\}\} = 2,$$

$$\beta^2 = \min\{2, \; \max\{2x_1 - x_2 + x_3 - x_4 + x_5 | \; \boldsymbol{x} \in T_2, x_1 = 0, x_2 = 1\}\} = 1,$$

and

$$\beta^3 = \min\{2, \; \max\{2x_1 - x_2 + x_3 - x_4 + x_5 | \; \boldsymbol{x} \in T_3, x_2 = 0\}\} = 2$$

so that the conditional logic cut (13) becomes $2x_1 + 0x_2 + x_3 - x_4 + x_5 \leq 2$. Observe that the point $(x_1, x_2, x_3, x_4, x_5) = (\frac{1}{2}, 1, 1, 0, 1)$ is feasible to the set X but violates the newly-generated cut. Note that regardless of the selected variable index $j \in \{1, 2, 3, 4, 5\}$, no single-coefficient adjustment producing an inequality of the form (20) will strengthen $\boldsymbol{\alpha x} \leq \beta$.

Variable upper bounding is but one example of structure which can be leveraged in defining $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$. The crucial ingredient in deciding what structures to incorporate is the identification of inequalities $\boldsymbol{\delta}_r\boldsymbol{x} \leq \gamma_r$ for $r = 1, \ldots R$, that not only satisfy (12) for some chosen scalar κ, but also permit a strategic computation of the values β^r of (8) in accordance with the strict inequality $\boldsymbol{\delta}_r\boldsymbol{x} < \gamma_r$. Such is the case with our above inequalities $x_r < x_{r+1}$.

4 Conditional Logic Extensions

In Section 2 we presented a general conditional logic framework for tightening the continuous relaxation of S. We now discuss ways in which this unifying perspective may be extended even further.

Our conditional logic approach derives linear restrictions of the form (13) by appropriately surrogating quadratic inequalities (10) arising from conditional logic statements of the form (5). In the same spirit, it is possible to derive a richer family of valid linear inequalities by surrogating nonlinear polynomials arising from more complex, *compound conditional logic* statements for $\boldsymbol{x} \in S$ of the form

$$\text{if } \ \boldsymbol{\delta}^1\boldsymbol{x} < \gamma^1 \text{ and if } \boldsymbol{\delta}^2\boldsymbol{x} < \gamma^2 \text{ and if } \ldots \text{and if } \boldsymbol{\delta}^T\boldsymbol{x} < \gamma^T, \ \text{ then } \ \boldsymbol{\alpha x} \leq \beta_c^*$$

$$\text{where } \beta_c^* \equiv \max\{\boldsymbol{\alpha x} \mid \boldsymbol{x} \in S, \boldsymbol{\delta}^t\boldsymbol{x} < \gamma^t \ \forall \ t = 1, \ldots, T\},$$

and the subscript c is used to distinguish the associate higher-dimensional parameter from its quadratic counterpart. As in Section 2, the key is to define the nonlinear inequalities and associated multipliers so that when the inequalities are surrogated together, the resulting coefficients on the nonlinear terms are 0. To illustrate this procedure, we present the following example.

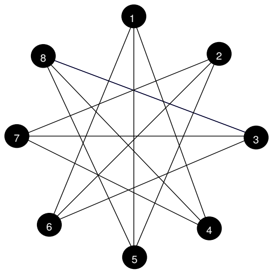

Figure 1: The web $W(8,3)$

Example 5. Consider the vertex packing polytope, S_P, of the graph depicted in Figure 1. The vertex packing polytope of a graph with vertex set V and edge set E is defined as the set of binary points in $|V|$-space satisfying $x_i + x_j \leq 1$ for all $(i,j) \in E$. In this example, the graph is a web $W(8,3)$ with $V = \{1,\ldots,8\}$ and $E = \{(i,j) | \, j = (i+3) \bmod 8, (i+4) \bmod 8, (i+5) \bmod 8 \;\forall\, i \in V\}$.

A valid, but redundant, inequality for S_P is $\sum_{j=1}^{8} x_j \leq 4$, obtained by summing all the defining inequalities of the polytope and dividing by three. While this inequality can be strengthen using "if, then" implications as described in Section 2, still stronger inequalities can be derived using "if and if, then" implications as we now show.

Let $R = 2$, $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ be $-x_1 \leq 0$, $\boldsymbol{\delta}_2 \boldsymbol{x} \leq \gamma_2$ be $x_1 \leq 1$, and $\boldsymbol{\alpha x} \leq \beta$ be $\sum_{j=1}^{8} x_j \leq 4$. As explained in Section 2, the conditional logic statements for $\boldsymbol{x} \in S_P$,

$$\text{if } \boldsymbol{\delta}_r \boldsymbol{x} < \gamma_r \text{ then } \boldsymbol{\alpha x} \leq \beta^r \text{ for } r = 1,2,$$

promote quadratic inequalities, $x_1(\beta^1 - \sum_{j=1}^{8} x_j) \geq 0$, and $(1-x_1)(\beta^2 - \sum_{j=1}^{8} x_j) \geq 0$, having $\beta^1 = \max\{\sum_{j=1}^{8} x_j | \boldsymbol{x} \in S_P, x_1 = 1\} = 3$, and $\beta^2 = \max\{\sum_{j=1}^{8} x_j | \boldsymbol{x} \in S_P, x_1 = 0\} = 3.5$ which, when surrogated with multipliers of 1, produce the (tighter) linear inequality

$$\sum_{j=1}^{8} x_j + 0.5x_1 \leq 3.5.$$

Example (continued). Now, in the same spirit, the compound conditional logic statements for $\boldsymbol{x} \in S_P$,

$$\text{if } \boldsymbol{\delta}_r^1 \boldsymbol{x} < \gamma_r^1 \text{ and if } \boldsymbol{\delta}_r^2 \boldsymbol{x} < \gamma_r^2 \quad \text{then} \quad \boldsymbol{\alpha x} \leq \beta_c^r$$

$$\text{where } \beta_c^r \equiv \max\{\boldsymbol{\alpha x} | \boldsymbol{x} \in S_P, \boldsymbol{\delta}_r^1 \boldsymbol{x} < \gamma_r^1, \boldsymbol{\delta}_r^2 \boldsymbol{x} < \gamma_r^2\} \ \forall \ r = 1, \ldots, R_c,$$

promote cubic inequalities of the form

$$(\gamma_r^1 - \boldsymbol{\delta}_r^1 \boldsymbol{x})(\gamma_r^2 - \boldsymbol{\delta}_r^2 \boldsymbol{x})(\beta_c^r - \boldsymbol{\alpha x}) \geq 0 \ \forall \ r = 1, \ldots, R_c. \tag{32}$$

Define $R_c = 4$ and let $\boldsymbol{\delta}_r^1 \boldsymbol{x} \leq \gamma_r^1$ and $\boldsymbol{\delta}_r^2 \boldsymbol{x} \leq \gamma_r^2$, $r = 1, \ldots, R_c$ be given as

$$\begin{array}{rcrl} -x_1 \leq 0 & \text{and} & -x_2 \leq 0 & r = 1, \\ -x_1 \leq 0 & \text{and} & x_2 \leq 1 & r = 2, \\ x_1 \leq 1 & \text{and} & -x_2 \leq 0 & r = 3, \text{ and} \\ x_1 \leq 1 & \text{and} & x_2 \leq 1 & r = 4, \end{array}$$

respectively. With these definitions, the cubic inequalities of the form (32) become

$$\begin{array}{ll} 4x_1x_2(\beta_c^1 - \sum_{j=1}^8 x_j) & \geq 0 \\ x_1(1-x_2)(\beta_c^2 - \sum_{j=1}^8 x_j) & \geq 0 \\ (1-x_1)x_2(\beta_c^3 - \sum_{j=1}^8 x_j) & \geq 0 \\ (1-x_1)(1-x_2)(\beta_c^4 - \sum_{j=1}^8 x_j) & \geq 0 \end{array} \tag{33}$$

having

$$\begin{array}{lll} \beta_c^1 & = \max\{\sum_{j=1}^8 x_j \,|\, \boldsymbol{x} \in S_P, x_1 = 1, x_2 = 1\} & = 3 \\ \beta_c^2 & = \max\{\sum_{j=1}^8 x_j \,|\, \boldsymbol{x} \in S_P, x_1 = 1, x_2 = 0\} & = 3 \\ \beta_c^3 & = \max\{\sum_{j=1}^8 x_j \,|\, \boldsymbol{x} \in S_P, x_1 = 0, x_2 = 1\} & = 3 \\ \beta_c^4 & = \max\{\sum_{j=1}^8 x_j \,|\, \boldsymbol{x} \in S_P, x_1 = 0, x_2 = 0\} & = 3. \end{array}$$

Surrogating these inequalities (33) with multipliers of 1 produces

$$\sum_{j=1}^{8} x_j \leq 3, \tag{34}$$

a facet of the polyhedra S_P [22].

In defining $\delta_r^t \boldsymbol{x} \leq \gamma_r^t$, $r = 1, \ldots, R$, $t = 1, \ldots, T$ there is great flexibility in choosing not only which inequalities to surrogate together, but also in choosing which inequalities to pair together. Example 6 exploits the four possible combinations associated with strictly satisfying a bound on each of two variables. This leads to the four inequalities (33), all of which are cubic. In general, not all the conditional logic inequalities need to have the same degree. Nor is it necessary for all r conditional logic statements to be tightening the same $\boldsymbol{\alpha x} \leq \beta$ inequality. The following example illustrates these freedoms using implications from inequalities other than the simple variable bounds.

Example 6. Let the sets S and X be defined as in (1) and (2) respectively, and suppose $\boldsymbol{\alpha x} \leq \beta$, $\boldsymbol{\delta}^1 \boldsymbol{x} \leq \gamma^1$, and $\boldsymbol{\delta}^2 \boldsymbol{x} \leq \gamma^2$ are three inequalities in X. For the sake of simplicity assume $\boldsymbol{\delta}^1 \geq \mathbf{0}$ and $\boldsymbol{\delta}^2 \geq \mathbf{0}$, and denote the two (nonempty) index sets of positive coefficients by $C_1 = \{j | \ \delta_j^1 > 0\}$, and $C_2 = \{j | \ \delta_j^2 > 0\}$, respectively. Assume without loss of generality that the indices are labeled such that $C_1 \cup C_2 = \{1, \ldots, q\}$ for some integer q satisfying $2 \leq q \leq |C_1| + |C_2|$. Surrogating the following $q+2$ conditional logic promoted nonlinear inequalities

Example (continued).

$$
\begin{aligned}
&x_r(\gamma^2 - \boldsymbol{\delta}^2 \boldsymbol{x})(\beta - \boldsymbol{\alpha x}) && \geq 0 \ \forall \ r \in C_1 \\
&x_r(\beta^r - \boldsymbol{\alpha x}) && \geq 0 \ \forall \ r \in C_2 \\
&(\gamma^1 - \boldsymbol{\delta}^1 \boldsymbol{x})(\gamma^2 - \boldsymbol{\delta}^2 \boldsymbol{x})(\beta_c^{q+1} - \boldsymbol{\alpha x}) && \geq 0 \\
&(\gamma^1 - \boldsymbol{\delta}^1 \boldsymbol{x})(\gamma^* - \boldsymbol{\delta}^2 \boldsymbol{x}) && \geq 0
\end{aligned}
$$

where

$$
\begin{aligned}
\beta^r \quad &= \max\{\boldsymbol{\alpha x} | \ \boldsymbol{x} \in S, x_r = 1\} && \forall \ r \in C_2 \\
\beta_c^{q+1} \quad &= \max\{\boldsymbol{\alpha x} | \ \boldsymbol{x} \in S, \boldsymbol{\delta}^1 \boldsymbol{x} < 1, \boldsymbol{\delta}^2 \boldsymbol{x} < 1\} \\
\gamma^* \quad &= \max\{\boldsymbol{\delta}^2 \boldsymbol{x} | \ \boldsymbol{x} \in S, \boldsymbol{\delta}^1 \boldsymbol{x} < \gamma^1\}
\end{aligned}
$$

with the nonnegative multipliers $\delta_r^1/(\gamma^1\gamma^2) \ \forall \ r \in C_1$, $\delta_r^2/\gamma^2 \ \forall \ r \in C_2$, $1/(\gamma^1\gamma^2)$, and $(\beta - \beta_c^{q+1})/(\gamma^1\gamma^2)$, respectively, produces the linear inequality

$$
\boldsymbol{\alpha x} + \frac{(\beta_c^1 - \beta)(\gamma^2 - \gamma^*)}{\gamma^1\gamma^2} \boldsymbol{\delta}^1 \boldsymbol{x} + \sum_{r \in C_2} \frac{(\beta - \beta^r)}{\gamma^2} \delta_r^2 x_r \leq \beta_c^{q+1} + (\beta - \beta_c^{q+1}) \left(\frac{\gamma^*}{\gamma^2}\right). \quad (35)
$$

If $\gamma^* = \gamma^2$, then (35) reduces to the coefficient increasing cut (29) discussed in Section 3.3 where $C = C_2$. However, if $\gamma^* < \gamma^2$, then additional strength is gained as the nonzero coefficients of C_1 and the righthand-side are decreased from their values in the coefficient increasing case.

We have shown how a conditional logic argument can be employed to construct strengthened formulations for the family of mixed 0-1 linear programs. Our approach naturally extends to the class of mixed integer linear programs. To illustrate the incorporation of general integer variables into our conditional logic framework, we provide the numerical example below. In a manner analogous to the way in which the lower bounds of 0 and upper bounds of 1 for binary variables were utilized, we now exploit general lower and upper bounds to accommodate integer variables.

Example 7. Suppose $S \equiv \{\boldsymbol{x} \in \mathcal{R}^2 | \ \boldsymbol{x} \in X, x_1, x_2 \text{ integer}\}$ where $X = \{\boldsymbol{x} \in \mathcal{R}^2 | \ 5x_1 + 7x_2 \leq 23, 1 \leq x_1 \leq 3, 0 \leq x_2 \leq 3\}$. Let $\boldsymbol{\alpha x} \leq \beta$ be the inequality $5x_1 + 7x_2 \leq 23$ and further let $R = 2$ with $\boldsymbol{\delta}_1 \boldsymbol{x} \leq \gamma_1$ given by the upper bounding restriction $x_1 \leq 3$, and $\boldsymbol{\delta}_2 \boldsymbol{x} \leq \gamma_2$ given by the lower bounding restriction $-x_1 \leq -1$. Define $T_1 = T_2 = S$ and compute the values β^1 and β^2 as in (8). Since in the presence of integer restrictions $x_1 \leq 3$ implies $x_1 \leq 2$ and $-x_1 \leq -1$ implies $-x_1 \leq -2$, we have

$$\beta^1 = \min\{23, \ \max\{5x_1 + 7x_2 | \ 5x_1 + 7x_2 \leq 23, 1 \leq x_1, 0 \leq x_2 \leq 3, \boldsymbol{x} \text{ integer}, x_1 \leq 2\}\}$$

and

$$\beta^2 = \min\{23, \ \max\{5x_1 + 7x_2 | \ 5x_1 + 7x_2 \leq 23, x_1 \leq 3, 0 \leq x_2 \leq 3, \boldsymbol{x} \text{ integer}, -x_1 \leq -2\}\}$$

so that $\beta^1 = 19$ and $\beta^2 = 22$. Now, let $\lambda_1 = \lambda_2 = 1$ and $\kappa = 2$ so that $1(3 - x_1) + 1(-1 + x_1) = 2$ with (12) satisfied. Inequality (13) then becomes (after multiplying by $\kappa = 2$ for simplification)

$$7x_1 + 14x_2 \leq 35 \tag{36}$$

which together with the constraints $1 \leq x_1 \leq 3$ and $0 \leq x_2$ in X gives $conv(S)$.

5 Conclusions

We derived a general conditional-logic framework for tightening the standard continuous relaxation of a mixed 0-1 linear program. The approach first generates quadratic inequalities that are strengthened via conditional logic, and then surrogates these restrictions to obtain linear inequalities that tighten the convex hull approximation. Our methodology generalizes a variety of published cut-strengthening techniques, including various single and multiple coefficient increasing and decreasing methods, and sequential lifting. Considerable effort was expended to show how mathematical structures such as special ordered sets, plant location and capacity expansion restrictions, variable upper bounds, and logical implications can be exploited, and how published results that employ such structures can be appropriately categorized. We then extended the underlying theory to incorporate compound conditional logic, which promotes a potentially richer family of restrictions through the surrogation of higher-order polynomial inequalities. We also showed via example how the continuous relaxations of general mixed integer programs can be similarly tightened.

Insofar as computational experience is concerned, we are now initiating a study that will embed strategically computed conditional logic inequalities within an enumerative algorithm to solve mixed 0-1 linear programs. We plan to report our findings in a companion paper in the near future. Based on the successes enjoyed by various researchers in implementing special cases of these inequalities, we feel our approach has great potential.

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