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A Polytope for a Product of Real Linear Functions in 0/1 Variables

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Abstract

In the context of integer programming, we develop a polyhedral method for linearizing a product of a pair of real linear functions in 0/1 variables. As an example, by writing a pair of integer variables in binary expansion, we have a technique for linearizing their product.

Introduction

We assume familiarity with polyhedral methods of linear integer programming (see [NW88], for example). There is a well-known method of linear integer programming for modeling the product (i.e., logical *and*) of a pair of binary variables. Specifically, the 0/1 solutions of $y = x^1 x^2$ are, precisely, the extreme points of the polytope in $\mathbb{R}^3$ that is the solution set of

$$y \geq 0 \; ; \tag{1}$$

$$y \geq x^1 + x^2 - 1 \; ; \tag{2}$$

$$y \leq x^1 \; ; \tag{3}$$

$$y \leq x^2 \; . \tag{4}$$

There has been extensive interest in the development of linear integer programming methods for handling and/or exploiting products of many 0/1

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variables (for example, see [Pad89], [DS90], [DS90], [DS96], [LPR97], [DL97], [LP96], [LP95], [DL94], [DSDL94], [DL93], [DL92c], [DLP92], [DL92b], [DL92a], [DGL91], [SA90], [AS86], [LS90], [LS91], [BH93], [HP72], [HR69], [HLL98]). In particular, quite a lot is known about the facets of the convex hull of the 0/1 solutions to: $y^{ij} = x^i x^j$, $1 \le i < j \le n$ (the *boolean quadric polytope*). Of course, we know less and less about the totality of the facets of the polytope as n increases, since optimization over these 0/1 solutions is NP-hard.

Rather than considering pairwise products of many 0/1 variables, we consider a single product of a pair of real linear functions in 0/1 variables. For $l = 1, 2$, let k_l be a positive integer, let $K_l = \{1, 2, \ldots, k_l\}$, let $\mathbf{a}^l$ be k_l-vector of positive real numbers a_i^l, let $\mathbf{x}^l$ be a k_l vector of binary variables x_i^l, and let $x^l = \sum_{i \in K_l} a_i^l x_i^l$. Note that if we had any $a_i^l = 0$, then we could just delete such x_i^l, and if we had any $a_i^l < 0$, then we could just complement such x_i^l and apply our methods to the nonlinear part of the resulting product.

Now, we let $P(a^1, a^2)$ be the convex hull of the solutions in $\mathbb{R}^{k_1+k_2+1}$ to

$$y = \left(\sum_{i \in K_1} a_i^1 x_i^1\right)\left(\sum_{j \in K_2} a_j^2 x_j^2\right) ; \qquad (5)$$

$$x_i^l \in \{0, 1\}, \text{ for } i \in K_l,\ l = 1, 2 . \qquad (6)$$

We note that $P((1), (1))$ is just the solution set of (1–4). Our goal is to investigate the polytope $P(a^1, a^2)$ generally.

In Section 1, we describe an application to modeling a product of a pair of nonnegative integer variables using binary expansion. In Section 2, we describe a linear integer formulation. In Section 3, we investigate which of our inequalities are facet describing. In Section 4, we determine a complete polyhedral characterization. In Section 5, we demonstrate how to solve the separation problem for the facet describing inequalities. In Section 6, we investigate some topological properties of *real* points in the polytope that satisfy (5). results.

1 Products of Nonnegative Integer Variables

Let x^1 and x^2 be a pair of nonnegative integer variables. This differs from our earlier use of x^1 and x^2, but our reasons for re-using these will become apparent. We can look directly at the convex hull of the integer solutions of $y = x^1 x^2$. This convex set, in $\mathbb{R}^3$ contains integer points that do not correspond to solutions of $y = x^1x^2$. For example, $x^1 = x^2 = y = 0$ is in this set, as is $x^1 = x^2 = 2, y = 4$, but the average of these two points is not a solution of $y = x^1x^2$. More concretely, the convex hull of the integer solutions to:

$$y = x^1x^2 \ ;$$
$$0 \le x^1 \le 2 \ ;$$
$$0 \le x^2 \le 2$$

is precisely the solution set of

$$\begin{aligned} y &\ge 0 \ ; \\ y &\ge 2x^1 + 2x^2 - 4 \ ; \\ y &\le 2x^1 \ ; \\ y &\le 2x^2 \ . \end{aligned}$$

If we use these latter linear inequalities to model the product y, and then seek to maximize y subject to these constraints and a side constraint $x^1 + x^2 \le 2$, we find the optimal solution $x^1 = 1$, $x^2 = 1$, $y = 2$, which does not satisfy $y = x^1x^2$. Therefore, this naïve approach is inadequate in the context of linear integer programming.

We adopt an approach that avoids the problem above. Specifically, we assume that, for practical purposes, x^1 and x^2 can be bounded above. So we can write the x^l in binary expansion: $x^l = \sum_{i \in K_l} 2^{i-1} x_i^l$, for $l = 1, 2$. That is, we let $a_i^l = 2^{i-1}$. The only integer points in $P(a^1, a^2)$ are the solutions of (5–6). Therefore, we avoid the problem that we encountered when we did not use the binary expansions of x^1 and x^2.

2 Linear Integer Formulation

Obviously, for $l = 1, 2$, the simple bound inequalities are valid for $P(a^1, a^2)$:

$$x_i^l \ge 0, \quad i \in K_l \ ; \tag{7}$$
$$x_i^l \le 1, \quad i \in K_l \ . \tag{8}$$

We view the remaining inequalities that we present as lower and upper bounds on the product variable y. In the sequel, H is a subset of $K_l \times K_{\bar{l}}$, where l is either 1 or 2, and $\bar{l} = 3 - l$.

Consider the following lower bound inequalities for y:

$$y \geq \sum_{(i,j)\in H} a_i^1 a_j^2 \left(x_i^1 + x_j^2 - 1\right) \ . \tag{9}$$

Proposition 1 *The inequalities (9) are valid for $P(a^1, a^2)$.*

Proof:

$$\begin{aligned} y &= \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^1 x_j^2 \\ &\geq \sum_{(i,j)\in H} a_i^1 a_j^2 x_i^1 x_j^2 \\ &\geq \sum_{(i,j)\in H} a_i^1 a_j^2 \left(x_i^1 + x_j^2 - 1\right) \ . \end{aligned}$$

□

We derive upper bounds on y by considering certain transformations ϕ^l, $l = 1, 2$, of the polytope $P(a^1, a^2)$:

$$\begin{aligned} \tilde{x}_i^l &= 1 - x_i^l \ , \quad \text{for } i \in K_l \ ; \\ \tilde{x}_j^{\bar{l}} &= x_j^{\bar{l}} \ , \quad \text{for } j \in K_{\bar{l}} \ ; \\ \tilde{y} &= \sum_{i\in K_l}\sum_{j\in K_{\bar{l}}} a_i^l a_j^{\bar{l}} \tilde{x}_i^l \tilde{x}_j^{\bar{l}} \ . \end{aligned}$$

Proposition 2 *The transformation ϕ^l, $l = 1, 2$, is an affine involution.*

Proof: Clearly $\phi^l \circ \phi^l$ is the identity transformation. To see that it is affine, we need just check that $\tilde{y}$ is an affine function of the original variables.

$$\begin{aligned} \tilde{y} &= \sum_{i\in K_l}\sum_{j\in K_{\bar{l}}} a_i^l a_j^{\bar{l}} \tilde{x}_i^l \tilde{x}_j^{\bar{l}} \\ &= \sum_{i\in K_l}\sum_{j\in K_{\bar{l}}} a_i^l a_j^{\bar{l}} (1 - x_i^l) x_j^{\bar{l}} \\ &= \sum_{i\in K_l}\sum_{j\in K_{\bar{l}}} a_i^l a_j^{\bar{l}} x_j^{\bar{l}} - \sum_{i\in K_l}\sum_{j\in K_{\bar{l}}} a_i^l a_j^{\bar{l}} x_i^l x_j^{\bar{l}} \\ &= \left(\sum_{i\in K_l} a_i^l\right) x^{\bar{l}} - y \ . \end{aligned}$$

□

Our upper bound inequalities take the form:

$$y \quad \leq \quad \left(\sum_{i \in K_l} a_i^l \right) x^{\bar{l}} + \sum_{(i,j) \in H} a_i^l a_j^{\bar{l}} \left(x_i^l - x_j^{\bar{l}} \right) \; . \tag{10}$$

Proposition 3 *The inequalities (10) are valid for $P(a^1, a^2)$.*

Proof: We apply the lower bound inequalities (9) to the variables transformed by ϕ^l, to obtain

$$\left(\sum_{i \in K_l} a_i^l \right) x^{\bar{l}} - y = \tilde{y} \geq \sum_{(i,j) \in H} a_i^l a_j^{\bar{l}} \left(\tilde{x}_i^l + \tilde{x}_j^{\bar{l}} - 1 \right) \; .$$

Solving for y, and rewriting this in terms of x^l and $x^{\bar{l}}$, we obtain the upper bound inequalities (10). □

Proposition 4 *The points satisfying (6) and (9–10) for all cross products H are precisely the points satisfying (5–6).*

Proof: By Propositions 1 and 3, we need only show that every point satisfying (6) and (9–10) for all cross products H also satisfies (5). Let $(\mathbf{x}^1, \mathbf{x}^2, y)$ be a point satisfying (6) and (9–10). Letting

$$H = \{i \in K_1 \; : \; x_i^1 = 1\} \times \{j \in K_2 \; : \; x_j^2 = 1\} \; ,$$

we obtain a lower bound inequality (9) that is satisfied as an equation by the point $(\mathbf{x}^1, \mathbf{x}^2, y)$. Similarly, letting

$$H = \{i \in K_1 \; : \; x_i^1 = 0\} \times \{j \in K_2 \; : \; x_j^2 = 1\} \; ,$$

we obtain an upper bound inequality (10) that is satisfied as an equation by the point $(\mathbf{x}^1, \mathbf{x}^2, y)$. So $\mathbf{x}^1$ and $\mathbf{x}^2$ together with (9–10) for cross products H determine y. But by the definition of $P(a^1, a^2)$, the point $(\mathbf{x}^1, \mathbf{x}^2, \sum_{i \in K_1} \sum_{j \in K_2} a_i^1 a_j^2 x_i^1 x_j^2)$ is in $P(a^1, a^2)$, so (5) must be satisfied. □

3 Facets

For $i \in K_l$, $l = 1, 2$, we let $e^i \in \mathbb{R}^{k_l}$ denote the i-th standard unit vector.

Proposition 5 *For $k_1, k_2 > 0$, the polytope $P(a^1, a^2)$ is full dimensional.*

Proof: We prove this directly. The following $k_1 + k_2 + 2$ points in $P(a^1, a^2)$ are easily seen to be affinely independent:

- $(\mathbf{x}^1, \mathbf{x}^2, y) = (\mathbf{0}, \mathbf{0}, 0)$;
- $(\mathbf{x}^1, \mathbf{x}^2, y) = (e^j, \mathbf{0}, 0)$, for $j \in K_1$;
- $(\mathbf{x}^1, \mathbf{x}^2, y) = (\mathbf{0}, e^j, 0)$, for $j \in K_2$;
- $(\mathbf{x}^1, \mathbf{x}^2, y) = (e^1, e^1, a_1^1 a_1^2)$.

□

Proposition 6 *For $l = 1, 2$, the inequalities (7) describe facets of $P(a^1, a^2)$ when $k_l > 1$.*

Proof: Again, we proceed directly. For both $l = 1$ and $l = 2$, the following $k_1 + k_2$ points in $P(a^1, a^2)$ satisfy (7) as an equation and are easily seen to be affinely independent:

- $(\mathbf{x}^1, \mathbf{x}^2, y) = (\mathbf{0}, \mathbf{0}, 0)$;
- $(\mathbf{x}^1, \mathbf{x}^2, y)$ defined by $\mathbf{x}^l = e^j$, $\mathbf{x}^{\bar{l}} = \mathbf{0}$, $y = 0$, for $j \in K_l \setminus \{i\}$;
- $(\mathbf{x}^1, \mathbf{x}^2, y)$ defined by $\mathbf{x}^l = \mathbf{0}$, $\mathbf{x}^{\bar{l}} = e^j$, $y = 0$, for $j \in K_{\bar{l}}$;
- $(\mathbf{x}^1, \mathbf{x}^2, y)$ defined by $\mathbf{x}^l = e^m$, for some $m \in K_l \setminus \{i\}$, $\mathbf{x}^{\bar{l}} = e^1$, $y = a_m^l a_1^{\bar{l}}$.

□

Proposition 7 *For $l = 1, 2$, the inequalities (8) describe facets of $P(a^1, a^2)$ when $k_l > 1$.*

Proof: The affine transformation ϕ^l is an involution from the face of $P(a^1,a^2)$ described by the simple lower bound inequality $x_i^l \geq 0$ to the face of $P(a^1,a^2)$ described by the simple upper bound inequality $x_i^l \leq 1$. Since the affine transformation is invertible, it is dimension preserving. Therefore, by Proposition 6, the result follows. $\square$

One might suspect from Proposition 4 that the only inequalities of the form (9–10) that yield facets have H being a cross product. This is not the case. Let $\mathcal{H}(k_1,k_2)$ denote a particular set of subsets of $K_1 \times K_2$. Each H in $\mathcal{H}(k_1,k_2)$ arises by choosing a permutation of the variables

$$\{x_i^1 \ : \ i \in K_1\} \cup \{x_j^2 \ : \ j \in K_2\} \ .$$

If x_i^1 precedes x_j^2 in the permutation, which we denote by $x_i^1 \prec x_j^2$, then (i,j) is in H.

For example, if $S_1 \subset K_1$ and $S_2 \subset K_2$, then we get the cross product $H = S_1 \times S_2 \in \mathcal{H}(k_1,k_2)$ by choosing any permutation of the variables having $\{x_j^2 \ : \ j \in K_2 \setminus S_2\}$ first, followed by $\{x_i^1 \ : \ i \in S_1\}$, followed by $\{x_j^2 \ : \ j \in S_2\}$, followed by $\{x_i^1 \ : \ i \in K_1 \setminus S_1\}$.

As another example, let $k_1 = 3$ and $k_2 = 2$, and consider the permutation: $x_2^1, x_2^2, x_1^1, x_3^1, x_1^2$, which yields

$$H = \{(2,2),(2,1),(1,1),(3,1)\} \ .$$

This choice of H is not a cross product. However, this H yields the lower bound inequality:

$$\begin{aligned} y \ \geq \ & a_2^1 a_2^2 (x_2^1 + x_2^2 - 1) \\ + \ & a_2^1 a_1^2 (x_2^1 + x_1^2 - 1) \\ + \ & a_1^1 a_1^2 (x_1^1 + x_1^2 - 1) \\ + \ & a_3^1 a_1^2 (x_3^1 + x_1^2 - 1) \ . \end{aligned}$$

We demonstrate that this inequality describes a facet by displaying $k_1+k_2+1 = 6$ affinely independent points in $P(a^1,a^2)$ that satisfy the inequality as an equation: We display the points as rows of the matrix below, where we have permuted the columns, in a certain manner, according to the permutation of the variables that led to H, and we have inserted a column of 1's. It suffices to check that the square "caterpillar matrix" obtained by deleting the y column is nonsingular.

x_2^2	x_1^2	1	x_2^1	x_1^1	x_3^1	y
0	0	1	1	1	1	$0 = (a_3^1 + a_1^1 + a_2^1)(0)$
0	1	1	1	1	1	$(a_3^1 + a_1^1 + a_2^1)(a_1^1)$
0	1	1	1	1	0	$(a_1^1 + a_2^1)(a_1^1)$
0	1	1	1	0	0	$(a_2^1)(a_1^2)$
1	1	1	1	0	0	$(a_2^1)(a_1^2 + a_2^2)$
1	1	1	0	0	0	$0 = (0)(a_1^2 + a_2^2)$

For $m \geq 3$, an order-m *caterpillar matrix* is obtained by choosing $k_1, k_2 \geq 1$ with $k_1 + k_2 + 1 = m$. The first row of such a matrix is $(\mathbf{1} \in \mathbb{R}^{k_2}, 1, \mathbf{0} \in \mathbb{R}^{k_1})$ and the last row is $(\mathbf{0} \in \mathbb{R}^{k_2}, 1, \mathbf{1} \in \mathbb{R}^{k_1})$. Each row is obtained from the one above it by either flipping the right-most 1 to 0, or flipping the 0 immediately to the left of the left-most 1 to 1. So the "caterpillar" moves from right to left by either pushing its head forward a bit or pulling its tail forward a bit. Since caterpillar matrices have their 1's in each row consecutively appearing, they are totally unimodular (see [HK56]). That is, the determinant of a caterpillar matrix is in $\{0, \pm 1\}$. In fact, we have the following result.

Proposition 8 *The determinant of a caterpillar matrix is in* $\{\pm 1\}$.

Proof: The proof is by induction on m. The only order-3 caterpillar matrices are

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is easy to check that these have determinant in $\{\pm 1\}$.

Now, suppose that we have a caterpillar matrix of order $m \geq 4$. Depending on the bit flip that produces the last row from the one above it, the matrix has the form

$$\left(\begin{array}{cccc|c|ccccc} 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\ & & & & \vdots & & & & & \vdots \\ & & & & 1 & & & & & 0 \\ & & & & 1 & & & & & 0 \end{array}\right)$$

or

$$\left(\begin{array}{ccccc|c|cccc} 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ & & & & \vdots & \vdots & & & & \\ & & & & 1 & 1 & & & & \\ & & & & 1 & 1 & & & & \end{array}\right).$$

In the first case, we can expand the determinant along the last column and we obtain a caterpillar matrix of order $m-1$ with the same determinant as the original matrix, up to the sign. In the second case, we subtract the first row from the second row (which does not affect the determinant), and then we expand along the second row of the resulting matrix. Again, we obtain a caterpillar matrix of order $m-1$ with the same determinant as the original matrix, up to the sign. In either case, the result follows by induction. $\square$

Proposition 9 *An inequality of the form (9) describes a facet of $P(a^1, a^2)$ when $H \in \mathcal{H}(k_1, k_2)$.*

Proof: By Proposition 1, these inequalities are valid for $P(a^1, a^2)$. It suffices to exhibit $k_1 + k_2 + 1$ affinely independent points that satisfy (9) as an equation. Let Ψ be the permutation that gives rise to H. It is easy to check that $(\mathbf{x}^1, \mathbf{x}^2, y) = (\mathbf{0}, \mathbf{1}, 0)$ is a point of $P(a^1, a^2)$ which satisfies (9) as an equation. We generate the remaining $k_1 + k_2$ points by successively flipping bits in the order of the permutation Φ. We simply need to check that each bit flip preserves equality in (9). If a variable x_i^1 is flipped from 0 to 1, the increase in y (i.e., the left-hand side of (9)) and in $\sum_{(i,j)\in H} a_i^1 a_j^2 \left(x_i^1 + x_j^2 - 1\right)$ (i.e., the right-hand side of (9)) is precisely $a_i^1 \sum_{x_j^2 \prec x_i^1} a_j^2$. Similarly, if a variable x_j^2 is flipped from 1 to 0, the decrease in both of these quantities is precisely $a_j^2 \sum_{x_i^1 \prec x_j^2} a_i^1$.

Next, we take these $k_1 + k_2 + 1$ points and use them to generate a caterpillar matrix of order $k_1 + k_2 + 1$. The successive rows are determined by the successively generated points. A point $(\mathbf{x}^1, \mathbf{x}^2, y)$ yields the row $(\mathbf{x}_\Phi^1, 1, \mathbf{x}_\Phi^2)$, where $\mathbf{x}_\Phi^l$ is just $\mathbf{x}^l$ permuted according to the order of the x_i^l in Φ. Clearly this defines a caterpillar matrix, which is nonsingular by Proposition 8. Hence, the generated points are affinely independent, so (9) describes a facet when $H \in \mathcal{H}(k_1, k_2)$. $\square$

Proposition 10 *An inequality of the form (10) describes a facet of $P(a^1, a^2)$ when $H \in \mathcal{H}(k_1, k_2)$.*

Proof: Using the transformation ϕ^l, this follows from Proposition (9). $\square$

4 Inequality Characterization

In this section, we demonstrate how every facet of $P(a^1, a^2)$ is one of the ones described in Section 3.

Proposition 11 *The polytope $P(a^1, a^2)$ is precisely the set of real points satisfying (7–10).*

Proof: By Propositions 1 and 3, it suffices to demonstrate that every facet of $P(a^1, a^2)$ is described by an inequality of the form (7–10).

First, we consider facet describing inequalities that do not involve the variable y. Suppose that

$$\sum_{i \in K_1} b_i^1 x_i^1 + \sum_{j \in K_2} b_j^2 x_j^2 \leq c \tag{$*$}$$

describes a facet of $P(a^1, a^2)$. Let $l = 1$ or 2. For all $i \in K_l$, we choose an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(*)$ as an equation and has $x_i^l = 1$. If there is no such point, then $(*)$ would describe the same facet as is described by (7). Decreasing this variable from 1 to 0, and adjusting y to satisfy (5), implies that $b_i^l \geq 0$. Next, for all $i \in K_l$, we can choose an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(*)$ as an equation and has $x_i^l = 0$. If there is no such point, then $(*)$ would describe the same facet as is described by (8). Increasing this variable from 0 to 1, and adjusting y to satisfy (5), implies that $b_i^l \leq 0$. Therefore $b_i^l = 0$, for all $i \in K_l$. Therefore, depending on c, either $(*)$ describes the empty face or $P(a^1, a^2)$ itself. In either case, we have a contradiction.

Therefore, any facet described by an inequality that does not involve the variable y is also described by an inequality of the form (7–8).

Any facet describing inequality that *does* involve y, can be written as an upper or lower bound on y. Suppose that a facet describing inequality for $P(a^1, a^2)$ has the form of a lower bound on y. We can take it to have the form:

$$y \geq \sum_{i \in K_1} b_i^1 a_i^1 x_i^1 + \sum_{j \in K_2} b_j^2 a_j^2 x_j^2 - c \tag{$**$}$$

We will demonstrate that $(**)$ is a lower bound inequality (9).

First, we develop some simple relationships between the coefficients of $(**)$ and extreme points of $P(a^1, a^2)$ that satisfy $(**)$ as an equation. For

$l = 1, 2$, for arbitrary $i \in K_l$, choose an extreme point $p = (\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation and has $x_i^l = 0$. There must be such a point since otherwise we would have $x_i^l = 1$ for all extreme points of $P(a^1, a^2)$ satisfying $(**)$ as an equation. Increasing x_i^l from 0 to 1, and increasing y by $a_i^l x^{\bar{l}}$, we get an extreme point p' of $P(a^1, a^2)$. Therefore, we must have $b_i^l \leq x^{\bar{l}}$ for any extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation and has $x_i^l = 0$. Moreover, if $b_i^l = x^{\bar{l}}$, then the point p' also satisfies $(**)$ as an equation. Now, if we instead choose an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation and has $x_i^l = 1$, by decreasing x_i^l from 1 to 0, and decreasing y by $a_i^l x^{\bar{l}}$, we get an extreme point of $P(a^1, a^2)$. Therefore, we must have $b_i^l \geq x^{\bar{l}}$ for any extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation and has $x_i^l = 1$.

For an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation, we summarize this information:

$$b_i^l > x^{\bar{l}} \implies x_i^l = 1 \; ; \tag{11}$$
$$b_i^l = x^{\bar{l}} \implies x_i^l \text{ is arbitrary;} \tag{12}$$
$$b_i^l < x^{\bar{l}} \implies x_i^l = 0 \; . \tag{13}$$

Next, let π_1 be a permutation of K_1 so that $b^1_{\pi_1(1)} \geq b^1_{\pi_1(2)} \geq \cdots \geq b^1_{\pi_1(k_1)}$. Then, permuting the coordinates of $\mathbf{x}^1$, we define

$$\mathbf{x}^1_{\pi_1} = (x^1_{\pi_1(1)}, x^1_{\pi_1(2)}, \ldots, x^1_{\pi_1(k_1)}) \; .$$

Then, (11-13) implies that every extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation has

$$\mathbf{x}^1_{\pi_1} = (\underbrace{1, 1, \ldots, 1}_{b^1_{\pi_1(i)} > x^2}, \underbrace{?, ?, \ldots, ?}_{b^1_{\pi_1(i)} = x^2}, \underbrace{0, 0, \ldots, 0}_{b^1_{\pi_1(i)} < x^2}) \; .$$

Similarly, let π_2 be a permutation of K_2 so that $b^2_{\pi_2(1)} \leq b^2_{\pi_2(2)} \leq \cdots \leq b^2_{\pi_2(k_2)}$, and let

$$\mathbf{x}^2_{\pi_2} = (x^2_{\pi_2(1)}, x^2_{\pi_2(2)}, \ldots, x^2_{\pi_2(k_2)}) \; .$$

Then, (11-13) implies that every extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ has

$$\mathbf{x}^2_{\pi_2} = (\underbrace{0, 0, \ldots, 0}_{b^2_{\pi_2(i)} < x^1}, \underbrace{?, ?, \ldots, ?}_{b^2_{\pi_2(i)} = x^1}, \underbrace{1, 1, \ldots, 1}_{b^2_{\pi_2(i)} > x^1}) \; .$$

Next, we choose an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation. From what we have observed above, we can adjust $\mathbf{x}^2$ so that

$$\mathbf{x}^2_{\pi_2} = (\underbrace{0,0,\ldots,0}_{b^2_{\pi_2(i)}<x^1},\underbrace{0,0,\ldots,0}_{b^2_{\pi_2(i)}=x^1},\underbrace{1,1,\ldots,1}_{b^2_{\pi_2(i)}>x^1}) \ ,$$

and adjust y appropriately, and this point will still satisfy $(**)$ as an equation. Now, if we adjusted any bits of $\mathbf{x}^2$, this changes the value of x^2. With respect to this new values of x^2, $\mathbf{x}^1$ has

$$\mathbf{x}^1_{\pi_1} = (\underbrace{1,1,\ldots,1}_{b^1_{\pi_1(i)}>x^2},\underbrace{?,?,\ldots,?}_{b^1_{\pi_1(i)}=x^2},\underbrace{0,0,\ldots,0}_{b^1_{\pi_1(i)}<x^2}) \ .$$

As we did for $\mathbf{x}^2$, we can adjust $\mathbf{x}^1$ so that it has

$$\mathbf{x}^1_{\pi_1} = (\underbrace{1,1,\ldots,1}_{b^1_{\pi_1(i)}>x^2},\underbrace{1,1,\ldots,1}_{b^1_{\pi_1(i)}=x^2},\underbrace{0,0,\ldots,0}_{b^1_{\pi_1(i)}<x^2}) \ ,$$

and adjust y appropriately, and this point will still satisfy $(**)$ as an equation.

This adjustment may change the value of x^1, but, in any case, we have produced an extreme point $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfies $(**)$ as an equation and has the form of a caterpillar:

$$\mathbf{x}^2_{\pi_2}, 1, \mathbf{x}^1_{\pi_1} = (0,0,\ldots,\overset{\hat{\jmath}}{\overset{\downarrow}{0}},1,1,\ldots,1),1,(1,1,\ldots,\overset{\hat{\imath}}{\overset{\downarrow}{1}},0,0,\ldots,0) \ .$$

Next, we describe how this caterpillar moves to the left. Above, $\hat{\jmath}$ denotes the next bit position that may flip from 0 to 1, and $\hat{\imath}$ denotes the next bit position that may flip from 1 to 0. If $b^2_{\hat{\jmath}} = x^1$, then we can flip the $\hat{\jmath}$ bit (adjusting y appropriately), and the point will still satisfy $(**)$ as an equation. Similarly, if $b^1_{\hat{\imath}} = x^2$, then we can flip the $\hat{\imath}$ bit (adjusting y appropriately), and the point will still satisfy $(**)$ as an equation. So, the only situation that stops the caterpillar from moving to the left is when both $b^2_{\hat{\jmath}} < x^1$ and $b^1_{\hat{\imath}} > x^2$:

$$\mathbf{x}^2_{\pi_2}, 1, \mathbf{x}^1_{\pi_1} = (\underbrace{0,0,\ldots,\overset{\hat{\jmath}}{\overset{\downarrow}{0}}}_{b^2_{\pi_2(j)}<x^1},\underbrace{1,1,\ldots,1}_{b^2_{\pi_2(j)}\geq x^1}),1,(\underbrace{1,1,\ldots,\overset{\hat{\imath}}{\overset{\downarrow}{1}}}_{b^1_{\pi_1(i)}>x^2},\underbrace{0,0,\ldots,0}_{b^1_{\pi_1(i)}\leq x^2}) \ .$$

Now, if $b^2_{\hat{j}} < x^1$, we increase $b^2_{\hat{j}}$ up to x^1, and then we can continue by flipping the $\hat{j}$ bit from 0 to 1. Note that increasing $b^2_{\hat{j}}$ produces an inequality that is *not* valid for $P(a^1, a^2)$, since $(**)$ describes a facet of $P(a^1, a^2)$ and all points of $P(a^1, a^2)$ satisfy (7). However, all extreme points that we have already generated had $x^2_{\hat{j}} = 0$, so those points still satisfy $(**)$ as an equation. So the only remaining case to consider is if x^2 has all bits equal to 1 and $b^1_{\hat{i}} > x^2$:

$$\mathbf{x}^2_{\pi_2}, 1, \mathbf{x}^1_{\pi_1} = (1, 1, \ldots, 1), 1, (\underbrace{1, 1, \ldots, \overset{\overset{\hat{\imath}}{\downarrow}}{1}}_{b^1_{\pi_1(i)} > x^2}, \underbrace{0, 0, \ldots, 0}_{b^1_{\pi_1(i)} \leq x^2}) \ .$$

In this case, $b^1_{\hat{i}} > x^2$ for *all* extreme points $(\mathbf{x}^1, \mathbf{x}^2, y)$ of $P(a^1, a^2)$ that satisfy $(**)$ as an equation (since this particular extreme point makes x^2 as large as possible). Therefore, all extreme points of $P(a^1, a^2)$ that satisfy $(**)$ as an equation have $x^1_{\hat{i}} = 1$, which contradicts the hypothesis that $(**)$ describes a facet of $P(a^1, a^2)$.

So our initial caterpillar can crawl all the way to the left, possibly at the expense of increasing some coefficients $b^2_{\hat{j}}$, to reach the state $\mathbf{x}^2, 1, \mathbf{x}^1 = \mathbf{1}, 1, \mathbf{0}$. Similarly, the initial caterpillar can crawl all of the way to the right, producing extreme points of $P(a^1, a^2)$ that satisfy $(**)$ as an equation, possibly at the expense of increasing some coefficients $b^1_{\hat{i}}$.

Once all of this crawling is done, we will have produced a caterpillar matrix. This demonstrates that either $(**)$ is of the form (9) for some $H \in \mathcal{H}$, or, if we did increase some coefficients, we conclude that some inequality of the form (9) is not valid for all points of $P(a^1, a^2)$, in contradiction with Proposition 1.

Finally, the affine involution ϕ^l is dimension preserving, so we can also conclude that every facet describing upper bound inequality on y is also described by an inequality of the form (10). □

5 Separation

We can efficiently include all facet describing inequalities of $P(a^1, a^2)$ implicitly in a linear programming formulation, provided that we can separate on them in polynomial time (see [GLS84, GLS81, GLS93]). That is, provided we have a polynomial-time algorithm that determines whether a given point is in $P(a^1, a^2)$ and provides a violated facet describing inequality if the point is not in $P(a^1, a^2)$.

Separation for the simple lower and upper bound inequalities (7–8) is easily handled by enumeration. For a point $(\mathbf{x}^1, \mathbf{x}^2, y)$ satisfying (7–8), separation for the lower and upper bound inequalities (9–10) is also rather simple. For the lower bound inequalities (9), we simply let

$$H = \left\{(i,j) \in K_1 \times K_2 \ : \ x_i^1 + x_j^2 > 1\right\} \ ,$$

and then we just check whether (x^1, x^2, y) violates the lower bound inequality (9) for this choice of H. Similarly, for the upper bound inequalities (10), we let

$$H = \left\{(i,j) \in K_l \times K_{\bar{l}} \ : \ x_i^l - x_j^{\bar{l}} < 0\right\} \ ,$$

and then we just check whether $(\mathbf{x}^1, \mathbf{x}^2, y)$ violates the upper bound inequality (10) for this choice of H.

Using Propositions 9 and 10, we can see how this separation method yields *facet describing* violated inequalities. We develop a permutation of the variables

$$\{x_i^1 \ : \ i \in K_1\} \cup \{x_j^2 \ : \ j \in K_2\} \ ,$$

according to their values. Let $\delta_1 > \delta_2 > \cdots > \delta_p$ denote the distinct values of the $\{x_i^1 \ : \ i \in K_1\}$. For convenience, let $\delta_0 = 0$ and $\delta_{p+1} = 1$. We define the partition via $2p+1$ blocks, some of which may be empty. For, $t = 1, 2, \ldots, p$, block $2t$ consists of

$$\{x_i^1 \ : \ i \in K_1, \ x_i^1 = \delta_t\} \ .$$

For, $t = 0, 1, 2, \ldots, p$, block $2t+1$ consists of

$$\{x_j^2 \ : \ j \in K_2, \ 1 - \delta_t < x_j^2 \le 1 - \delta_{t+1}\} \ .$$

This permutation of the variables determines a subset H of $K_1 \times K_2$ as described in Section 3. This choice of H yields a facet describing lower bound inequality (9).

Similarly, for the upper bound inequalities (10), we let $\delta_1 < \delta_2 < \cdots < \delta_p$ denote the distinct values of the $\{x_i^l \ : \ i \in K_l\}$. As before, let $\delta_0 = 0$ and $\delta_{p+1} = 1$, and we define a partition via $2p+1$ blocks, some of which may be empty. For, $t = 1, 2, \ldots, p$, block $2t$ consists of

$$\{x_i^l \ : \ i \in K_l, \ x_i^l = \delta_t\} \ .$$

For, $t = 0, 1, 2, \ldots, p$, block $2t + 1$ consists of

$$\{x_j^{\bar{l}} \ : \ j \in K_{\bar{l}}, \ \delta_t < x_j^{\bar{l}} \leq \delta_{t+1}\} \ .$$

This permutation of the variables determines a subset H of $K_1 \times K_2$ as described in Section 3. This choice of H yields a facet describing upper bound inequality (10).

6 Ideal Points

For many combinatorial polytopes, it is natural to investigate adjacency of extreme points via edges (i.e., 1-dimensional faces). One motivation is that this notion of adjacency may prove useful in some local-search heuristics. In this section, we investigate a different notion of adjacency for extreme points — one that seems more natural for $P(a^1, a^2)$.

The point $(\mathbf{x}^1, \mathbf{x}^2, y) \in P(a^1, a^2)$ is *ideal* if it satisfies (5). Clearly, the extreme points of $P(a^1, a^2)$ are ideal. Also, $P(a^1, a^2)$ contains points that are not ideal. For example,

$$(x_1^1, x_1^2, y) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \frac{1}{2}(0,0,0) + \frac{1}{2}(1,1,1)$$

is in $P((1),(1))$, but it is not ideal since

$$\sum_{i \in K_1} \sum_{j \in K_2} a_i^1 a_j^2 x_i^1 x_j^2 = 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{2} \ .$$

Proposition 12 *Every pair of distinct extreme points of $P(a^1, a^2)$ is connected by a curve of ideal points of $P(a^1, a^2)$. Moreover, such a curve can be taken to be either a line segment or two line segments joined at another extreme point of $P(a^1, a^2)$.*

Proof: Let $(\mathbf{x}^{11}, \mathbf{x}^{12}, y^{11})$ and $(\mathbf{x}^{21}, \mathbf{x}^{22}, y^{22})$ be a pair of distinct extreme points of $P(a^1, a^2)$. If $\mathbf{x}^{11} = \mathbf{x}^{21}$ then we consider the curve obtained by letting

$$(\mathbf{z}^1, \mathbf{z}^2, y) \quad = \quad \lambda(\mathbf{x}^{11}, \mathbf{x}^{12}, y^{11}) + (1 - \lambda)(\mathbf{x}^{21}, \mathbf{x}^{22}, y^{22}),$$

as λ ranges between 0 and 1. For $\lambda = 1$ we have $(\mathbf{z}^1, \mathbf{z}^2, y) = (\mathbf{x}^{11}, \mathbf{x}^{12}, y^{11})$, and for $\lambda = 0$ we have $(\mathbf{z}^1, \mathbf{z}^2, y) = (\mathbf{x}^{21}, \mathbf{x}^{22}, y^{22})$. Clearly, the curve is a line segment entirely contained in the convex polytope $P(a^1, a^2)$, since we have defined each point on the curve as a convex combination of the pair of extreme points of P. So it remains to demonstrate that each point on the curve is ideal:

$$
\begin{aligned}
&\sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 z_i^1 z_j^2 \\
&\qquad = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 \left(\lambda x_i^{11} + (1-\lambda)x_i^{21}\right)\cdot\left(\lambda x_j^{12} + (1-\lambda)x_j^{22}\right) \\
&\qquad = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{11}\left(\lambda x_j^{12} + (1-\lambda)x_j^{22}\right) \\
&\qquad = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 \lambda x_i^{11} x_j^{12} + (1-\lambda)x_i^{21}x_j^{22} \\
&\qquad = \lambda y^{11} + (1-\lambda)y^{22} \\
&\qquad = y\ .
\end{aligned}
$$

Therefore the points on the curve are ideal.

Similarly, if $\mathbf{x}^{12} = \mathbf{x}^{22}$, we use the same line segment above to connect the points.

Suppose now that $\mathbf{x}^{11} \neq \mathbf{x}^{21}$ and $\mathbf{x}^{12} \neq \mathbf{x}^{22}$. We define a third point (x^{11}, x^{22}, y^{12}), where

$$y^{12} = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{11} x_j^{22}\ .$$

Then we connect this third point, in the manner above, to each of the points of the original pair. □

The curve of ideal points given to us by Proposition 12 is entirely contained in a 2-dimensional polytope, but it is not smooth in general. By allowing the curve to be contained in a 3-dimensional polytope, we can construct a smooth curve of ideal points connecting each pair of extreme points of $P(a^1, a^2)$.

Proposition 13 *Every pair of extreme points of $P(a^1, a^2)$ is connected by a smooth curve of ideal points of $P(a^1, a^2)$.*

Proof: Let $(\mathbf{x}^{11}, \mathbf{x}^{12}, y^{11})$ and $(\mathbf{x}^{21}, \mathbf{x}^{22}, y^{22})$ be a pair of distinct extreme points of $P(a^1, a^2)$. Our goal is to connect these points with a smooth curve of ideal points of $P(a^1, a^2)$. Toward this end, we consider two other points of $P(a^1,a^2)$, $(\mathbf{x}^{11},\mathbf{x}^{22},y^{12})$ and $(\mathbf{x}^{21},\mathbf{x}^{12},y^{21})$, which we obtain from the original pair by letting

$$y^{12} = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{11} x_j^{22}$$

and

$$y^{21} = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{21} x_j^{12} \ .$$

Now, we consider the curve obtained by letting

$$\begin{aligned}(\mathbf{z}^1,\mathbf{z}^2,y) \quad = \quad & \lambda^2(\mathbf{x}^{11},\mathbf{x}^{12},y^{11}) + (1-\lambda)^2(\mathbf{x}^{21},\mathbf{x}^{22},y^{22}) \\ & +\lambda(1-\lambda)(\mathbf{x}^{11},\mathbf{x}^{22},y^{12}) + \lambda(1-\lambda)(\mathbf{x}^{21},\mathbf{x}^{12},y^{21}) \ ,\end{aligned}$$

as λ ranges between 0 and 1. For $\lambda = 1$ we have $(\mathbf{z}^1,\mathbf{z}^2,y) = (\mathbf{x}^{11},\mathbf{x}^{12},y^{11})$, and for $\lambda = 0$ we have $(\mathbf{z}^1,\mathbf{z}^2,y) = (\mathbf{x}^{21},\mathbf{x}^{22},y^{22})$. Clearly, the curve is entirely contained in the convex polytope $P(a^1,a^2)$, since we have defined each point on the curve as a convex combination of extreme points of P. So it remains to demonstrate that each point on the curve is ideal:

$$\begin{aligned}&\sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 z_i^1 z_j^2 \\
&\quad = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 \left(\lambda^2 x_i^{11} + (1-\lambda)^2 x_i^{21} + \lambda(1-\lambda)x_i^{11} + \lambda(1-\lambda)x_i^{21}\right) \\
&\qquad\quad \cdot \left(\lambda^2 x_j^{12} + (1-\lambda)^2 x_j^{22} + \lambda(1-\lambda)x_j^{22} + \lambda(1-\lambda)x_j^{12}\right) \\
&\quad = \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 \left(\lambda x_i^{11} + (1-\lambda)x_i^{21}\right)\cdot\left(\lambda x_j^{12} + (1-\lambda)x_j^{22}\right) \\
&\quad = \lambda^2 \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{11} x_j^{12} + (1-\lambda)^2 \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{21} x_j^{22} \\
&\qquad\quad +\lambda(1-\lambda) \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{11} x_j^{22} + \lambda(1-\lambda) \sum_{i\in K_1}\sum_{j\in K_2} a_i^1 a_j^2 x_i^{21} x_j^{22} \\
&\quad = \lambda^2 y^{11} + (1-\lambda)^2 y^{22} + \lambda(1-\lambda)y^{12} + \lambda(1-\lambda)y^{21} \\
&\quad = y \ .\end{aligned}$$

Therefore the points on the curve are ideal. □

[Pad89]). a tree

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References

[AS86] Warren P. Adams and Hanif D. Sherali. A tight linearization and an algorithm for zero-one quadratic programming problems. *Management Sci.*, 32(10):1274–1290, 1986.

[BH93] Endre Boros and Peter L. Hammer. Cut-polytopes, Boolean quadratic polytopes and nonnegative quadratic pseudo-Boolean functions. *Math. Oper. Res.*, 18(1):245–253, 1993.

[DGL91] Michel Deza, Viatcheslav Grishukhin, and Monique Laurent. The symmetries of the cut polytope and of some relatives. In *Applied geometry and discrete mathematics*, pages 205–220. Amer. Math. Soc., Providence, RI, 1991.

[DL92a] Michel Deza and Monique Laurent. Facets for the cut cone. I. *Math. Programming, Ser. A*, 56(2):121–160, 1992.

[DL92b] Michel Deza and Monique Laurent. Facets for the cut cone. II. Clique-web inequalities. *Math. Programming, Ser. A*, 56(2):161–188, 1992.

[DL92c] Michel Deza and Monique Laurent. New results on facets of the cut cone. *J. Combin. Inform. System Sci.*, 17(1–2):19–38, 1992.

[DL93] Michel Deza and Monique Laurent. The cut cone: simplicial faces and linear dependencies. *Bull. Inst. Math. Acad. Sinica*, 21(2):143–182, 1993.

[DL94] Michel Deza and Monique Laurent. Applications of cut polyhedra. I, II. *J. Comput. Appl. Math.*, 55(2):191–216, 217–247, 1994.

[DL97] Michel Deza and Monique Laurent. *Geometry of cuts and metrics.* Springer-Verlag, Berlin, 1997.

[DLP92] Michel Deza, Monique Laurent, and Svatopluk Poljak. The cut cone. III. On the role of triangle facets. *Graphs Combin.*, 8(2):125–142, 1992.

[DS90] Caterina De Simone. Lifting facets of the cut polytope. *Oper. Res. Lett.*, 9(5):341–344, 1990.

[DS96] Caterina De Simone. A note on the Boolean quadric polytope. *Oper. Res. Lett.*, 19(3):115–116, 1996.

[DS90] Caterina De Simone. The cut polytope and the Boolean quadric polytope. *Discrete Math.*, 79(1):71–75, 1989/90.

[DSDL94] Caterina De Simone, Michel Deza, and Monique Laurent. Collapsing and lifting for the cut cone. *Discrete Math.*, 127(1–3):105–130, 1994. Graph theory and applications (Hakone, 1990).

[GLS81] Martin Grötschel, László Lovász, and Alexander Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[GLS84] Martin Grötschel, László Lovász, and Alexander Schrijver. Corrigendum to our paper: "The ellipsoid method and its consequences in combinatorial optimization". *Combinatorica*, 4(4):291–295, 1984.

[GLS93] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization.* Springer-Verlag, Berlin, second edition, 1993.

[HK56] Alan J. Hoffman and Joseph B. Kruskal. Integral boundary points of convex polyhedra. In *Linear inequalities and related systems*, pages 223–246. Princeton University Press, Princeton, N. J., 1956. Annals of Mathematics Studies, no. 38.

[HLL98] Jill Hardin, Jon Lee, and Janny Leung. On the Boolean-quadric packing uncapacitated facility-location polytope. *Ann. Oper. Res.*, 83:77–94, 1998. Models and algorithms for planning and scheduling problems (Wernigerode, 1995).

[HP72] Peter L. Hammer and Uri N. Peled. On the maximization of a pseudo-Boolean function. *J. Assoc. Comput. Mach.*, 19:265–282, 1972.

[HR69] Peter L. Hammer and Sergiu Rudeanu. Pseudo-Boolean programming. *Operations Res.*, 17:233–261, 1969.

[LP95] Monique Laurent and Svatopluk Poljak. On a positive semidefinite relaxation of the cut polytope. *Linear Algebra Appl.*, 223/224:439–461, 1995. Special issue honoring Miroslav Fiedler and Vlastimil Pták.

[LP96] Monique Laurent and Svatopluk Poljak. Gap inequalities for the cut polytope. *European J. Combin.*, 17(2–3):233–254, 1996. Discrete metric spaces (Bielefeld, 1994).

[LPR97] Monique Laurent, Svatopluk Poljak, and Franz Rendl. Connections between semidefinite relaxations of the max-cut and stable set problems. *Math. Programming, Ser. B*, 77(2):225–246, 1997. Semidefinite programming.

[LS90] Laszlo Lovász and Alexander Schrijver. Matrix cones, projection representations, and stable set polyhedra. In *Polyhedral combinatorics (Morristown, NJ, 1989)*, pages 1–17. Amer. Math. Soc., Providence, RI, 1990.

[LS91] Laszlo Lovász and Alexander Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optim.*, 1(2):166–190, 1991.

[NW88] George L. Nemhauser and Laurence A. Wolsey. *Integer and combinatorial optimization.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1988. A Wiley-Interscience Publication.

[Pad89] Manfred W. Padberg. The Boolean quadric polytope: Some characteristics, facets and relatives. *Math. Programming, Ser. B*, 45(1):139–172, 1989.

[SA90] Hanif D. Sherali and Warren P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM J. Discrete Math.*, 3(3):411–430, 1990.