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Multihomogeneous Newton Methods

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Abstract. We study analytic multihomogeneous functions and a multihomogeneous Newton's method for finding their zeros. We give a convergence result for this iteration and we study two examples : the evaluation map and the generalized eigenvalue problem.

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1. Introduction and Main Results.

Section 1.1. Introduction. In a series of papers, Shub [8] and Shub and Smale [9], [10], [11], [12], [13], studied a projective version of Newton's method for homogeneous systems. Their particular focus was the problem of finding zeros of systems of n homogeneous polynomial equations in $n+1$ unknowns. In this paper we study multihomogeneous functions and a multihomogeneous Newton's method for finding their zeros.

Here are three examples of multihomogeneous functions. Let $\mathcal{H}_d$ be the space of homogeneous polynomials of degree d defined on $\mathbb{C}^n$. Let $(d) = (d_1, \dots, d_m)$ and $\mathcal{H}_{(d)} = \prod_{i=1}^m \mathcal{H}_{d_i}$. So elements of $\mathcal{H}_{(d)}$ represent polynomial functions $f : \mathbb{C}^n \to \mathbb{C}^m$ where $f = (f_1, \dots, f_m)$ and f_i is homogeneous of degree d_i. The evaluation map

$$ev : \mathcal{H}_{(d)} \times \mathbb{C}^n \to \mathbb{C}^m$$

$ev(f, x) = f(x)$ is multihomogeneous. Each coordinate function of ev is linear in f and homogeneous of degree d_i in x.

A second example is given by the generalized eigenvalue problem. Let $A, B : \mathbb{C}^n \to \mathbb{C}^n$ be linear operators. Then

$$F_{(A,B)} : \mathbb{C}^2 \times \mathbb{C}^n \to \mathbb{C}^n, \quad F_{(A,B)}(\alpha, \beta, x) = (\alpha B - \beta A)(x),$$

is bilinear, i.e. it is linear in (α, β) and linear in x. The generalized eigenvalue problem is to find the zeros of $F_{(A,B)}$.

A third example is given by homogenization. If $f : \mathbb{E} \to \mathbb{F}$ is complex analytic then

$$\hat{f} : \mathbb{E} \times \mathbb{C}^\star \to \mathbb{F}, \quad \hat{f}(x, t) = f(x/t),$$

is complex analytic and homogeneous of degree 0.

In general let $\mathbb{E}_1, \dots, \mathbb{E}_k$ be complex or real vector spaces and $\mathbb{F} = \mathbb{C}^m$ *or* $\mathbb{R}^m$. Let $\mathbb{E} = \mathbb{E}_1 \times \dots \times \mathbb{E}_k$ and $((d)) = ((d_1), \dots, (d_k))$, $(d_i) = (d_{1i}, \dots, d_{ki})$ for $i = 1, \dots, m$. Then $f : \mathbb{E} \to \mathbb{F}$ is multihomogeneous of degree $((d))$ if and only if the i^{th} coordinate function satisfies

$$f_i(\lambda_1 x_1, \dots, \lambda_k x_k) = \prod_{j=1}^k \lambda_j^{d_{ji}} f_i(x_1, \dots, x_k)$$

for $(x_1, \dots, x_k) \in \mathbb{E}$ and $(\lambda_1, \dots, \lambda_k)$ a $k-$uple of scalars, i.e $(\lambda_1, \dots, \lambda_k) \in \mathbb{G} = \mathbb{C}^k$ *or* $\mathbb{R}^k$ as the case may be.

We assume throughout that f is analytic. The domain of f may be an open subset of $\mathbb{E}$, but with abuse of notation we continue to write $f : \mathbb{E} \to \mathbb{F}$.

The multihomogeneous projective Newton iteration we define below is defined on $\mathbb{E}$ but is invariant under the natural identifications which define the product of the projective spaces $\mathbb{P}(\mathbb{E}_1) \times \dots \times \mathbb{P}(\mathbb{E}_k)$. Indeed this is much of our motivation in defining Newton's iteration as we do, but it is important to keep in mind that implementations of the method reside in $\mathbb{E}$ itself !

For the rest of this paper we will assume that $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ are complex and finite dimensional vector spaces and that $\mathbb{E}_i$ has an Hermitian product $< \, , \, >_i$. For the case where $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ are real we would replace the Hermitian product by an inner product. Also, we denote

$$\mathbb{E}^\star = (\mathbb{E}_1 \setminus \{0\}) \times \dots \times (\mathbb{E}_k \setminus \{0\}).$$

If $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{G}$ we define

$$\times\lambda : \mathbb{E} \to \mathbb{E}$$

by

$$\times\lambda x = (\lambda_1 x_1, \dots, \lambda_k x_k).$$

Then $\mathbb{P}(\mathbb{E}_1) \times \dots \times \mathbb{P}(\mathbb{E}_k)$ is the quotient of $\mathbb{E}^\star$ by the action of $\mathbb{G}^\star = (\mathbb{C} \setminus \{0\}) \times \dots \times (\mathbb{C} \setminus \{0\})$ (k times). For $x \in \mathbb{E}^\star$, $x = (x_1, \dots, x_k)$, we let $x_i^\perp$ be the Hermitian complement of x_i in $\mathbb{E}_i$,

$$x^\perp = \prod_{i=1}^k x_i^\perp \subset \mathbb{E} \quad and \quad V_x = (x^\perp)^\perp \subset \mathbb{E}.$$

Notice that V_x is also the subspace of $\mathbb{E}$ spanned by the vectors $(0, \ldots, x_i, \ldots, 0)$, $i = 1, \ldots, k$. The dimension of V_x is k since $x \in \mathbb{E}^\star$. For each i, $x_i^\perp$ is a natural representative of the tangent space $T_{x_i}\mathbb{P}(\mathbb{E}_i)$ and hence $x^\perp$ is a natural representative of the tangent space

$$T_x\left(\prod_{i=1}^{k} \mathbb{P}(\mathbb{E}_i)\right) = \prod_{i=1}^{k} T_{x_i}(\mathbb{P}(\mathbb{E}_i)).$$

If $x = \times\lambda y$ for $\lambda \in \mathbb{C}^\star$ and $v \in y^\perp$ then $\times\lambda v \in x^\perp$ represents the same tangent vector in $T_x\left(\prod \mathbb{P}(\mathbb{E}_i)\right)$.

We now define an Hermitian structure on $\mathbb{E}$ depending on x and hence on $x^\perp$ by

$$< v, w >_x = \sum_{i=1}^{k} \frac{< v_i, w_i >_i}{< x_i, x_i >_i}$$

for $x \in \mathbb{E}^\star$, v and $w \in \mathbb{E}$. If $\lambda \in \mathbb{C}^\star$ then $\times\lambda$ maps $x^\perp$ onto $(\times\lambda x)^\perp$ and

$$(*) \qquad < \times\lambda v, \times\lambda w >_{(\times\lambda x)} = < v, w >_x$$

so $< \, , \, >_x$ defines an Hermitian product on $T_x(\mathbb{P}(\mathbb{E}_1) \times \ldots \times \mathbb{P}(\mathbb{E}_k))$. Condition $(*)$ says that $\times\lambda$ is an isometry from $x^\perp$ to $(\times\lambda x)^\perp$ as well as $\mathbb{E}$ to $\mathbb{E}$ with their given Hermitian products.

We are now ready to define the multihomogeneous projective Newton iteration for f. We denote this map as $N_f : \prod_i \mathbb{P}(\mathbb{E}_i) \hookleftarrow$.

Definition 1.

$$N_f(x) = f(x) - (Df(x)|_{x^\perp})^\dagger f(x).$$

Here $(Df(x)|_{x^\perp})^\dagger$ is the Moore-Penrose inverse of the restriction of $Df(x)$ to $x^\perp$. We recall that if $A : V_1 \to V_2$ is a linear map between two finite dimensional complex vector spaces with Hermitian products, then the Moore-Penrose inverse of A maps V_2 to V_1 and is the composition of two maps

$$A^\dagger : V_2 \to V_1, \quad A^\dagger = i\Pi$$

where Π is the Hermitian projection of V_2 onto imA and $i : imA \to V_1$ is the right inverse of A whose image in V_1 is the Hermitian complement of $kerA$. If A is surjective then $A^\dagger = A^\star(AA^\star)^{-1}$ where $A^\star$ is the adjoint of A. In this paper we only take Moore-Penrose inverses of surjective linear maps, unless otherwise noted.

N_f is of course naturally defined on $\mathbb{E}$, we use N_f to denote this map as well. From the context it should be clear which map we mean and whether we mean Newton's iteration, projective Newton's iteration or multihomogeneous projective Newton's iteration.

Proposition 1. N_f is well-defined i.e. if $y = \times\lambda x$ for $x \in \mathbb{E}^\star$ and $\lambda \in \mathbb{C}^\star$ then $N_f(y) = \times\lambda N_f(x)$.

For the proof we use a lemma which will be useful later. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$ where $\Lambda_i = \prod_{j=1}^{k} \lambda_j^{d_{ji}}$ and f has degree $((d))$. Then

Lemma 1. 1. $f(\times\lambda x) = \times\Lambda f(x)$,
2. $Df(\times\lambda x) \times \lambda = \times\Lambda Df(x)$,
3. $D^l f(\times\lambda x)(\times\lambda, \ldots, \times\lambda) = \times\Lambda D^l f(x)$,
4. $(\times\lambda)^{-1}(Df(\times\lambda x)|_{(\times\lambda x)^\perp})^\dagger = (Df(x)|_{x^\perp})^\dagger(\times\Lambda)^{-1}$.

Proof of Lemma 1. 1 is the definition of multihomogeneity. 2 and 3 then follow from the chain rule, 4 follows from 2 since $(\times\lambda)$ is an isometry which maps $kerDf(x)$ to $kerDf(\times\lambda x)$ and hence $im(Df(x)|_{x^\perp})^\dagger$ to $im(Df(\times\lambda x)|_{(\times\lambda x)^\perp})^\dagger$. □

Proof of Proposition 1. We have

$$(Df(\times\lambda x)|_{(\times\lambda x)^\perp})^\dagger f(\times\lambda x) = (\times\lambda)(Df(x)|_{x^\perp})^\dagger(\times\Lambda)^{-1}(\times\Lambda)f(x) = (\times\lambda)(Df(x)|_{x^\perp})^\dagger f(x)$$

by 4 and 1 of Lemma 1. □

Our analysis of the multihomogeneous Newton method closely follows Smale [14]. There are three important quantities associated to f and x which we now define.

Definition 2. 1. $\gamma(f,x) = \max(1, \sup_{k\geq 2} \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(x)}{k!} \right\|_x^{\frac{1}{k-1}}),$
2. $\beta(f,x) = \|(Df(x)|_{x^\perp})^\dagger f(x)\|_x$,
3. $\alpha(f,x) = \beta(f,x)\gamma(f,x)$.

In the definition of $\gamma(f,x)$, $\| \ \|_x$ is the operator norm with respect to $< \, , \, >_x$.
We now verify that $\alpha(f,x)$, $\beta(f,x)$ and $\gamma(f,x)$ are defined on $I\!P(I\!E_1) \times \ldots \times I\!P(I\!E_k)$.

Proposition 2. For any $x \in I\!E^\star$ and $\lambda \in \mathbb{C}^\star$ we have $\star(f,x) = \star(f, \times\lambda x)$ with $\star \in \{\alpha, \beta, \gamma\}$.

Proof of Proposition 2. By Lemma 1

$$(\times\lambda)(Df(x)|_{x^\perp})^\dagger f(x) = (Df(\times\lambda x)|_{(\times\lambda x)^\perp})^\dagger f(\times\lambda x)$$

as in Proposition 1 and

$$(Df(x)|_{x^\perp})^\dagger D^k f(x) = (\times\lambda)^{-1}(Df(\times\lambda x)|_{(\times\lambda x)^\perp})^\dagger D^k f(\times\lambda x)(\times\lambda, \ldots, \times\lambda).$$

Since $\times\lambda$ is an isometry we obtain the required result. □

We recall that for $i = 1, \ldots, k$ the Riemannian distance in $I\!P(I\!E_i)$ is given by

$$d_R(x_i, y_i) = arccos \frac{| < x_i, y_i >_i |}{\|x_i\|_i \|y_i\|_i},$$

and in $I\!P(I\!E_1) \times \ldots \times I\!P(I\!E_k)$ by

$$d_R(x,y) = \left(\sum_{i=1}^k d_R(x_i, y_i)^2 \right)^{1/2}$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k) \in I\!E^\star$. Here and throughout we confuse $x_i \in I\!E_i \setminus \{0\}$ and $x \in I\!E^\star$ with their equivalence classes in $I\!P(I\!E_i)$ and $I\!P(I\!E_1) \times \ldots \times I\!P(I\!E_k)$ respectively.

Our main Theorems concerning the convergence of the multihomogeneous Newton iteration are summarized in the following lines and proved after.

Section 1.2. α-Theorem.

Theorem 1. There is a universal constant $\alpha_u > 0$ with the following property : for any multihomogeneous system $f : I\!E \to I\!F$ and $x \in I\!E^\star$, if $\alpha(f,x) \leq \alpha_u$ and $Df(x)|_{x^\perp}$ (the restriction of $Df(x)$ to $x^\perp$) is onto, then the multihomogeneous Newton sequence

$$x_0 = x, \quad x_{k+1} = x_k - (Df(x_k)|_{x_k^\perp})^\dagger f(x_k),$$

satisfies

$$\|x_{k+1} - x_k\|_{x_k} \leq \left(\frac{1}{2}\right)^{2^k - 1} \beta(f,x)$$

for any $k \geq 0$. This sequence converges to a zero $\zeta \in I\!E^\star$ of f and

$$d_R(\zeta, x_k) \leq \sigma \left(\frac{1}{2}\right)^{2^k - 1} \beta(f,x)$$

with

$$\sigma = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{2^i - 1} = 1.6328\ldots$$

We can take $\alpha_u = 1/137$.

α-Theorems are available in several different contexts. This approach of Newton's methods finds its origins in a paper by S. Smale [12] for analytic functions $f : \mathbb{E} \to \mathbb{F}$ with $\mathbb{E}$ and $\mathbb{F}$ Banach spaces. Sharpened results are given by Royden [7], Shub-Smale [9] and Wang [16].

Newton's method can be generalized to search for zeros of maps $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, using the Moore-Penrose inverse of the derivative : $N_f^{MP}(x) = x - Df(x)^\dagger f(x)$. This method appears in the book of Allgower and Georg [1]. An α-Theorem is given in this context by Shub and Smale in [12].

Projective Newton's method has been proposed by Shub in [8] for homogeneous systems $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$ and is defined by $N_f^P(x) = x - Df(x)|_{x^\perp}^{-1} f(x)$. An α-Theorem has been given by Malajovich in [6]. In the same paper this author also studies Moore-Penrose Projective Newton's iteration $N_f^{MPP}(x) = x - Df(x)^\dagger f(x)$ for such homogeneous systems.

Section 1.3. γ-Theorem.

Theorem 2. There are universal constants γ_u *and* $c_u > 0$ *with the following properties : let* $\zeta \in \mathbb{E}^\star$ *be a zero of* f *with* $Df(\zeta)$ *onto and* $x \in \mathbb{E}^\star$. *If*

$$||x - \zeta||_\zeta \gamma(f, \zeta) \leq \gamma_u$$

then the multihomogeneous Newton sequence converges to a zero $\zeta' \in \mathbb{E}^\star$ *of* f *and*

$$d_R(\zeta', x_k) \leq \sigma \left(\frac{1}{2}\right)^{2^k - 1} \beta(f, x).$$

Moreover

$$d_R(\zeta', x) \leq 3||x - \zeta||_\zeta$$

and

$$d_R(\zeta', N_f(x)) \leq c_u \gamma(f, \zeta) ||x - \zeta||_\zeta^2.$$

Corollary 1. There is a universal constant δ_u *with the following property : let* $\zeta \in \mathbb{E}^\star$ *be a zero of* f *with* $Df(\zeta)$ *onto and* $x \in \mathbb{E}^\star$. *If*

$$d_R(x, \zeta)\gamma(f, \zeta) \leq \delta_u$$

then multihomogeneous Newton sequence converges to a zero $\zeta' \in \mathbb{E}^\star$ *of* f *and*

$$d_R(\zeta', x_{k+1}) \leq \left(\frac{1}{2}\right)^{2^k} d_R(\zeta, x).$$

This theorem gives the size of the attraction basin around a given zero of the system f. The affine case is treated by Shub-Smale in [9] and in [12] for overdetermined systems and Moore-Penrose Newton's iteration. For homogeneous systems $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$ see Blum-Cucker-Shub-Smale [2] Chapter 14, Theorem 1. The γ-Theorem is the main ingredient to prove complexity results for path-following methods. It will be used in the other sections.

Section 1.4. Newton's method for the evaluation map.

Let $\mathcal{H}_d$ be the space of homogeneous polynomials of degree d defined on $\mathbb{C}^n$. Let $(d) = (d_1, \ldots, d_m)$ and $\mathcal{H}_{(d)} = \prod_{i=1}^m \mathcal{H}_{d_i}$. The evaluation map

$$ev : \mathcal{H}_{(d)} \times \mathbb{C}^n \to \mathbb{C}^m,$$

$ev(f,x) = f(x)$ is bihomogeneous : each coordinate function $ev(f_i, x)$ is linear in f_i and homogeneous of degree d_i in x.

The Hermitian structure over $\mathcal{H}_{(d)}$ is the product structure : for $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$ we define

$$< f, g >= \sum_{i=1}^{m} < f_i, g_i >$$

and

$$< f_i, g_i >= \sum_{|\alpha|=d_i} \binom{d_i}{\alpha}^{-1} a_{i,\alpha} \bar{b}_{i,\alpha}$$

with $f_i(z) = \sum_{|\alpha|=d_i} a_{i,\alpha} z^\alpha$, $g_i(z) = \sum_{|\alpha|=d_i} b_{i,\alpha} z^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\binom{d_i}{\alpha} = \frac{d_i!}{\alpha_1! \ldots \alpha_n!}$.

Let us denote

$$V = \{(g, y) \in \mathcal{H}_{(d)} \times \mathbb{C}^n \ : \ ev(g, y) = 0\}.$$

For any $(f, x) \in \mathcal{H}_{(d)}$ close enough to V, multihomogeneous Newton's method constructs a sequence $N_{ev}^{(k)}(f, x)$ which converges quadratically to a unique element in V denoted by $M_{ev}(f, x)$. This defines a function which projects a neighborhood of V onto V itself. By Theorem 2, the size of this neighborhood is controled by

$$\gamma(ev, V) = \max_{(g,y)\in V} \gamma(ev, g, y).$$

We have obtained the following estimation

Theorem 3. $\gamma(ev, V) \leq \dfrac{D(D-1)}{2}(1 + \sqrt{m})$ *with* $D = \max d_i$.

The properties of $M_{ev}(f, x)$ are summarized in the following

Theorem 4. Let $(f, x) \in \mathcal{H}_{(d)} \times \mathbb{C}^n$ *be such that*

$$d_R((f, x); V) \leq \frac{2\delta_u}{D(D-1)(1+\sqrt{m})}.$$

Let $(g, y) \in V$ *which satisfies* $d_R((f, x); V) = d_R((f, x); (g, y))$. *Then, multihomogeneous Newton's sequence* $N_{ev}^{(k)}(f, x)$ *converges to* $M_{ev}(f, x) \in V$. *Moreover*

$$d_R(M_{ev}(f, x); (f, x)) \leq 3 \left(\frac{||f-g||^2}{||g||^2} + \frac{||x-y||^2}{||x||^2} \right)^{1/2}$$

and

$$d_R(M_{ev}(f, x); N_{ev}^{(k)}(f, x)) \leq \left(\frac{1}{2}\right)^{2^k} \left(\frac{||f-g||^2}{||g||^2} + \frac{||x-y||^2}{||x||^2} \right)^{1/2}.$$

Section 1.5. Path-Following.

In the following Theorem we analyse the complexity of a path-following method to solve approximately a system of equations. The context we deal with is the following : for any $t \in [0,1]$ let $f_t : \mathbb{E} \to \mathbb{F}$ be a multihomogeneous system depending smoothly on t. We also suppose that for $x \in \mathbb{E}^\star$, $\dim \mathbb{F} = \dim x^\perp$, that is, after disregarding the homogenizing directions, the number of equations and unknowns are the same. Let ζ_t be a smooth curve in $\mathbb{E}^\star$ such that $f_t(\zeta_t) = 0$ and $Df_t(\zeta_t)|_{\zeta_t^\perp}$ is an isomorphism. We associate to a subdivision $0 = t_0 < t_1 < \ldots < t_p = 1$ a sequence x_i defined by

$$x_0 = \zeta_0 \ \ \textit{and} \ \ x_{i+1} = N_{f_{t_{i+1}}}(x_i).$$

When the subdivision size $\max|t_{i+1}-t_i|$ is small enough then

$$d_R(x_i,\zeta_{t_i})\gamma(f_{t_i},\zeta_{t_i}) \leq \delta_u$$

so that, by Theorem 2, x_i may be taken as the starting point for a multihomogeneous Newton sequence $N^k_{f_{t_i}}(x_i)$ converging quadratically towards ζ_{t_i}.

The complexity of this path-following method is given by p, the number of points in the subdivision. Before to state our result we have to introduce more invariants :

Definition 3.

$$\gamma = \max_{0\leq t\leq 1} \gamma(f_t, x_t),$$

$$\mu = \max_{0\leq t\leq 1} \|Df_t(\zeta_t)^\dagger\|_{\zeta_t},$$

and L the length of the curve $t\in[0,1]\to f_t$.

μ is the condition number of the curve $t\in[0,1]\to(f_t,\zeta_t)$. Our main result asserts that the complexity of this path-following method depends mainly on the product $\mu\gamma L$.

Theorem 5. There is a partition $0=t_0<t_1<\ldots<t_p=1$ with

$$p = \left\lceil \frac{2}{\delta_u}\gamma\mu L \right\rceil$$

such that, for each $i=0\ldots p$ the sequence defined by

$$x_0 = \zeta_0 \quad and \quad x_{i+1} = N_{f_{t_{i+1}}}(x_i)$$

satisfies

$$d_R(x_i,\zeta_{t_i})\gamma(f_{t_i},\zeta_{t_i}) \leq \delta_u.$$

There is a considerable literature concerning path-following methods. The book of Allgower and Georg [1] is a good introduction to this subject. We follow here the lines of Shub and Smale : [9] for the affine case, [12] for the affine underdetermined case. The case of sparse polynomial systems is studied by Dedieu in [4].

Section 1.6. Newton's Method for the Generalized Eigenvalue Problem.

Let $(A,B)\in\mathcal{M}_n(\mathbb{C})\times\mathcal{M}_n(\mathbb{C})$ be a matrix pair. A pair is called singular when the homogeneous polynomial $P_{(A,B)}(\alpha,\beta)=det(\beta A-\alpha B)$ is identically 0. Otherwise it is said to be regular. In such a case this polynomial has degree n and its zeros consist in n lines through the origin. These lines are the eigenvalues of the pair (A,B) and the nontrivial solutions $x\in\mathbb{C}^n$ of the equation

$$(\beta A-\alpha B)x=0$$

are the corresponding eigenvectors.

In order to compute approximately the eigenvalues and eigenvectors of this matrix pair we introduce

$$F_{(A,B)}:\mathbb{C}^2\times\mathbb{C}^n\to\mathbb{C}^n,\quad F_{(A,B)}(\alpha,\beta,x)=(\beta A-\alpha B)x$$

which is a bihomogeneous polynomial with degree 1 in each variable. Multihomogeneous Newton's iterate is thus equal to

$$N_{F_{(A,B)}}(\alpha,\beta,x)=(\alpha,\beta,x)-DF_{(A,B)}(\alpha,\beta,x)|^\dagger_{(\alpha,\beta,x)^\perp}(\beta A-\alpha B)x.$$

A more precise description of this iterate is given in Section 2.6.

Our objective is here to describe the complexity of a path-following method to compute approximately an eigenpair (i.e. an eigenvalue, eigenvector pair) associated with a matrix pair. Let (A_0, B_0) and (A_1, B_1) be two regular matrix pairs. We consider two smooth curves

$$t \in [0,1] \to (1-t)(A_0, B_0) + t(A_1, B_1) = (A_t, B_t)$$

and

$$t \in [0,1] \to (\alpha_t, \beta_t, x_t) \in \mathbb{C}^2 \times \mathbb{C}^n$$

so that

$$(\beta_t A_t - \alpha_t B_t) x_t = 0.$$

We also suppose that (α_t, β_t) is a simple eigenvalue for the pair (A_t, B_t). The path following method consists in the following :

$$0 = t_0 < t_1 < \ldots < t_p = 1$$

is a given subdivision and

$$(a_0, b_0, z_0) = (\alpha_0, \beta_0, x_0),$$

$$(a_{i+1}, b_{i+1}, z_{i+1}) = N_{i+1}(a_i, b_i, z_i), \quad i = 0 \ldots p-1.$$

Here N_i is the multihomogeneous Newton's iterate associated with the matrix pair (A_{t_i}, B_{t_i}) . Starting from the eigenpair (α_0, β_0, x_0) of (A_0, B_0) we obtain an approximate eigenpair (a_p, b_p, z_p) for (A_1, B_1). Approximate means here $\alpha(F_{(A_1,B_1)}, (a_p, b_p, z_p) \le \alpha_u$ so that, by Theorem 1, the sequence $N_p^k(a_p, b_p, z_p))$, $k \ge 1$, converges quadratically to (α_1, β_1, x_1).

Our main theorem in this section gives a bound for a sufficient p in terms of the condition number of the path. This last quantity is defined by

Definition 4.

$$\mu = \max_{0 \le t \le 1} \mu(A_t, B_t, \alpha_t, \beta_t, x_t),$$

with

$$\mu(A, B, \alpha, \beta, x) = \|DF_{(A,B)}(\alpha, \beta, x)|^{\dagger}_{(\alpha,\beta,x)^{\perp}}\|_{(\alpha,\beta,x)}.$$

$\mu(A, B, \alpha, \beta, x)$ is the condition number for the generalized eigenvalue problem and μ the condition number for the path.

Theorem 6. There is a partition $0 = t_0 < t_1 < \ldots < t_p = 1$ *with* $p = \lceil \frac{1}{\delta_u} \max(r, s) \rceil$,

$$r = 2\mu \left(\|A_0 - A_1\|_F^2 + \|B_0 - B_1\|_F^2 \right)^{1/2},$$

$$s = \mu^2 \max\left((\|A_0\|^2 + \|B_0\|^2)^{1/2}, (\|A_1\|^2 + \ |B_1\|^2)^{1/2} \right) \left(\|A_0 - A_1\|_F^2 + \|B_0 - B_1\|_F^2 \right)^{1/2},$$

such that (a_p, b_p, z_p) *is an approximate eigenpair for* (A_1, B_1).

Here $\|A\|$ is the spectral norm and $\|A\|_F$ the Frobenius norm.

Remark. Such a path-following method might be combined with a "divide and conquer" strategy as in Li [5] :

$$A_0 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

and similarly for B_0 and B_1. See, also, Li's discussion of the number of solutions of $(\beta A - \alpha B)x = 0$ considered as a quadratic or a bihomogeneous system of equations. The bihomogeneous context seems more natural.

This paper is organized as follow : in Section 2.1 we give some results about the angle between two subspaces in an Euclidean or Hermitian space. These results will be useful later. We present them in a separate section to make their lecture easier. In Section $2.i$, $2 \le i \le 6$, we give the proofs of the theorems presented in Section $1.i$.

2. Proofs of Theorems.

Section 2.1. Angles between subspaces in an Hermitian space.

We denote by $\mathbb{E}$ a complex Hermitian space or a real Euclidean space. To measure the distance between two vector subspaces V and W in $\mathbb{E}$ it is useful to consider the following quantity :

Definition 5.

$$d(V,W) = \max_{v \in V^*} \min_{w \in W} \frac{\|v - w\|}{\|v\|}.$$

This number is the maximum of the sine of a given vector $v \in V$ with its orthogonal projection on W. It also has the following characterizations (Π_X denotes the orthogonal projection on X) :

Proposition 3. 1. $d(V,W) = \|\Pi_{W^\perp}\Pi_V\|$,
2. $d(V,W) = d(W^\perp, V^\perp)$,
3. $d(V,W) = d(V \cap (V \cap W)^\perp, W \cap (V \cap W)^\perp)$.

Proof. 1 goes as follows

$$d(V,W) = \max_{\substack{v \in V \\ \|v\|=1}} \|(id - \Pi_W)v\| = \max_{\substack{v \in V \\ \|v\| \le 1}} \|\Pi_{W^\perp} v\| = \max_{\|v\|=1} \|\Pi_{W^\perp}\Pi_V v\| = \|\Pi_{W^\perp}\Pi_V\|.$$

2 is a consequence of 1 since the norms of an operator and its transpose are equal. Let us prove the third assertion. For any $v \in V$ let us write it

$$v = v_1 + v_2 \in (V \cap W) \oplus (V \cap (V \cap W)^\perp).$$

Then

$$\Pi_W v = w_1 + w_2 \in (V \cap W) \oplus (W \cap (V \cap W)^\perp)$$

with $w_1 = v_1$ and $w_2 = \Pi_{W \cap (V \cap W)^\perp}(v_2)$. □

The proof of Proposition 3. 1. may be found in Stewart-Sun [15] with other useful properties of $d(V,W)$. This number measures the distance of V from W but is not stricto sensu a distance because in general $d(V,W)$ and $d(W,V)$ are not equal. For this reason it is convenient to define

$$\delta(V,W) = \max(d(V,W), d(W,V)).$$

δ is a (true) distance in the set of vector subspaces in $\mathbb{E}$. We also have

Proposition 4. 1. $0 \le d(V,W) \le 1$,
2. $d(V,W) = 0$ *if and only if* $V \subset W$,
3. $d(V,W) < 1$ *if and only if* $V \cap W^\perp = \{0\}$,
4. $d(V_1,V_3) \le d(V_1,V_2) + d(V_2,V_3)$,
5. If $V_1 \subset V_2$ *then* $d(V_1,W) \le d(V_2,W)$ *and if* $W_1 \subset W_2$ *then* $d(V,W_2) \le d(V,W_1)$,
6. $d(V, W_1 + W_2) \le \min(d(V,W_1), d(V,W_2))$,
7. If V_1 *and* V_2 *are orthogonal then* $d(V_1 \oplus V_2, W) \le d(V_1,W) + d(V_2,W)$ *and* $d(V_1 \oplus V_2, W) \le \sqrt{2}\max(d(V_1,W), d(V_2,W))$,
8. If $dimV = dimW$ *then* $d(V,W) = d(W,V)$.

These properties and more precisely 2, 4 and 8 show that $d(V,W)$ defines a distance (sticto sensu) on the Grassmaniann manifold $\mathbb{G}_{n,p}$ of p-dimensional vector subspaces in $\mathbb{C}^n$. When $dimV \neq dimW$ then, by 3, $\delta(V,W) = 1$, while, when $dimV = dimW$, $d(V,W) = d(W,V) = \delta(V,W)$. In the sequel we only use $d(V,W)$.

Proof. 1 to 7.1 are staightforward. We now prove 7.2. If v_1 and v_2 are orthogonal then $\|v_1\| + \|v_2\| \leq \sqrt{2}\|v_1 + v_2\|$. So if V_1 and V_2 are orthogonal

$$d(V_1 \oplus V_2, W) = \|\Pi_{W^\perp}(v_1 + v_2)\| \leq \|\Pi_{W^\perp} v_1\| + \|\Pi_{W^\perp} v_2\| \leq d(V_1, W)\|v_1\| + d(V_2, W)\|v_2\| \leq$$

$$\max(d(V_1, W), d(V_2, W))(\|v_1\| + \|v_2\|) \leq \sqrt{2}\max(d(V_1, W), d(V_2, W))\|v_1 + v_2\|.$$

To prove 8 we first remark that $d(V,W)$ is the largest singular value of $\Pi_{W^\perp}\Pi_V = (id - \Pi_W)\Pi_V = \Pi_V - \Pi_W\Pi_V$ and similarly $d(W,V)$ is the largest singular value of $\Pi_W - \Pi_V\Pi_W$. Let us introduce a unitary trasformation Q such that $Q^2 = id$ and $QV = W$. The existence of such an involution will be proved at the end of this section. We have $\Pi_W = Q\Pi_V Q$ so that

$$\Pi_{W^\perp}\Pi_V = \Pi_V - \Pi_W\Pi_V = \Pi_V - Q\Pi_V Q\Pi_V$$

and similarly

$$\Pi_{V^\perp}\Pi_W = Q(\Pi_V - Q\Pi_V Q\Pi_V)Q.$$

Thus $\Pi_{W^\perp}\Pi_V$ and $\Pi_{V^\perp}\Pi_W$ have the same singular values and $d(V,W) = d(W,V)$. □

Appendix.

Let V and W be two vector subspaces in $\mathbb{E}$ with the same dimension n. The proof of Proposition 4.8 requires the existence of an involution Q in $\mathbb{E}$ which sends V onto W. The existence of such an involution may be well known, but we have not find it in the literature. A proof of the fact may be derived from the CS decomposition for partitionned unitary matrices, see Stewart-Sun [15]. We give here a concise and elegant construction due to A. J. Hoffman.

We only consider the case $\mathbb{E} = \mathbb{C}^{2n}$, $V \cap W = \{0\}$ and $V \oplus W = \mathbb{C}^{2n}$. The general case is easily deduced from this one. We also suppose that V is spanned by the n first vectors of the canonical basis in $\mathbb{C}^{2n}$. Let us introduce two $2n \times n$ matrices :

$$S = \begin{pmatrix} I_n \\ O \end{pmatrix} \quad and \quad T = \begin{pmatrix} A \\ C \end{pmatrix}$$

such that the columns of T span W and T is orthonormal. Notice that S spans V. Let us write $AU = H$ the polar decomposition of A : U is unitary and H positive semidefinite; $TU = \begin{pmatrix} H \\ B^\star \end{pmatrix}$ also spans W. We remark now that $B^\star$ is nonsingular : if $B^\star x = 0$ then $TUx = \begin{pmatrix} Hx \\ 0 \end{pmatrix}$ so that $TUx \in V \cap W = \{0\}$. This gives $x = 0$ since U is unitary and T orthonormal. B is also nonsingular. Let us now consider the following $2n \times 2n$ matrix

$$Q = \begin{pmatrix} H & B \\ B^\star & -B^{-1}HB \end{pmatrix}.$$

We have

$$H^2 + BB^\star = \begin{pmatrix} H & B \end{pmatrix} \begin{pmatrix} H \\ B^\star \end{pmatrix} = U^\star T^\star TU = I_n$$

so that

$$HBB^\star = H(I_n - H^2) = (I_n - H^2)H = BB^\star H.$$

This yealds $B^{-1}HB = B^\star HB^{-\star}$ and consequently Q is Hermitian. Using the same argument we see easily that $Q^2 = I_{2n}$ so that Q is an involution. To complete the proof we remark that $QS = \begin{pmatrix} H \\ B^\star \end{pmatrix} = TU$ spans W. □

Section 2.2. α-Theorem.

In this section we give a proof of Theorem 1. It is shared into fourteen different lemmas. We first recall some notations and introduce some new ones. We let $x_i^\perp$ be the Hermitian complement of x_i in $\mathbb{E}_i$,

$$x^\perp = \prod_{i=1}^k x_i^\perp \subset \mathbb{E} \ \ and \ \ V_x = (x^\perp)^\perp \subset \mathbb{E}.$$

We also introduce

$$W_x = im(Df(x)|_{x^\perp})^\dagger \ \ and \ \ W_x^\perp = V_x \oplus (kerDf(x) \cap x^\perp)$$

where the $\oplus$ is orthogonal. We also use frequently for $x, \zeta \in \mathbb{E}^\star$ and $y \in \mathbb{E}$

$$u_x = ||y - x||_x \gamma(f, x), \quad u_\zeta = ||y - \zeta||_\zeta \gamma(f, \zeta)$$

and the function

$$\psi(u) = 2u^2 - 4u + 1, \quad 0 \le u \le 1 - \frac{\sqrt{2}}{2}.$$

This function is decreasing from 1 at $u = 0$ to 0 at $u = 1 - \sqrt{2}/2$. We first start with a linear algebra lemma.

Lemma 2.a. Let X and Y be Hermitian spaces and $A, B : X \to Y$ linear operators with B onto. If

$$||B^\dagger(B - A)|| \le \lambda < 1$$

then A is onto and

$$||A^\dagger B|| < \frac{1}{1 - \lambda}.$$

Proof. Let us denote $C = B - A$. We have $||B^\dagger C|| \le \lambda < 1$ so that $id_X - B^\dagger C$ is nonsingular and

$$||(id_X - B^\dagger C)^{-1}|| < \frac{1}{1 - \lambda}$$

by a classical argument. Because B is onto we have $BB^\dagger = id_Y$ so that

$$(B - C)^\dagger B(id_X - B^\dagger C) = (B - C)^\dagger (B - C) = \Pi$$

with Π the orthogonal projection on $(kerA)^\perp$. Thus

$$||A^\dagger B|| = ||(B - C)^\dagger B|| = ||\Pi(id_X - B^\dagger C)^{-1}|| \le ||\Pi|| \ ||(id_X - B^\dagger C)^{-1}|| < \frac{1}{1 - \lambda}.$$

Moreover

$$A = B - C = B - BB^\dagger C = B(id_X - B^\dagger C)$$

is the composition of B onto and $(id_X - B^\dagger C)$ nonsingular. Thus A is onto and we are done. $\square$

Lemma 2.b. Let $x \in \mathbb{E}^\star$ and $y \in \mathbb{E}$ be given such that $Df(x)|_{x^\perp}$ is onto and $u_x < 1 - \frac{\sqrt{2}}{2}$. Then $Df(y)|_{x^\perp}$ is onto and

$$||(Df(y)|_{x^\perp})^\dagger Df(x)|_{x^\perp}||_x \le \frac{(1 - u_x)^2}{\psi(u_x)}.$$

Proof. $Df(y) = Df(x) + \sum_{k \ge 2} k \dfrac{D^k f(x)}{k!}(y - x)^{k-1}$ so that

$$(Df(x)|_{x^\perp})^\dagger (Df(y)|_{x^\perp} - Df(x)|_{x^\perp}) = \sum_{k \ge 2} k (Df(x)|_{x^\perp})^\dagger \frac{D^k f(x)}{k!}(y - x)^{k-1}|_{x^\perp}.$$

If we take the operator norm of both sides we get

$$\|(Df(x)|_{x^\perp})^\dagger (Df(y)|_{x^\perp} - Df(x)|_{x^\perp})\|_x \le \sum_{k\ge 2} k\gamma(f,x)^{k-1}\|y-x\|_x^{k-1} = \sum_{k\ge 2} k u_x^{k-1} = \frac{1}{(1-u_x)^2} - 1$$

and this number is < 1 since $u_x < 1 - \frac{\sqrt{2}}{2}$. By Lemma 2.a $Df(y)|_{x^\perp}$ is onto and

$$\|(Df(y)|_{x^\perp})^\dagger Df(x)|_{x^\perp}\|_x \le \frac{1}{1-\left(\frac{1}{(1-u_x)^2} - 1\right)} = \frac{(1-u_x)^2}{\psi(u_x)}. \quad \square$$

Lemma 3.a. We have $f(\zeta) = 0$ *if and only if* $V_\zeta \subset ker Df(\zeta)$. *In this case*

$$W_\zeta^\perp = ker Df(\zeta) = V_\zeta \oplus (ker Df(\zeta) \cap \zeta^\perp)$$

and

$$W_\zeta = im(Df(\zeta)|_{\zeta^\perp})^\dagger = im Df(\zeta)^\perp.$$

Proof. Since $f_i : \mathbb{E} \to \mathbb{C}$ is multihomogeneous with degrees $d_{1i}, \ldots, d_{ki}$ then (Euler formula)

$$Df_i(x)(\times\lambda x) = f_i(x)\sum_{j=1}^{k} d_{ji}\lambda_j$$

and this proves the first assertion. Since $W_\zeta = im(Df(\zeta)|_{\zeta^\perp})^\dagger$ is the orthogonal complement in $\zeta^\perp$ of $ker Df(\zeta)|_{\zeta^\perp}$ we have

$$W_\zeta = \zeta^\perp \cap (\zeta^\perp \cap ker Df(\zeta))^\perp = \zeta^\perp \cap (V_\zeta \oplus ker Df(\zeta)^\perp)$$

so that

$$W_\zeta^\perp = V_\zeta \oplus (\zeta^\perp \cap ker Df(\zeta)).$$

Since $V_\zeta \subset ker Df(\zeta)$ this gives $W_\zeta^\perp = ker Df(\zeta)$ and $W_\zeta = ker Df(\zeta)^\perp = im Df(\zeta)^\perp$. $\square$

Lemma 3.b. If $f(\zeta) = 0$ *and* $u_\zeta < 1 - \frac{\sqrt{2}}{2}$ *then*

$$d_\zeta(ker Df(y) \cap y^\perp, ker Df(\zeta) \cap \zeta^\perp) \le \frac{1}{(1-u_\zeta)^2} - 1 + u_\zeta.$$

Proof. We have $\zeta^\perp = (ker Df(\zeta) \cap \zeta^\perp) \oplus W_\zeta$. Given $v \in ker Df(y) \cap y^\perp$

$$0 = Df(y)v = Df(\zeta)v + \sum_{k\ge 2} k\frac{D^k f(\zeta)}{k!}(y-\zeta)^{k-1}v.$$

Now let $v = v_1 + v_2$ with $v_1 \in W_\zeta^\perp$ and $v_2 \in W_\zeta$. Then $Df(\zeta)^\perp Df(\zeta)v = v_2$ and

$$-v_2 = \sum_{k\ge 2} k Df(\zeta)^\dagger \frac{D^k f(\zeta)}{k!}(y-\zeta)^{k-1}v$$

so that

$$\|v - v_1\|_\zeta = \|v_2\|_\zeta \le \left(\frac{1}{(1-u_\zeta)^2} - 1\right)\|v\|_\zeta$$

as in the proof of Lemma 2.b. By Lemma 3.a, the component of v_1 in V_ζ is also the component of v in V_ζ and its norm is bounded by

$$d_\zeta(y^\perp, \zeta^\perp)\|v\|_\zeta = d_\zeta(V_\zeta, V_y)\|v\|_\zeta \le u_\zeta\|v\|_\zeta,$$

the last inequality is a consequence of Lemma 4.a, and the equality from Proposition 3.2. □

Lemma 3.c. If $f(\zeta)=0$, $\dfrac{1}{(1-u_\zeta)^2}-1+u_\zeta \le 1$ *and* $Df(\zeta)$ *is onto then* $Df(y)|_{y^\perp}$ *is onto.*

Proof. By Lemma 3.b

$$d_\zeta(ker Df(y)\cap y^\perp, ker Df(\zeta)\cap\zeta^\perp) < 1$$

thus, by Proposition 4.3

$$(ker Df(y)\cap y^\perp)\cap(ker Df(\zeta)\cap\zeta^\perp)^\perp = \{0\}$$

so that

$$dim\ ker Df(y)\cap y^\perp \le dim\ ker Df(\zeta)\cap\zeta^\perp \le dim\ ker Df(\zeta)$$

and we are done. □

Lemma 4.a. $d_x(V_x, V_y) \le \|x-y\|_x \le u_x$.

Proof. Let $v\in V_x$ be given with $v = \times\lambda x$. With $w=\times\lambda y$ we have

$$\|v-w\|_x = \left(\sum_i \frac{\|\lambda_i(x_i-y_i)\|_i^2}{\|x_i\|_i^2}\right)^{1/2} \le \sum_i \frac{\|\lambda_i(x_i-y_i)\|_i}{\|x_i\|_i} \le \left(\sum_i |\lambda_i|^2\right)^{1/2}\|x-y\|_x = \|v\|_x\|x-y\|_x.$$

This gives $d_x(V_x,V_y)\le\|x-y\|_x$ and this last quantity is $\le u_x$ since $\gamma(f,x)\ge 1$. □

Lemma 4.b. If $f(\zeta)=0$ *and* $u_\zeta < 1-\frac{\sqrt{2}}{2}$ *then*

$$d_\zeta(ker Df(y), W_\zeta^\perp) \le u_\zeta\phi(u_\zeta), \quad with \quad \phi(u)=\frac{2-u}{2\psi(u)}.$$

Proof. By lemma 3.a $ker Df(\zeta)\oplus W_\zeta$ is under the hypothesis an orthogonal direct sum decomposition of $\mathbb{E}$. Let $v\in ker Df(y)$ and $v=v_1+v_2$ where $v_1\in ker Df(\zeta)$ and $v_2\in W_\zeta$. $Df(y)v=0$ implies that $Df(y)v_1=-Df(y)v_2$ and that $Df(\zeta)^\dagger Df(y)v_1 = -Df(\zeta)^\dagger Df(y)v_2$. Now we estimate the norms of both sides of the equation :

$$\|Df(\zeta)^\dagger Df(y)v_1\|_\zeta = \|Df(\zeta)^\dagger(Df(\zeta)+\sum_{k\ge2}k\frac{D^kf(\zeta)}{k!}(y-\zeta)^{k-1})v_1\|_\zeta \le (\frac{1}{(1-u_\zeta)^2}-1)\|v_1\|_\zeta,$$

$$\|Df(\zeta)^\dagger Df(y)v_2\|_\zeta = \|Df(\zeta)^\dagger(Df(\zeta)+\sum_{k\ge2}k\frac{D^kf(\zeta)}{k!}(y-\zeta)^{k-1})v_2\|_\zeta \ge \|v_2\|_\zeta-(\frac{1}{(1-u_\zeta)^2}-1)\|v_2\|_\zeta.$$

Let $u\phi_1(u)=\dfrac{1}{(1-u)^2}-1$ then it follows that

$$\|v_1\|_\zeta \ge \frac{1-u_\zeta\phi_1(u_\zeta)}{u_\zeta\phi_1(u_\zeta)}\|v_2\|_\zeta$$

and that

$$d_\zeta(ker Df(y), W_\zeta^\perp) \le \frac{u_\zeta\phi_1(u_\zeta)}{1-u_\zeta\phi_1(u_\zeta)} = u_\zeta\phi(u_\zeta).$$ □

It remains to estimate $d_\zeta(W_\zeta^\perp, W_y^\perp)$ which we do in the next lemma with ϕ as above.

Lemma 4.c. If $f(\zeta)=0$ then

$$d_\zeta(W_\zeta^\perp, W_y^\perp) \le \sqrt{2} u_\zeta \phi(u_\zeta).$$

Proof.

$$d_\zeta(W_\zeta^\perp, W_y^\perp) = d_\zeta(V_y \oplus (ker Df(y) \cap y^\perp), W_\zeta^\perp)$$

which by Proposition 4.7

$$\le \sqrt{2}\max(d_\zeta(V_y, W_\zeta^\perp), d_\zeta(ker Df(y) \cap y^\perp, W_\zeta^\perp))$$

and by Lemma 4.b

$$\le \sqrt{2}\max(u_\zeta, u_\zeta\phi(u_\zeta)) = \sqrt{2} u_\zeta \phi(u_\zeta). \quad \square$$

Lemma 5. If $d_\zeta(ker Df(\zeta), W_y^\perp) \le \frac{\sqrt{3}}{2}$ and $Df(y)|_{y^\perp}$ is onto then

$$||(Df(y)|_{y^\perp})^\dagger Df(y)||_\zeta \le 2.$$

Proof. If $w_1 \in ker Df(y)^\perp$ then

$$(Df(y)|_{y^\perp})^\dagger Df(y) w_1 = w_2$$

where $w_2 \in W_y$ and there is a $v \in ker Df(y)$ such that $w_1 + v = w_2$. Then $\Pi_{ker Df(y)} w_2 = v$ and by Proposition 3 and the hypotheses $||v||_\zeta \le \frac{\sqrt{3}}{2}||w_2||_\zeta$ so

$$||w_2||_\zeta^2 = ||w_1||_\zeta^2 + ||v||_\zeta^2 \le ||w_1||_\zeta^2 + \frac{3}{4}||w_2||_\zeta^2$$

and $||w_2||_\zeta \le 2||w_1||_\zeta$ which was to be proved. $\square$

Lemma 6. For any $x,\ y \in \mathbb{E}^\star$ let us define $c(x,y) = \max_i \dfrac{||x_i||_i}{||y_i||_i}$ and $c(y,x) = \max_i \dfrac{||y_i||_i}{||x_i||_i}$. Then
1. $||v||_y \le c(x,y)||v||_x$ for any $v \in \mathbb{E}$,
2. $||B||_y \le c(x,y)c(y,x)^k||B||_x$ for any k-linear operator $B : \mathbb{E}^k \to \mathbb{E}$,
Moreover when $u_x < 1$ we have
3. $c(x,y) \le \dfrac{1}{1-u_x}$ and $c(y,x) \le 1 + u_x$.

Proof. Assertions 1 and 2 are easy. To prove 3 notice that

$$c(x,y) = \max_i \frac{||x_i||_i}{||y_i||_i} \le \max_i \frac{1}{1 - \frac{||x_i - y_i||_i}{||x_i||_i}} \le \frac{1}{1-u_x}$$

since

$$\frac{||x_i - y_i||_i}{||x_i||_i} \le ||y - x||_x \le u_x < 1.$$

Moreover

$$c(y,x) = \max_i \frac{||y_i||_i}{||x_i||_i} \le ||y||_x \le 1 + ||y-x||_x \le 1 + u_x. \quad \square$$

Let $x \in \mathbb{E}^\star$ be given such that $Df(x)|_{x^\perp}$ is onto. We use affine α-Theory (see Shub-Smale [12] Theorem 1.4) to conclude, if $\alpha(f,x) \le \alpha_0$, the existence of a zero ζ of f such that the Newton iterates

$$x_0 = x, \quad x_{k+1} = x_k - (Df(x_k)|_{x^\perp})^\dagger f(x_k)$$

are such that $Df(x_k)|_{x^\perp}$ is onto, converge to a zero ζ of f and for all $k \geq 1$

$$\|x_{k+1} - x_k\|_x \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x_1 - x_0\|_x.$$

With

$$\sigma = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{2^i - 1} = 1.63284\ldots$$

Here α_0 is a universal constant. According to Shub-Smale [9] and [12] we can take $\alpha_0 = (13 - 3\sqrt{17})/4 = 0.15767\ldots$. We have the following

Lemma 7. Let $x \in \mathbb{E}^\star$ be given such that $Df(x)|_{x^\perp}$ is onto and $\alpha(f,x) \leq \alpha_0$. Let us denote $y = x_1 = x - (Df(x)|_{x^\perp})^\dagger f(x)$. Then

$$\|\zeta - x\|_x \leq \sigma \|y - x\|_x = \sigma\beta(f,x),$$

$$\|\zeta - y\|_x \leq (\sigma - 1)\|y - x\|_x = (\sigma - 1)\beta(f,x).$$

The proof is easy and left to the reader. □

Lemma 8. With the same hypotheses as in Lemma 7, there is a universal constant $\tau > 0$, approximately equal to $23.78463\ldots$, such that $u_\zeta \leq \tau u_x$.

Proof. Since $f(\zeta) = 0$ we have $V_\zeta \subset ker Df(\zeta)$ so that $(Df(\zeta)|_{\zeta^\perp})^\dagger = Df(\zeta)^\dagger$ is the minimum norm right inverse of $Df(\zeta)$. Thus

$$\gamma(f,\zeta) = \max(1, \sup_{k\geq 2} \left\| (Df(\zeta)|_{\zeta^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_\zeta^{\frac{1}{k-1}}) \leq \max(1, \sup_{k\geq 2} \left\| (Df(\zeta)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_\zeta^{\frac{1}{k-1}})$$

and by Lemma 6

$$\leq \max(1, \sup_{k\geq 2} \left(c(x,\zeta) c(\zeta,x)^k \left\| (Df(\zeta)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_x \right)^{\frac{1}{k-1}}).$$

Moreover

$$\left\| (Df(\zeta)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_x \leq \left\| (Df(\zeta)|_{x^\perp})^\dagger Df(x) \right\|_x \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_x.$$

Let us denote $v = \|x - \zeta\|_x \gamma(f,x)$. We have by Lemma 7 $v \leq \sigma\alpha(f,x) \leq \sigma\alpha_0 < 1 - \sqrt{2}/2$ and by Lemma 2

$$\left\| (Df(\zeta)|_{x^\perp})^\dagger Df(x) \right\|_x \leq \frac{(1-v)^2}{\psi(v)}.$$

We also have

$$\left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|_x \leq \sum_{l\geq 0} \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^{k+l} f(x)}{k!l!} \right\|_x \|x - \zeta\|_x^l$$

$$\leq \sum_{l\geq 0} \|\frac{(k+l)!}{k!l!} \gamma(f,x)^{k+l-1} \|x - \zeta\|_x^l = \frac{\gamma(f,x)^{k-1}}{(1-v)^{k+1}}.$$

Since by Lemma 6 $c(x,\zeta) \leq 1/(1-v)$ and $c(\zeta,x) \leq 1+v$ we obtain

$$\gamma(f,\zeta) \leq \max(1, \sup_{k\geq 2} \left(\frac{(1+v)^k}{1-v} \frac{(1-v)^2}{\psi(v)} \frac{\gamma(f,x)^{k-1}}{(1-v)^{k+1}} \right)^{\frac{1}{k-1}}) \leq \left(\frac{1+v}{1-v}\right)^2 \frac{\gamma(f,x)}{\psi(v)}$$

so that, by Lemmas 6 and 7

$$u_\zeta = ||\zeta - y||_\zeta \gamma(f,\zeta) \le c(x,\zeta)||\zeta - y||_x \gamma(f,\zeta) \le \frac{1}{1-v}(\sigma-1)||x-y||_x \left(\frac{1+v}{1-v}\right)^2 \frac{\gamma(f,x)}{\psi(v)} \le \tau u_x$$

with

$$\tau = \frac{\sigma-1}{1-v}\left(\frac{1+v}{1-v}\right)^2 \frac{1}{\psi(v)}|_{v=\sigma\alpha_0}$$

since $v \le \sigma\alpha_0$. From this expression we are able to deduce a numerical value for τ. □

Lemma 9. With the same hypotheses as in Lemma 7, there is a universal constant $0 < \alpha_1 \le \alpha_0$ such that, if $\alpha(f,x) \le \alpha_1$ then

$$\beta(f,y) \le \frac{2(1+\sigma\alpha_0)}{\psi(\alpha_0)}\beta(f,x)^2\gamma(f,x).$$

We can take $\alpha_1 = 0.009$.

Proof. We start from the following inequality

$$\beta(f,y) = ||(Df(y)|_{y^\perp})^\dagger f(y)||_y \le c(x,y)\ ||(Df(y)|_{y^\perp})^\dagger Df(y)|_{x^\perp}||_x \times$$

$$||(Df(y)|_{x^\perp})^\dagger Df(x)||_x\ ||(Df(x)|_{x^\perp})^\dagger f(y)||_x = \mathbf{1}\times\mathbf{2}\times\mathbf{3}\times\mathbf{4}.$$

We have $\mathbf{1} \le \frac{1}{1-u_x}$ by Lemma 6, $\mathbf{3} \le \frac{(1-u_x)^2}{\psi(u_x)}$ by Lemma 2 since $Df(x)|_{x^\perp}$ is onto and $u_x = \alpha(f,x) \le \alpha_1 < 1-\sqrt{2}/2$. By Lemmas 5 and 6 we also have

$$\mathbf{2} \le c(\zeta,x)||(Df(y)|_{y^\perp})^\dagger f(y)||_\zeta \le 2(1+v) \le 2(1+\sigma\alpha_0)$$

as in the proof of Lemma 8. This is accomplished when $d_\zeta(ker Df(\zeta), W_y^\perp) \le \sqrt{3}/2$ and $Df(y)|_{y^\perp}$ onto (Lemma 5). The first inequality is satisfied when $u_\zeta < 1-\sqrt{2}/2$ and $u_\zeta\phi(u_\zeta) \le \sqrt{3}/2$ (Lemma 4.b), $Df(y)|_{y^\perp}$ is onto when $\frac{1}{(1-u_\zeta)^2} - 1 + u_\zeta < 1$. Since by Lemma 8 $u_\zeta \le \tau u_x$ these inequalities are satisfied when $\tau\alpha_1 < 1-\sqrt{2}/2$, $\tau\alpha_1\phi(\tau\alpha_1) \le \sqrt{3}/2$ and $\frac{1}{(1-\tau\alpha_1)^2} - 1 + \tau\alpha_1 < 1$. This is accomplished with $\alpha_1 = 0.009$. Let us now give a bound for $\mathbf{4}$. We start from

$$f(y) = f(x) + \sum_{k\ge1}\frac{D^k f(x)}{k!}(y-x)^k.$$

Since $y = N_f(x)$ we have $f(x) + Df(x)(y-x) = 0$ so that

$$\mathbf{4} = ||(Df(x)|_{x^\perp})^\dagger f(y)||_x \le \sum_{k\ge2}||(Df(x)|_{x^\perp})^\dagger\frac{D^k f(x)}{k!}||_x||y-x||_x^k \le \frac{u_x}{1-u_x}||y-x||_x.$$

Puting these inequalities togother gives the requiered result. □

Lemma 10. With the same hypotheses as in Lemma 9, then

$$\gamma(f,y) \le 2\left(\frac{1+\alpha_0}{1-\alpha_0}\right)^2\frac{1+\sigma\alpha_0}{\psi(\alpha_0)}\gamma(f,x).$$

Proof. Similar to the proof of Lemma 9 :

$$\gamma(f,y) = \max(1, \sup_{k\ge2}\left\|(Df(y)|_{y^\perp})^\dagger\frac{D^k f(y)}{k!}\right\|_y^{\frac{1}{k-1}}).$$

To bound this quantity in terms of $\gamma(f,x)$ we start from

$$||(Df(y)|_{y^\perp})^\dagger \frac{D^k f(y)}{k!}||_y \le c(x,y)c(y,x)^k \times \mathbf{2} \times \mathbf{3} \times ||(Df(x)|_{x^\perp})^\dagger \frac{D^k f(y)}{k!}||_x \le$$

$$\frac{(1+u_x)^k}{1-u_x} 2(1+\sigma\alpha_0)\frac{(1-u_x)^2}{\psi(u_x)} \frac{\gamma(f,x)^{k-1}}{(1-u_x)^{k+1}},$$

the last inequality may be proved as in Lemma 8. This gives

$$\gamma(f,y) \le \left(\frac{1+u_x}{1-u_x}\right)^2 \frac{2(1+\sigma\alpha_0)}{\psi(u_x)}\gamma(f,x)$$

and we are done. □

Proof of Theorem 1. We start from $x_0 = x$ with $Df(x)|_{x^\perp}$ onto and $\alpha(f,x) \le \alpha_u = 1/137$. By Lemmas 9 and 10 and since $\alpha_u \le \alpha_1$ we have $\alpha(f,y) \le \tau_1\alpha(f,x)^2$ where

$$\tau_1 = \left(2\frac{1+\alpha_0}{1-\alpha_0}\frac{1+\sigma\alpha_0}{\psi(\alpha_0)}\right)^2.$$

We have $\tau_1 = 63.03684\ldots$ so that $2\tau_1\alpha(f,x) \le 1$. We obtain, by induction over k

$$\alpha(f,x_k) \le \left(\frac{1}{2}\right)^{2^k-1} \alpha(f,x).$$

Using Lemma 9 again we get

$$\beta(f,y) \le \tau_2\alpha(f,x)\beta(f,x)$$

with

$$\tau_2 = \frac{1+\sigma\alpha_0}{\psi(\alpha_0)} = 6.00162\ldots.$$

By induction over k and since $2\tau_2\alpha_u \le 1$ we obtain

$$||x_{k+1} - x_k||_{x_k} \le \beta(f,x_k) \le \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x).$$

We now notice that

$$d_R(x_{k+1},x_k) \le ||x_{k+1} - x_k||_{x_k}$$

because $x_{k+1} - x_k \in T_{x_k}\Pi_i\mathbb{P}(\mathbb{E}_i)$. The rest of the proof is easy. □

Section 3. γ-Theorem.

In this section we give a proof of Theorem 2. According to Theorem 1, x will be an approximate zero if $Df(x)|_{x^\perp}$ is onto and $\alpha(f,x) \le \alpha_u$. Let us denote $w = ||x-\zeta||_\zeta\gamma(f,\zeta)$. By Lemma 3.c $Df(x)|_{x^\perp}$ is onto when $\frac{1}{(1-w)^2} - 1 + w < 1$. This is accomplished when $w \le 0.24512\ldots$ Let us now compute a bound for $\alpha(f,x)$. We have by Lemma 6

$$\beta(f,x) = ||(Df(x)|_{x^\perp})^\dagger f(x)||_x \le \frac{1}{1-w}||(Df(x)|_{x^\perp})^\dagger f(x)||_\zeta$$

$$\le \frac{1}{1-w}||(Df(x)|_{x^\perp})^\dagger Df(\zeta)||_\zeta ||(Df(\zeta)|_{\zeta^\perp})^\dagger f(x)||_\zeta = \mathbf{1} \times \mathbf{2} \times \mathbf{3}.$$

By Lemma 2 we get $\mathbf{2} \le \dfrac{(1-w)^2}{\psi(w)}$ since $w < 1 - \sqrt{2}/2$. Moreover

$$f(x) = Df(\zeta)(x-\zeta) + \sum_{k\ge 2} \frac{D^k f(\zeta)}{k!}(x-\zeta)$$

so that $\mathbf{3} \le \dfrac{\|x-\zeta\|_\zeta}{1-w}$. This gives

$$\beta(f,x) \le \frac{\|x-\zeta\|_\zeta}{\psi(w)}.$$

To get a bound for $\gamma(f,x)$ we use an argument similar to Lemma 8 and we obtain

$$\gamma(f,x) \le \frac{(1+w)^2}{(1-w)^2\psi(w)}\gamma(f,\zeta).$$

Thus

$$\alpha(f,x) \le \left(\frac{(1+w)}{(1-w)\psi(w)}\right)^2 \gamma(f,\zeta)\|x-\zeta\|_\zeta.$$

This quantity is $\le \alpha_u$ when

$$\gamma_u \le \alpha_u \left(\frac{(1+w)}{(1-w)\psi(w)}\right)^{-2}.$$

According to the bound $w \le 0.24512\ldots$ and the value $\alpha_u = 1/137$ we can take $\gamma_u = 0.00005\ldots$ but such a value is pessimistic. A better bound might be obtained by a direct proof of this γ-Theorem. According to Theorem 1 there is a zero ζ' of f such that Newton's sequence x_k converges to ζ' and

$$d_R(\zeta', x_k) \le \sigma \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x).$$

With $k=1$ and by the previous estimation for β we obtain

$$d_R(\zeta', x) \le \frac{\sigma}{\psi(w)}\|x-\zeta\|_\zeta \le 2.86543\ldots\|x-\zeta\|_\zeta \le 3\|x-\zeta\|_\zeta.$$

As has been proved before, we have $\psi(\gamma_u)\beta(f,x) \le \|x-\zeta\|_\zeta$. We also have noticed that $\alpha(f,x) \le \alpha_u$ so that, by Theorem 1,

$$d_R(\zeta', x_k) \le \sigma \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x)$$

for some root ζ' of f. This inequality also applies with $N_f(x)$ instead of x and gives, using Lemma 9,

$$d_R(\zeta', N_f(x)) \le \sigma\beta(f,y) \le c_1\beta(f,x)^2\gamma(f,x).$$

We now use the estimation $\gamma(f,x) \le c_2\gamma(f.x)$ obtained previously to obtain

$$d_R(\zeta', N_f(x)) \le c_u\gamma(f,\zeta)\|\zeta - x\|_\zeta^2. \ \square$$

Proof of Corollary 1. Let us first remark that $d_R(x,y) \le \|x-y\|_x$ and say $\frac{5}{6}\|x-y\|_x \le d_R(x,y)$ if we take representatives such that $x - y \in x^\perp$ and if $d_R(x,y)$ is small enough. This property is supposed to be satisfied when

$$d_R(x,y) \le d_R(x,y)\gamma(f,\zeta) \le \delta_u.$$

If we take representatives of x and ζ satisfying $\zeta - x \in \zeta^{\perp}$ we obtain

$$\|\zeta - x\|_{\zeta}\gamma(f,\zeta) \leq \frac{6}{5}\delta_u \leq \gamma_u$$

so that, Theorem 2 gives a Newton's sequence converging to a zero ζ' of f with

$$d_R(\zeta', x_k) \leq \sigma \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x).$$

This inequality also apply with $N_f(x)$ instead of x and gives

$$d_R(\zeta', x_{k+1}) \leq \sigma \left(\frac{1}{2}\right)^{2^k-1} \beta(f, N_f(x)).$$

We now use Lemma 9 to bound $\beta(f, N_f(x))$ in terms of $\beta(f,x)$. Then we use the estimations of $\gamma(f,x)$ in terms of $\gamma(f,\zeta)$ and $\beta(f,x)$ in terms of $\|x-\zeta\|_{\zeta}$ obtained in the proof of Theorem 2 to obtain

$$d_R(\zeta', x_{k+1}) \leq c_1 \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x)^2\gamma(f,x) \leq c_2 \left(\frac{1}{2}\right)^{2^k-1} \gamma(f,\zeta)\|x-\zeta\|_{\zeta}^2$$

$$\leq c_3\frac{6}{5}\delta_u \left(\frac{1}{2}\right)^{2^k-1} \|x-\zeta\|_{\zeta} \leq \left(\frac{1}{2}\right)^{2^k} d_R(x-\zeta)$$

if $c_3\frac{72}{25}\delta_u \leq 1$. □

Section 2.4. Newton's method for the evaluation map.

In this section we give the proofs of Theorems 3 and 4. We first describe Newton's iterate $N_{ev}(f,x)$. In a first Lemma we study the derivative of the evaluation map.

Lemma 11. For any $(f,x) \in \mathcal{H}_{(d)} \times \mathbb{C}^n$ we have

$$Dev(f,x)(\dot{f},\dot{x}) = \dot{f}(x) + Df(x)\dot{x}.$$

Moreover $Dev(f,x)$ is always onto. □

Let us denote by $f^{\perp}$ and $x^{\perp}$ the vector subspaces in $\mathcal{H}_{(d)}$ and $\mathbb{C}^n$ orthogonal complements of $\mathbb{C}f$ and $\mathbb{C}x$. In the sequel we suppose that $\|f\| = \|x\| = 1$. In such a case, the Hermitian structure $\| \ \|_{(f,x)}$ coincides with the usual product structure on $\mathcal{H}_{(d)} \times \mathbb{C}^n$. The following representation lemma will be useful later. A proof may be found in Blum-Cucker-Shub-Smale [2] Section 14.1.

Lemma 12. For any $f_i \in \mathcal{H}_{d_i}$ and $x \in \mathbb{C}^n$ we have

$$f_i(x) = < f_i(z), < z, x >^{d_i} >, \quad \| < z, x >^{d_i} \| = \|x\|^{d_i}$$

and

$$|f_i(x)| \leq \|f_i\|\|x\|^{d_i}. \ \square$$

Newton's iterate for eval is given by

$$N_{ev}(f,x) = (f,x) - \left(Dev(f,x)|_{f^{\perp}\times x^{\perp}}\right)^{\dagger} f(x).$$

In the following lemma we compute this Moore-Penrose inverse. We first have to introduce some notations. $f(x)$ is a column vector, $Df_i(x)$ is a row vector with entries $\frac{\partial f_i}{\partial z_j}(x)$, $1 \le j \le n$ and $Df(x)$ is the $m \times n$ matrix with rows $Df_i(x)$, $1 \le i \le m$. For any matrix A we denote by $A^\star$ its adjoint. With these notations, the usual scalar product in $\mathbb{C}^n$ is given by $< x, y >= y^\star x$ and the value at $\dot{x}$ of the derivative $Df(x)$ is also the product $Df(x)\dot{x}$ of the matrix $Df(x)$ by the column vector $\dot{x}$.

Lemma 13. For any $\mu \in \mathbb{C}^m$ we have $\left(Dev(f,x)|_{f^\perp \times x^\perp}\right)^\dagger \mu = (\dot{f}, \dot{x})$ with

$$\dot{f}_j(z) = \lambda_j < z, x >^{d_j} - \left(\sum_{i=1}^{m} \lambda_i \overline{f_i(x)}\right) f_j(z),$$

$$\dot{x} = Df(x)^\star \lambda - < Df(x)^\star \lambda, x > x,$$

$$\lambda = \left(I_m + Df(x)Df(x)^\star - \mathcal{D}f(x)f(x)^\star \mathcal{D}\right)^{-1} \mu,$$

where I_m is the identity matrix and $\mathcal{D}$ the diagonal matrix with entries $1 + d_i$, $1 \le i \le m$.

Proof. We have $(\dot{f}, \dot{x}) \in kerDev(f,x)$ if and only if $\dot{f}(x) + Df(x)\dot{x} = 0$ or, according to Lemma 12,

$$< \dot{f}_i(z), < z, x >^{d_i} > + < \dot{x}, Df_i(x)^\star >= 0, \quad 1 \le i \le m,$$

or, in terms of the $\mathcal{H}_{(d)}$ scalar product

$$< \dot{f}(z), (0, \ldots, < z, x >^{d_i}, \ldots, 0) > + < \dot{x}, Df_i(x)^\star >= 0, \quad 1 \le i \le m.$$

Thus $kerDev(f,x)^\perp$ consists in the $(m+1)-$uples

$$(\lambda_1 < z, x >^{d_1}, \ldots, \lambda_m < z, x >^{d_m}, Df(x)^\star \lambda), \quad \lambda \in \mathbb{C}^m.$$

In fact $\left(Dev(f,x)|_{f^\perp \times x^\perp}\right)^\dagger$ involves the orthogonal complement in $f^\perp \times x^\perp$ of $kerDev(f,x)|_{f^\perp \times x^\perp}$. This subspace is equal to

$$\left(kerDev(f,x) \cap (f^\perp \times x^\perp)\right)^\perp \cap (f^\perp \times x^\perp) = \left(kerDev(f,x)^\perp + (\mathbb{C}f \times \mathbb{C}x)\right) \cap (f^\perp \times x^\perp).$$

Any element in this subspace is equal to some

$$(\dot{f}, \dot{x}) = (\alpha f_1 + \lambda_1 < z, x >^{d_1}, \ldots, \alpha f_m + \lambda_m < z, x >^{d_m}, \beta x + Df(x)^\star \lambda)$$

with $< \dot{f}, f >= 0$ and $< \dot{x}, x >= 0$. Consequently

$$\alpha = -\sum_i \lambda_i \overline{f_i(x)} \quad and \quad \beta = - < Df(x)^\star \lambda, x >$$

since $||f|| = ||x|| = 1$. Let us now compute

$$\left(Dev(f,x)|_{f^\perp \times x^\perp}\right)^\dagger \mu = (\dot{f}, \dot{x}).$$

We have $(\dot{f}, \dot{x})$ as before and $\dot{f}(x) + Df(x)\dot{x} = \mu$ so that

$$\left(I_m + Df(x)Df(x)^\star - \mathcal{D}f(x)f(x)^\star \mathcal{D}\right) \lambda = \mu$$

using Euler equation for f_i i.e $Df_i(x)x = d_i f_i(x)$, and we are done. □

Lemma 14. For any $x \in \mathbb{C}^n$ with $||x|| = 1$, $\dot{x}_1, \ldots, \dot{x}_k \in \mathbb{C}^n$ we have

$$|D^k f_i(x)(\dot{x}_1, \ldots, \dot{x}_k)| \le d_i(d_i - 1)\ldots(d_i - k + 1)||f_i||||\dot{x}_1|| \ldots ||\dot{x}_k||.$$

The proof of this lemma may be found in Blum-Cucker-Shub-Smale [2] Section 14.1. □

Lemma 15. $D^k ev(f_i,x)(\dot f_1,\dot x_1,\ldots,\dot f_k,\dot x_k) = D^k f_i(x)(\dot x_1,\ldots,\dot x_k) + \sum_{j=1}^k D^{k-1}\dot f_j(x)(\dot x_1,\ldots,\underline{\dot x_j},\ldots,\dot x_k)$ *where* $\underline{\dot x_j}$ *indicates that* $\dot x_j$ *is missing.*

Lemma 16. $\dfrac{\|D^k ev(f_i,x)\|}{k!} \le \binom{d_i}{k}\|f_i\| + \binom{d_i}{k-1}$.

The proof of this lemma is an easy consequence of Lemmas 14 and 15. □

Lemma 17. When $f(x)=0$ *then*

$$\| \left(Dev(f,x)|_{f^\perp\times x^\perp}\right)^\dagger \| = \|(I+Df(x)Df(x)^\star)^{-1}\| \le 1.$$

Proof. When $f(x)=0$ we have by Lemma 13 : $\left(Dev(f,x)|_{f^\perp\times x^\perp}\right)^\dagger \mu = (\dot f,\dot x)$ with

$$\dot f_i(z) = \lambda_i < z, x >^{d_i}, \quad \dot x = Df(x)^\star\lambda \quad and \quad \lambda = (I+Df(x)Df(x)^\star)^{-1}\mu.$$

Thus

$$\| \left(Dev(f,x)|_{f^\perp\times x^\perp}\right)^\dagger \|^2 = \max\|\lambda\|^2 + \|Df(x)^\star\lambda\|^2$$

where the maximum is taken for $\|\mu\|=1$. This maximum is also equal to

$$\max\lambda^\star(I+Df(x)Df(x)^\star)\lambda = \max\mu^\star(I+Df(x)Df(x)^\star)^{-1}\mu = \|(I+Df(x)Df(x)^\star)^{-1}\|.$$

This quantity is always less than or equal to 1 since adding to the identity matrix a psd matrix increases the eigenvalues. □

Proof of Theorem 3. For any $(f,x)\in V$ we have

$$\| \left(Dev(f,x)|_{f^\perp\times x^\perp}\right)^\dagger \frac{D^k ev(f,x)}{k!}\| \le \| \left(Dev(f,x)|_{f^\perp\times x^\perp}\right)^\dagger \| \, \|\frac{D^k ev(f,x)}{k!}\|.$$

By Lemma 17 the first term is bounded by 1 and by Lemma 16 we get

$$\le \left(\sum_{i=1}^m \left(\binom{d_i}{k}\|f_i\| + \binom{d_i}{k-1}\right)^2\right)^{1/2} \le \binom{D}{k}(1+\sqrt{m}).$$

We now have to take the $(k-1)$−th root of this expression and its maximum for $k\ge 2$. The well known inequality

$$\binom{D}{k}^{\frac{1}{k-1}} \le \binom{D}{2}, \quad k\ge 2,$$

gives the required result. □

Proof of Theorem 4. This theorem is a consequence of Theorem 2, Corollary 1 and Theorem 3. □

Section 2.5. Path-Following.

In this section we give the proof of Theorem 5. We choose the subdivision t_i, $0\le i\le p$, such that all the distances $d_R(\zeta_{t_{i+1}},\zeta_{t_i})$ are equal. Then we have

$$pd_R(\zeta_{t_{i+1}},\zeta_{t_i}) = \sum_i d_R(\zeta_{t_{i+1}},\zeta_{t_i}) \le L_\zeta$$

with L_ζ the length of the curve $t \in [0,1] \to \zeta_t$ in the Riemannian distance

$$L_\zeta = \int_0^1 ||\dot{\zeta}_t||_{\zeta_t} dt$$

and $\dot{\zeta}_t$ the derivative with respect to t. Since $\dot{\zeta}_t = \dfrac{d\zeta_t}{df_t}\dfrac{df_t}{dt}$ and $\left\| \dfrac{d\zeta_t}{df_t} \right\|_{\zeta_t} \le \mu$ we obtain $L_\zeta \le \mu L$ so that

$$d_R(\zeta_{t_{i+1}}, \zeta_{t_i})\gamma \le \frac{\gamma \mu L}{p} \le \frac{\delta_u}{2}.$$

We have to prove that

$$d_R(x_i, \zeta_{t_i})\gamma(f_{t_i}, \zeta_{t_i}) \le \delta_u$$

by Corollary 1. This will be accomplished if

$$d_R(x_i, \zeta_{t_i})\gamma \le \frac{\delta_u}{2}.$$

We prove this inequality by induction over i. The case $i = 0$ is easy since $\zeta_0 = x_0$. We have by Lemma 6

$$d_R(x_i, \zeta_{t_{i+1}})\gamma \le d_R(\zeta_{t_{i+1}}, \zeta_{t_i}) + d_R(\zeta_{t_i}, x_i) \le \delta_u.$$

By Corollary 1 this gives

$$d_R(x_{i+1}, \zeta_{t_{i+1}}) = d_R(N_{f_{i+1}}(x_i), \zeta_{t_{i+1}}) \le d_R(x_i, \zeta_{t_{i+1}})/2 \le \delta_u/2$$

and we are done. □

Section 2.6. Newton's Method for the Generalized Eigenvalue Problem.

In this section we first give a precise description of multihomogeneous Newton's iterate for the generalized eigenvalue problem (gep), we compute its condition number and we prove Theorem 6. We have introduced previously

$$F_{(A,B)} : \mathbb{C}^2 \times \mathbb{C}^n \to \mathbb{C}^n, \quad F_{(A,B)}(\alpha, \beta, x) = (\beta A - \alpha B)x$$

whose zeros are the eigenpairs of (A, B). Multihomogeneous Newton's iterate is equal to

$$N_{F_{(A,B)}}(\alpha, \beta, x) = (\alpha, \beta, x) - DF_{(A,B)}(\alpha, \beta, x)|^{\dagger}_{(\alpha,\beta,x)^\perp}(\beta A - \alpha B)x.$$

This iterate is computed in the next proposition

Proposition 5. Let (A, B) be a regular matrix pair, $(\alpha, \beta) \in \mathbb{C}^2$ and $x \in \mathbb{C}^n$ both non-zero. If (α, β) is not an eigenvalue of the pair (A, B) then multihomogeneous Newton's iterate is given by

$$N_{F_{(A,B)}}(\alpha, \beta, x) = (\alpha + \dot{\lambda}\bar{\beta}, \beta - \dot{\lambda}\bar{\alpha}, x - \dot{x}),$$

$$\dot{\lambda} = \frac{< x, x >}{< (\beta A - \alpha B)^{-1}(\bar{\alpha} A + \bar{\beta} B)x, x >},$$

$$\dot{x} = (\beta A - \alpha B)^{-1}((\beta - \dot{\lambda}\bar{\alpha})A - (\alpha + \dot{\lambda}\bar{\beta})B)x.$$

Proof. The subspace $(\alpha, \beta, x)^\perp$ consists in those couples $(\dot{\alpha}, \dot{\beta}, \dot{x}) \in \mathbb{C}^2 \times \mathbb{C}^n$ satisfying $< \dot{x}, x >= 0$ and $(\dot{\alpha}, \dot{\beta}) = \dot{\lambda}(-\bar{\beta}, \bar{\alpha})$. Thus $(\alpha, \beta, x)^\perp$ has dimension n. We also have

$$DF_{(A,B)}(\alpha, \beta, x)(\dot{\alpha}, \dot{\beta}, \dot{x}) = (\dot{\beta} A - \dot{\alpha} B)x + (\beta A - \alpha B)\dot{x}.$$

To compute Newton's iterate we have to solve the following system :

$$(\dot{\beta}A-\dot{\alpha}B)x+(\beta A-\alpha B)\dot{x}=(\beta A-\alpha B)x,$$

$$(\dot{\alpha},\dot{\beta})=\dot{\lambda}(-\bar{\beta},\bar{\alpha}) \quad and \quad <\dot{x},x>=0.$$

Since (α,β) is not an eigenvalue, the matrix $(\beta A-\alpha B)$ is nonsingular. Multiplying the first equation by $(\beta A-\alpha B)^{-1}$ and then taking the scalar product with x gives

$$\dot{\lambda}<(\beta A-\alpha B)^{-1}(\bar{\alpha}A+\bar{\beta}B)x,x>=<x,x>$$

and we are done. $\square$

We now compute the condition number for the gep. According to the definition 4, when (α,β,x) is an eigenpair of (A,B) then

$$\mu(A,B,\alpha,\beta,x)=\|DF_{(A,B)}(\alpha,\beta,x)|_{(\alpha,\beta,x)^{\perp}}^{\dagger}\|_{(\alpha,\beta,x)}.$$

In Dedieu [2] a similar computation is given but the condition number of the eigenvalue and the condition number for the eigenvector are computed separately. We prove here that the condition number for the eigenpair is equal to the maximum of 1 and the condition number for the eigenvector.

Proposition 6. Let (A,B) be a regular matrix pair, $(\alpha,\beta)\in\mathbb{C}^2$ and $x\in\mathbb{C}^n$ both non-zero with

$$(\beta A-\alpha B)x=0 \quad and \quad Ax\neq 0.$$

Suppose that (α,β) is a simple eigenvalue of the pair (A,B) i.e. a simple root of the polynomial $det(\beta A-\alpha B)$. Then

$$\mu(A,B,\alpha,\beta,x)=\max\left(1,\|\left(\Pi_{(Ax)^{\perp}}(\beta A-\alpha B)|_{x^{\perp}}\right)^{-1}\|\ (|\alpha|^2+|\beta|^2)^{1/2}\right)$$

with $\Pi_{(Ax)^{\perp}}$ the orthogonal projection over $(Ax)^{\perp}$.

Before proving this proposition we make some comments.
1. For an eigenpair (α,β,x) of (A,B) we always have $Ax\neq 0$ or $Bx\neq 0$ since the pair (A,B) is regular. When both are nonzero then$(Ax)^{\perp}=(Bx)^{\perp}$ since $\beta Ax=\alpha Bx$.
2. $\mu(A,B,\alpha,\beta,x)$ is invariant under scaling on both the eigenvector and the eigenvalue. It is, in fact, defined on the variety

$$\mathcal{V}=\{(A,B,\alpha,\beta,x)\in\mathcal{M}_n(\mathbb{C})^2\times\mathbb{P}(\mathbb{C}^2)\times\mathbb{P}(\mathbb{C}^n):(\beta A-\alpha B)x=0\}.$$

This is a consequence of Lemma 1 and the definition of μ.
3. The condition number is invariant under unitary transformations :

$$\mu(A,B,\alpha,\beta,x)=\mu(V^{\star}AU,V^{\star}BU,\alpha,\beta,U^{\star}x)$$

for any unitary matrices U and V. We do not prove this fact here, it is a consequence of the definition of μ, of the chain rule and of the unitary invariance of the spectral norm for matrices.
4. The linear operator $\Pi_{(Ax)^{\perp}}(\beta A-\alpha B)|_{x^{\perp}}$ is nonsingular if and only if (α,β) is a simple eigenvalue of the pair (A,B). See Dedieu [3] Lemma 4.1 for a proof of this fact.

Proof of Proposition 6. By the invariance property under scaling we can suppose that $|\alpha|^2+|\beta|^2=1$ and $\|x\|=1$. By the unitary invariance property, using the Schur decomposition for a matrix pair (see Dedieu [3] or Stewart-Sun [15] Chap. 6, Theorem 1.9) we may suppose that $x=e_1$ the first basis vector in $\mathbb{C}^n$ and

$$A=\begin{pmatrix}\alpha & a^{\star}\\ 0 & \tilde{A}\end{pmatrix} \quad and \quad B=\begin{pmatrix}\beta & b^{\star}\\ 0 & \tilde{B}\end{pmatrix}.$$

We have

$$DF_{(A,B)}(\alpha,\beta,x)(\dot\alpha,\dot\beta,\dot x) = (\dot\beta A - \dot\alpha B)x + (\beta A - \alpha B)\dot x.$$

When $(\dot\alpha,\dot\beta) \perp (\alpha,\beta)$ and $\dot x \perp x$ we can write

$$(\dot\alpha,\dot\beta) = \dot\lambda(-\bar\beta,\bar\alpha) \ \ and \ \ \dot x = \begin{pmatrix} 0 \\ \dot y \end{pmatrix}$$

so that

$$DF_{(A,B)}(\alpha,\beta,x)(\dot\alpha,\dot\beta,\dot x) = \begin{pmatrix} \dot\lambda \\ (\beta\tilde A - \alpha\tilde B)\dot y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta\tilde A - \alpha\tilde B \end{pmatrix}\begin{pmatrix} \dot\lambda \\ \dot y \end{pmatrix}.$$

The condition number is equal to

$$\|DF_{(A,B)}(\alpha,\beta,x)|^{-1}_{(\alpha,\beta,x)^\perp}\|_{(\alpha,\beta,x)}.$$

Since $|\alpha|^2+|\beta|^2=1$ and $\|x\|=1$ the endomorphism norm involved in this definition is the usual spectral norm so that

$$\mu(A,B,\alpha,\beta,x) = \max\left(1, \|(\beta\tilde A - \alpha\tilde B)^{-1}\|\right).$$

To conclude we just notice that

$$\beta\tilde A - \alpha\tilde B = \Pi_{(Ax)^\perp}(\beta A - \alpha B)|_{x^\perp}. \ \square$$

We now give an estimation for $\gamma(F_{(A,B)},\alpha,\beta,x)$ when (α,β) is a simple eigenvalue of the pair (A,B).

Proposition 7. Under the hypothesis above

$$\gamma(F_{(A,B)},\alpha,\beta,x) \le \max\left(1, \frac{1}{2}\mu(A,B,\alpha,\beta,x)\left(\|A\|^2+\|B\|^2\right)^{1/2}\right).$$

Proof. We first suppose that $|\alpha|^2+|\beta|^2=1$ and $\|x\|=1$ since, by Proposition 2, γ is invariant under scaling. We have

$$D^2F(\alpha,\beta,x)(\dot\alpha_1,\dot\beta_1,\dot x_1)(\dot\alpha_2,\dot\beta_2,\dot x_2) = (\dot\beta_2 A - \dot\alpha_2 B)\dot x_1 + (\dot\beta_1 A - \dot\alpha_1 B)\dot x_2$$

so that, when $(\dot\alpha_i,\dot\beta_i) = \dot\lambda_i(-\bar\beta,\bar\alpha)$, $i=1,2$,

$$= (\bar\alpha A + \bar\beta B)(\dot\lambda_2\dot x_1 + \dot\lambda_1\dot x_2).$$

Since (α,β) and x are normalized we have

$$\|D^2F(\alpha,\beta,x)\|_{(\alpha,\beta,x)} = \max_{|\dot\lambda_i|^2+\|\dot x_i\|^2=1}\|(\bar\alpha A + \bar\beta B)(\dot\lambda_2\dot x_1 + \dot\lambda_1\dot x_2)\| = \|\bar\alpha A + \bar\beta B\|$$

because

$$\|\dot\lambda_2\dot x_1 + \dot\lambda_1\dot x_2\| \le |\dot\lambda_2|\|\dot x_1\| + |\dot\lambda_1|\|\dot x_2\| \le (|\dot\lambda_1|^2+\|\dot x_1\|^2)^{1/2}(|\dot\lambda_2|^2+\|\dot x_2\|^2)^{1/2} = 1.$$

This gives

$$\gamma(F_{(A,B)},\alpha,\beta,x) = \max\left(1, \frac{1}{2}\|(DF(\alpha,\beta,x)|^{-1}_{(\alpha,\beta,x)^\perp}D^2F(\alpha,\beta,x)\|_{(\alpha,\beta,x)}\right) \le$$

$$\max\left(1, \frac{1}{2}\mu(A,B,\alpha,\beta,x)\ \|\bar\alpha A + \bar\beta B\|\right) \le \max\left(1, \frac{1}{2}\mu(A,B,\alpha,\beta,x)\left(\|A\|^2+\|B\|^2\right)^{1/2}\right). \ \square$$

Proof of Theorem 6. We put together Theorem 5, Proposition 6 and Proposition 7 to obtain the requiered estimation. $\square$

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