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Ratios of Zerofree Balanced Ternary Integers

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Abstract

The integer $k = 247$ is not the ratio of two integers whose balanced ternary representations contain no nonleading zeroes.

The *balanced ternary* representation of an integer is the base 3 representation using the digits 0,1 and $\overline{1}(= -1)$. By the *digits* of an integer, we mean the digits of its balanced ternary representation. Call a *zerofree balanced ternary integer* an integer with no nonleading 0 digits.

John Selfridge and Carole Lacampagne [1] asked whether every $k \equiv \pm 1 \pmod 3$ could be represented as the ratio of two zerofree balanced ternary integers. We show that $k = 247$ cannot be so represented, and give a handful of other counterexamples.

Suppose we have such a nonzero m such that both m and km are zerofree balanced ternary integers. For each $n \geq 1$ set

- a_n = integer represented by n least significant digits of m
- b_n = integer represented by n least significant digits of km (= n least significant digits of ka_n)
- c_n = integer represented by highest digits of ka_n
- d_n = digit multiplying 3^n in m
- e_n = digit multiplying 3^n in km

We have

$$\begin{aligned} ka_n &= 3^n c_n + b_n \\ a_{n+1} &= a_n + 3^n d_n \\ b_{n+1} &= b_n + 3^n e_n \end{aligned}$$

Define the set

$$\Gamma = \{\pm 44, \pm 46, \pm 48, \pm 50, \pm 60, \pm 62, \pm 66, \pm 68, \pm 82, \pm 98, \pm 102, \pm 104, \pm 110, \pm 116\}.$$

Verify two properties of Γ:

1. Γ is the smallest set containing $\pm(247-1)/3 = \pm 82$ such that if $c \in \Gamma$ and $d, e \in \{-1, +1\}$ and $\gamma = (c + 247d - e)/3$ is an integer, then $\gamma \in \Gamma$.

2. Γ contains no zerofree balanced ternary integers.

We will prove by induction that for $n \geq 1$, the balanced ternary representation of m has at least n digits, and $c_n \in \Gamma$. For the base case, $n = 1$, nonzero m has at least 1 digit; $a_1 = b_1 = \pm 1$, $c_1 = (247a_1 - b_1)/3 = \pm 82$, so that $c_1 \in \Gamma$. Proceed by induction. Suppose that $n \geq 1$, the balanced ternary representation of m has at least n digits, and $c_n \in \Gamma$. By property 2, c_n is not a zerofree balanced ternary integer; therefore neither is $ka_n = 3^n c_n + b_n$, so that $a_n \neq m$, and m has at least $n+1$ digits. From

$$\begin{aligned} 247a_n &= 3^n c_n + b_n \\ 247a_{n+1} &= 3^{n+1} c_{n+1} + b_{n+1} \\ a_{n+1} &= a_n + 3^n d_n \\ b_{n+1} &= b_n + 3^n e_n \end{aligned}$$

one calculates

$$c_{n+1} = (c_n + 247d_n - e_n)/3,$$

with $d_n, e_n \in \{-1, +1\}$. So by property 1, the integer c_{n+1} is in Γ. The induction is complete. For all $n \geq 1$ we have $c_n \in \Gamma$ and m has at least n digits, so that $|m| > 3^{n-1}/2$, contradicting the finiteness of m.

By the same method of proof one calculates that the smallest positive integers, other than multiples of three, which are not the ratios of zerofree balanced ternary integers are: 247, 277, 967, 977, 1211, 1219, 1895, 1937, 1951, 1961, 2183, 2191, 2911, 2921, 3029, 3641, 3649, 5095, 5111, 5815, 5839, 5849, 7271, 7279, 8177, 8743, 8753, 10931, 10939 and 11681.

References

[1] Richard K. Guy, *Unsolved Problems in Number Theory*, Second Edition, Springer-Verlag, 1994. Problem F31, pages 266-267.