RC 20726 (91831) 2/10/97
Computer Science/Mathematics 34 pages

Research Report

Effecting Postponement through Standardization and Process Sequencing

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August 15, 1996

Abstract

Companies have found that delaying product differentiation is an effective means of reducing the costs of complexity resulting from increased product variety. In this paper, we develop models to study the impact of delaying product differentiation through standardization of the components and processes, and through re-sequencing the processes. We also develop models for the joint standardization and process sequence optimization problems. We show that the the resulting integer programs can be solved optimally by linear programming-based algorithms.

1 Introduction

It has been found that as much as 80% of the costs, 50% of quality, and about 80% of business complexity can be influenced through product and process design (Child et al. [2]). Since product and process design form the basis of many of the business operations, the bottom-up approach of focusing first on product and process design has been found to be very effective in many instances (Gingrich and Metz [13]).

Product and process design strategies can result in greater efficiency in manufacturing, logistics, and distribution. To describe this concept, Lee [18] coined the term *designing for supply chain management.* Since designing for supply chain management involves more than just manufacturing-related issues, it is broader than designing for manufacturability and designing for assembly.

In this paper we focus on postponement, a product and process design strategy that is increasing being employed as a means of reducing business complexity and increasing the efficiency of the supply chain. This research is motivated by opportunities for application of this strategy at companies like HP, Motorola, and IBM (Lee et al [17], Garg [9]).

Postponement can be defined as the strategy of delaying product differentiation until as late as is cost-effective. In a production line manufacturing several end-products, these end-products share common components and processes in their initial stages. However, products are customized into their specific identities when specialized processes are performed or when specialized components are added. The point at which the first of such customization stages is performed is called the point of product differentiation. Delaying the point of product differentiation means deferring the customization point to a later stage in the manufacturing system. Therefore, the term *postponement* is synonymous with the terms *delaying product differentiation* and *late customization.*

Typically, one delays the point of product differentiation by

- Standardizing components and/or processes at the point of product differentiation, thereby deferring the customization process;
- Changing the sequence of operations so that one defers customization to a point

later in the production line.

In this paper, we develop models to derive qualitative insights on postponement through standardization and process sequencing.

Research in the area of postponement has focused on both the qualitative and the quantitative aspects. The concept of postponement was first introduced in marketing by Alderson [1]. Alderson [1] observed that every differentiation that makes a product more suitable for a specified market segment, makes it less suitable for other segments. Therefore, moving the point of differentiation nearer to the time of purchase, where demand would be more predictable, can reduce the costs of uncertainty through lower inventory levels.

Zinn and Bowersox [25] detail four ways of delaying the point of product differentiation. These ways of effecting postponement correspond to delaying differentiation in the four primary stages of the production process. The four types of postponement are: Labeling postponement, Packaging postponement, Assembly postponement, and Manufacturing postponement.

There is a wealth of literature on applications of postponement in different companies from various industries. In the computer and high-technology industry, the Hewlett-Packard Company has pioneered designing products for localization. Its efforts in this area have resulted in successful products like the Deskjet printer (Lee, Billington and Carter [17]). They found that product and process design can affect the degree of localization, the site at which localization can be performed, and the cost of such localization. Delaying customization of products via design for localization increases the flexibility with which a company can respond to changes in the mix of demands from different market segments.

In the food industry, Gingrich and Metz [13] describe how Kellogg limited the costs of complexity resulting from increased product proliferation in cereal manufacturing. Ealey and Mercer [6] describe how the advent of active suspensions in automobiles has delayed differentiation in some cars. Active suspensions have improved ride and handling characteristics by replacing suspension springs and shock absorbers with computer-driven hydraulic rams capable of adjusting wheel and body position in real time. However, the real advantage of these systems comes from the fact that one can tune all suspensions by simply reprogramming their computers for each type of car.

As mentioned earlier, changing the process sequence can also be a means of effecting time and form postponement. Benetton's case (Dapiran [4], and Lee and Tang [16]) is particularly illustrative. In order to cope with the uncertainties about the demands for different colors of sweaters, Benetton adopted a *knit now, dye later* strategy. According to this strategy, all sweaters are knitted out of a white yarn. These sweaters are dyed later when the demands for different colors are known with greater accuracy.

Although researchers (Lee and Billington [15], and Pine [20]) have developed conceptual frameworks to study issues involved in delaying the point of product differentiation, there are very few quantitative models to help determine the value of delaying the point of product differentiation.

Schwarz [21] has studied a similar problem in the design of a distribution system. He assesses the value of introducing a centralized warehouse to pool the risk over external supplier lead times. However, this model has not focused on product design issues. Lee [19], one of the first papers to do so, proposes product and process redesign to delay product differentiation. His models are motivated by applications encountered in industry and show that delayed product differentiation results in lower inventory costs.

Lee and Tang [16] have developed a more general model to study the effects of delayed product differentiation. Lee [19] has considered inventories stocked either at the point of product differentiation or as finished goods. Lee and Tang model a multi-stage production system with intermediate buffer stocks preceding each stage. Their model further generalizes previous research by studying the impact of various costs on postponement decisions. The costs their model considers include: inventory savings, cost of investments for standardization and process redesign, and the difference in processing costs resulting from delaying product differentiation.

Lee and Tang [16] assume that there is a constant common production lead time of one period at each stage. In this research, we generalize that model by allowing arbitrary lead times at each stage of the production system. We use our model to study the effect of delaying product differentiation by standardizing components and by redesigning processes.

Additional work in this area includes Garg and Tang [10], and Garg and Lee [11]. Garg and Tang developed models to analyze postponement strategies in systems with multiple points of differentiation. The main objective of their work is to derive insights into the relative marginal benefits of postponement at each point of differentiation through standardization. Garg and Lee [11] study the impact of process sequencing decisions on supply chain management.

We refer the reader to the work of Gerchak et al [12] and the references therein for work in the area of part commonality. The main focus of the work in this area is to show that increased part commonality results in lower inventory levels.

The main focus of the past research has been on quantifying the savings in inventory resulting from component commonality. Researchers have not considered the joint effect of standardization and process sequencing. In our models, inventory savings resulting from standardization are one of the components of the cost equation.

Our work contributes to research in this area by

- Deriving qualitative insights into the effects of postponement and process sequencing under conditions more general than that considered previously.
- Developing models for studying the combined effects of standardization and process sequencing.

The rest of the paper is organized as follows: in Section 2 we develop the framework for modeling divergent flow systems; Section 3 presents a model for analyzing the effects of postponement through standardization of components and/or processes; Section 4 develops models for studying the joint effect and interactions between standardization and process sequencing decisions; and finally, in Section 5, we conclude this paper by describing directions for future research.

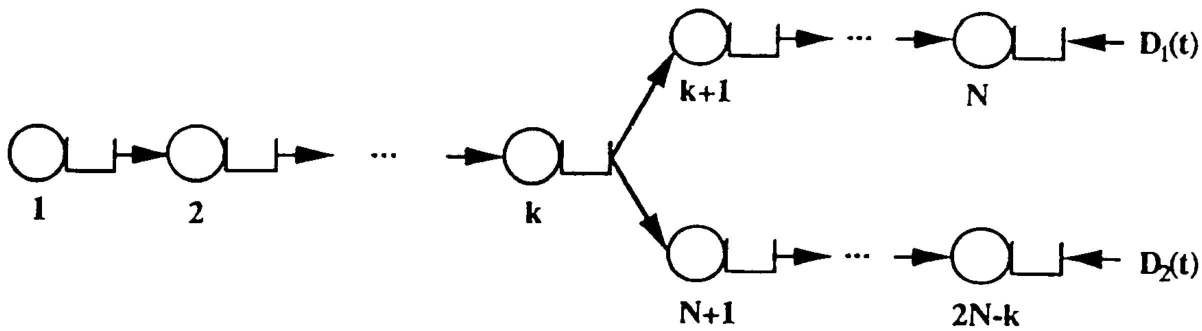

Figure 1: Schematic of the Divergent Flow System, $\mathcal{D}$.

2 The Divergent Flow System

2.1 Description of the Divergent System

Divergent inventory systems arise in pure distribution-type networks where a factory can supply several distribution centers, which in turn service all the retailers within their region. Divergent inventory systems also arise in models of multiple products within a product family. In this case, usually the same set of raw materials is processed through several initial stages. Product variety increases progressively down the system until all the products are differentiated and completed by their final stages.

Without any loss of generality, the divergent system, $\mathcal{D}$, we study has a pure serial flow for the first k stages, and thereafter, the flows diverge into two arms, each having $N-k$ stages. There are intermediate buffers after each production stage in which the output (work-in-process) from that stage is stored. We assume that there is an arbitrary, stage-dependent lead time between the stages. This type of production system can be characterized as a divergent feed-forward flow system. A schematic of this system is appears in Figure 1.

We consider a periodic review system in which the external demands are observed at the two end points. Demands in each period are finite and are *iid* random variables

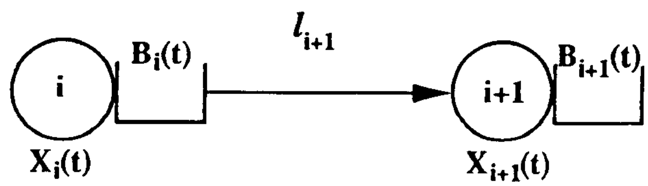

Figure 2: Material Flow between Stages

with the following characteristics: $\boldsymbol{ED}\ (t) = (\mu_1, \mu_2)^T$ and

$$\text{Cov}(\boldsymbol{D}\ (t), \boldsymbol{D}\ (t+s)) = \begin{cases} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

where ρ is the coefficient of correlation between the demands at the two end points.

The model assumes that lead time required to transfer the items from the output buffer of the preceding stage, complete their processing at stage i, and to transfer them to the output buffer of stage i, is l_i time periods (see Figure 2). Let $X_i(t)$ be the number of items that complete their processing at stage i and become available at the output buffer of that stage during period t; and $B_i(t)$ be the output buffer stock level at stage i at the beginning of period t before any production commitments are made. We assume that any unfilled demand is completely backlogged and there is no capacity restriction on the production level and the buffer size at each stage. However, we show that the production rule used in our model results in production quantities, $X_i(t)$, and buffer stock levels, $B_i(t)$, that are bounded for all time periods, t.

Define $Y_i(t)$ to be the inventory *position* at operation i at the beginning of time period t before any production decisions are made.

$$Y_i(t) = B_i(t) + \sum_{j=0}^{l_i-1} X_i(t+j); \quad i = 1, \ldots, 2N-k. \tag{1}$$

We assume that the initial inventory position at each operation i, $Y_i(1)$, is finite. One can write the inventory balance equations for each operation i, for $i \in \{1, \ldots, 2N-k\} \setminus \{k, N, 2N-k\}$, as

$$B_i(t+1) = B_i(t) + X_i(t) - X_{i+1}(t + l_{i+1}). \tag{2}$$

Using Equations (1) and (2) we have:

$$Y_i(t+1) = Y_i(t) + X_i(t + l_i) - X_{i+1}(t + l_{i+1}). \tag{3}$$

For $i = k$ we have:

$$Y_k(t+1) = Y_k(t) + X_k(t + l_k) - X_{k+1}(t + l_{k+1}) - X_{N+1}(t + l_{N+1}), \tag{4}$$

and for $i = N$ we have:

$$Y_N(t+1) = Y_N(t) + X_N(t + l_N) - D_1(t), \tag{5}$$

and finally for $i = 2N - k$ we have:

$$Y_{2N-k}(t+1) = Y_{2N-k}(t) + X_{2N-k}(t + l_{2N-k}) - D_2(t). \tag{6}$$

Note that $B_i(t)$, $X_i(t)$ and $Y_i(t)$ are all random variables. We can now describe the production policy used to control the divergent system described above.

2.2 Production Control Policy

Most of the research on divergent inventory systems has focused on two-stage distribution systems. The problem most researchers have attempted to solve is not that of determining the optimal control policies, but of employing simple control policies to determine the control parameters like safety stocks, reorder points, order up-to points, or lot sizes at each stage of the system.

Production control policies for divergent systems can be classified as *pull-* or *push-*type control systems. Pull-type control systems studied by researchers include METRIC (Sherbrooke [23]) and its variants. METRIC assumes that demands are generated by a stationary compound Poisson process and that the bases follow a one-for-one replenishment policy.

However, the results of METRIC are an approximation. Several researchers have developed exact expressions for METRIC, usually under some restrictive assumptions. However, METRIC and its variants cannot be used to model the divergent system described in Section 2.1 because our divergent system consists of more than two-echelons and it assumes a more general demand distribution.

For a pure serial system operating under a push-type control policy with a finite planning horizon, Clark and Scarf [3] developed the optimal periodic-review policy. However, this model is not optimal for divergent flow systems because of possible of imbalance of inventories. Federgruen and Zipkin [8] extended Clark and Scarf's model by deriving optimal policies for serial systems with an infinite planning horizon. This framework also does not yield analytical expressions for various performance measures.

Eppen and Schrage [7] modeled two-stage divergent systems operating under a push-type control policy. They assumed that the depot does not hold any stock but is merely an agency for centrally ordering parts with the external supplier and for allocating orders arriving from the supplier to the bases. Demands observed at the bases in each period are *iid* and are normally distributed. The depot uses an equal-fractile allocation policy to allocate stock to the bases. However, equal-fractile allocation may not be achievable in each period because of stock imbalance; therefore, this policy is also an approximation. Furthermore, this model only considers two-stage systems and it does not allow stocks at each echelon.

In order to model our divergent system, we extend the production control rule framework of Tang [24] and Denardo and Tang [5]. Tang [24] and Denardo and Tang [5] employed this production rule to model serial systems with a lead time of one review period. We extend their model to consider a divergent system with stage-dependent lead times. Unlike most other control policies, this production rule control policy also does not place any restrictions on the distribution of the demands for the end-products. However, like all other analytical models for divergent systems, this control policy is not guaranteed to be optimal. And, as we shall see, it is simple and yields results that are intuitive.

In a divergent system, the decision-maker has to determine the amount each stage should order from its upstream site in each period. The production control policy is a means of making that decision. The rationale behind this production control rule is to keep the inventory position at stage i as close as possible to its *target* stock level, T_i. One can view these target stocks as the ideal amount of inventory required to cope with uncertainties, analogous to order-up-to points in base-stock policy inventory systems. This production control rule may be represented as follows:

$$X_i(t + l_i) = \mu_1 + \mu_2 + \sum_{j=i}^{2N-k} (T_j - Y_j(t))\, r_j \quad \text{for } i = 1, \ldots, k, \tag{7}$$

$$X_i(t + l_i) = \mu_1 + \sum_{j=i}^{N} (T_j - Y_j(t))\, r_j \quad \text{for } i = k+1, \ldots, N, \tag{8}$$

$$X_i(t + l_i) = \mu_2 + \sum_{j=i}^{2N-k} (T_j - Y_j(t))\, r_j \quad \text{for } i = N+1, \ldots, 2N-k. \tag{9}$$

In the production rule presented above, the first term on the right-hand side represents the average level of production required to meet the expected demand while the second term represents the adjustments made to this average production level to compensate for the deviations of the downstream inventory positions from their target levels. r_i is the *restoration* coefficient for inventory position at stage i.

The restoration coefficient modulates the variations in production across periods. For example, if $0 < r_i < 1$, the restoration coefficient behaves like a smoothing constant by damping the effect of deviation of the inventory position at stage i from its target stock level on the production decision at upstream stages. On the other hand, if $r_i > 1$, the coefficient amplifies the effect of deviation of the inventory position at stage i from its target stock level on the production decision at upstream stages.

We can see that the production rule adjusts the production levels according to the inventory positions at downstream stages, hence this production rule is one from the class of pull-type control systems. The production decisions at each stage are based on echelon-level information as in Clark and Scarf [3]. Since the production rule determines the order quantities at each stage, we can characterize this production rule as an installation stock policy using echelon information.

Like Clark and Scarf [3], and Eppen and Schrage [7], we also assume that the effect of imbalance of inventories is negligible. Imbalance results in an overestimation of the service provided by the inventory system. There are two main sources of imbalance of inventories in divergent systems: lot size requirements at the bases and at the depot, and the non-availability of sufficient inventory at the depot for allocation to the bases. In our divergent system imbalance can occur only because of non-availability of sufficient stock at the depot for allocation to the bases.

2.3 The Operating Characteristics

The state of the production line can be characterized by the inventory positions at each operation, $Y_i(t)$, and the production quantities, $X_i(t)$. In this section we derive the operating characteristics of the system under repeated application of the production rule.

Define:

$\boldsymbol{B}\ (t) = (B_1(t), \ldots, B_{2N-k}(t))^T$,

$\boldsymbol{Y}\ (t) = (Y_1(t), \ldots, Y_{2N-k}(t))^T$,

$\hat{\boldsymbol{X}}\ (t) = (X_1(t + l_1), \ldots, X_{2N-k}(t + l_{2N-k}))^T$,

$\boldsymbol{T}\ = (T_1, \ldots, T_{2N-k})^T$,

$\boldsymbol{D}\ (t) = (D_1(t), D_2(t))^T$,

$\boldsymbol{e}_i\ =$ a $(2N - k)$-column vector with 1 in the ith row and zeros otherwise.

$\boldsymbol{V}\ = (\boldsymbol{e}_N\ , \boldsymbol{e}_{2N-k}\ )$, a $(2N - k) \times 2$-matrix.

$\boldsymbol{F}\ =$ a $(2N - k) \times (2N - k)$-matrix with elements f_{ij} such that:

$$f_{ij} = \begin{cases} 1 & \text{if } i = j + 1,\ j \neq N \\ 1 & \text{if } i = N + 1,\ j = k \\ 0 & \text{otherwise} \end{cases}$$

$\boldsymbol{C}\ =$ a $(2N - k) \times (2N - k)$-matrix with elements c_{ij} given by:

$$c_{ij} = \begin{cases} 1 & \text{if } i \geq j, j \leq k \\ 1 & \text{if } N \geq i \geq j,\ N \geq j > k \\ 1 & \text{if } 2N - k \geq i \geq j,\ 2N - k \geq j > N \\ 0 & \text{otherwise} \end{cases}$$

$\boldsymbol{d}$ = a $(2N-k)$-column vector comprising of elements d_i given by:

$$d_i = \begin{cases} \mu_1 + \mu_2 & \text{for } i = 1, \ldots, k \\ \mu_1 & \text{for } i = k+1, \ldots, N \\ \mu_2 & \text{for } i = N+1, \ldots, 2N-k \end{cases}$$

Combine and rewrite the inventory position balance equations, (3) through (6), as

$$\boldsymbol{Y}(t+1) = \boldsymbol{Y}(t) + (\boldsymbol{I} - \boldsymbol{F})^T \hat{\boldsymbol{X}}(t) - \boldsymbol{V}\boldsymbol{D}(t) \tag{10}$$

Similarly the rewrite production rule equations, (7) through (9), as

$$\hat{\boldsymbol{X}}(t) = \boldsymbol{d} + \boldsymbol{C}^T [r]_{dg} (\boldsymbol{T} - \boldsymbol{Y}(t)) \tag{11}$$

where $[r]_{dg}$ is a diagonal matrix whose diagonal element $r_{ii} = r_i$, $\forall i$.

Let $\boldsymbol{R} = \boldsymbol{I} - [r]_{dg}$. We assume that $\sum_0^\infty \boldsymbol{R}^i$ is finite. This assumption is required to guarantee that a divergent system operating under the production control policy will achieve covariance stationarity asymptotically. As a consequence of this assumption, it follows that $\lim_{n\to\infty} \boldsymbol{R}^n = 0$ and $|1-r_i| < 1$, or the range of values the restoration coefficients can take is $0 < r_i < 2$. Since $\sum_0^\infty \boldsymbol{R}^i = (\boldsymbol{I} - \boldsymbol{R})^{-1}$, it also implies that $[r]_{dg}$ is non-singular.

Although the production control policy assumes uncapacitated output buffers and production at each stage, we now show that this assumption will not result in unbounded inventory positions and production quantities if the demands in each period are finite.

Lemma 1 Application of the production rule results in inventory positions, $\boldsymbol{Y}(t)$, production quantities, $\boldsymbol{X}(t)$, and buffer stock levels, $\boldsymbol{B}(t)$, that are bounded for all periods t if the initial inventory positions are finite, i.e., $|\boldsymbol{Y}(1)| < \infty$, and demands in each period are finite, i.e., $\boldsymbol{D}(t) < \infty$, $\forall t$.

Proof: The proof appears in Garg [9]. ■

Let $f(x) = (1-(1-x)^2)^{-1}$, $g(x,y) = (1-(1-x)(1-y))^{-1}$ for $x < 1, y \le 1$ and $a = \sigma_1^2 r_N^2 f(r_N)$, $b = \rho\sigma_1\sigma_2 r_N r_{2N-k} g(r_N, r_{2N-k})$, $c = \sigma_2^2 r_{2N-k}^2 f(r_{2N-k})$, $\overline{r}_i = 1 - r_i$. Also, let $x \wedge y$ denote $\min(x,y)$, $\gamma_1(l_1, l_2, r) = \frac{\overline{r}}{r^2}\left[(l_1 \wedge l_2)r - \overline{r}^{|l_1 - l_2|}(1 - \overline{r}^{l_1 \wedge l_2})\right]$, $\gamma_2(l_1, l_2, r) = \frac{\overline{r}}{r^2}\left[(l_1 \wedge l_2)r - (1 - \overline{r}^{l_1 \wedge l_2})\right]$ and $F(l, r) = l + 1 + 2\gamma_1(l, l, r) + 2\overline{r}(1 - \overline{r}^l)/r$.

Lemma 2 For $0 < r_N, r_{2N-k} < 2$, the following hold:

(i) a is convex and increasing in r_N.

(ii) b is concave and increasing in r_N and r_{2N-k} if $\rho > 0$, and is convex and decreasing in r_N and r_{2N-k} otherwise.

(iii) c is convex and increasing in r_{2N-k}.

(iv) $\gamma(l, r)$ is convex and decreasing in r.

(v) $\sqrt{F(l,r)}$ is concave and increasing in l.

(vi) $\sqrt{F(l_1,r)}/l_1 \leq \sqrt{F(l_2,r)}/l_2$ for $l_1 \geq l_2$, or $\sqrt{F(l,r)}$ is sub-linear in l.

Proof: See Garg [9] for the proof of this lemma. ■

We now derive the steady-state operating characteristics of the production system under repeated application of the production rule. We have assumed that the system attains a steady state when the first two moments of the three processes, $\{\boldsymbol{Y}(t) : t > 0\}$, $\{\boldsymbol{X}(t) : t > 0\}$, and $\{\boldsymbol{B}(t) : t > 0\}$, all become stationary, i.e., covariance stationary.

Theorem 1 Repeated application of the production rule leads the system asymptotically to a covariance stationary state at a geometric rate with the following properties:

(i)

$$\lim_{t\to\infty} E(Y_i(t)) = T_i, \quad \forall i.$$

(ii)

$$\lim_{t\to\infty} E(X_i(t)) = \begin{cases} \mu_1 + \mu_2, & i = 1, \ldots, k, \\ \mu_1, & i = k+1, \ldots, N, \\ \mu_2, & i = N+1, \ldots, 2N-k. \end{cases}$$

(iii)

$$\lim_{t\to\infty} E(B_i(t)) = \begin{cases} T_i - l_i(\mu_1+\mu_2), & i=1,\ldots,k, \\ T_i - l_i\mu_1, & i=k+1,\ldots,N, \\ T_i - l_i\mu_2, & i=N+1,\ldots,2N-k. \end{cases}$$

(iv)

$$\lim_{t\to\infty} \mathrm{Cov}(Y_i(t),Y_j(t)) = \begin{cases} \sigma_1^2 f(r_N), & i=j=N, \\ \sigma_2^2 f(r_{2N-k}), & i=j=2N-k, \\ \rho\sigma_1\sigma_2 g(r_N, r_{2N-k}), & i=N, j=2N-k, \\ \rho\sigma_1\sigma_2 g(r_N, r_{2N-k}), & i=2N-k, j=N, \\ 0, & \text{otherwise.} \end{cases}$$

(v)

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = (a+b)r_N^{|l_i-l_j|} + (b+c)r_{2N-k}^{|l_i-l_j|}, \quad 1\le i\le k, 1\le j\le k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = \begin{cases} ar_N^{l_j-l_i} + br_{2N-k}^{l_j-l_i}, & \text{if } l_j \ge l_i, \\ (a+b)r_N^{l_i-l_j}, & \text{if } l_j < l_i, \end{cases} \quad 1\le i\le k, k+1\le j\le N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = \begin{cases} br_N^{l_j-l_i} + cr_{2N-k}^{l_j-l_i}, & \text{if } l_j \ge l_i, \\ (b+c)r_{2N-k}^{l_i-l_j}, & \text{if } l_j < l_i, \end{cases} \quad 1\le i\le k, N+1\le j\le 2N-k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = ar_N^{|l_j-l_i|}, \quad k+1\le i\le N, k+1\le j\le N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = \begin{cases} br_N^{l_j-l_i}, & \text{if } l_j \ge l_i, \\ br_{2N-k}^{l_i-l_j}, & \text{if } l_j < l_i, \end{cases} \quad k+1\le i\le N, N+1\le j\le 2N-k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_i(t),X_j(t)) = cr_{2N-k}^{|l_j-l_i|}, \quad N+1\le i,j\le 2N-k.$$

(vi)

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) = (a+2b+c)(l_i \wedge l_j) + (a+b)(\gamma_1(l_i, l_j, r_N) + \gamma_2(l_i, l_j, r_N)) + (b+c)(\gamma_1(l_i, l_j, r_{2N-k}) + \gamma_2(l_i, l_j, r_{2N-k})), \quad 1 \le i \le k, 1 \le j \le k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) = (a+b)(l_i \wedge l_j) + a\gamma_1(l_i, l_j, r_N) + b\gamma_1(l_i, l_j, r_{2N-k}) + (a+b)\gamma_2(l_i, l_j, r_N), \quad 1 \le i \le k, k+1 \le j < N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_N(t)) = (a+b)(l_i \wedge l_N) + a\gamma_1(l_i, l_N, r_N) + b\gamma_1(l_i, l_N, r_{2N-k}) + (a+b)\gamma_2(l_i, l_N, r_N) \\ (a+b)\bar{r}_N(1-\bar{r}_N^{l_i})/r_N^2, \quad 1 \le i \le k, j = N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) = (b+c)(l_i \wedge l_j) + b\gamma_1(l_i, l_j, r_N) + c\gamma_1(l_i, l_j, r_{2N-k}) + (b+c)\gamma_2(l_i, l_j, r_{2N-k}), \quad 1 \le i \le k, N+1 \le j < 2N-k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_{2N-k}(t)) = (b+c)(l_i \wedge l_{2N-k}) + b\gamma_1(l_i, l_{2N-k}, r_N) + c\gamma_1(l_i, l_{2N-k}, r_{2N-k}) + (b+c)\gamma_2(l_i, l_{2N-k}, r_{2N-k}) + (b+c)\bar{r}_{2N-k}(1-\bar{r}_{2N-k}^{l_i})/r_{2N-k}^2, \quad 1 \le i \le k, j = 2N-k;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) = a(l_i \wedge l_j) + a(\gamma_1(l_i, l_j, r_N) + \gamma_2(l_i, l_j, r_N)), \quad k+1 \le i, j < N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_N(t)) = a(l_i \wedge l_N) + a(\gamma_1(l_i, l_N, r_N) + \gamma_2(l_i, l_N, r_N)) + a\bar{r}_N(1-\bar{r}_N^{l_i})/r_N^2, \quad k+1 \le i < N, j = N;$$

$$\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) = b(l_i \wedge l_j) + b(\gamma_1(l_i, l_j, r_N) + \gamma_2(l_i, l_j, r_{2N-k})),$$
$$k+1 \le i < N, N+1 \le j < 2N-k;$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_{2N-k}(t)) &= b(l_i \wedge l_{2N-k}) + b(\gamma_1(l_i, l_{2N-k}, r_N) + \\ &\quad \gamma_2(l_i, l_{2N-k}, r_{2N-k})) \\ &\quad + b\bar{r}_{2N-k}(1 - \bar{r}_{2N-k}^{l_i})/r_{2N-k}^2, \\ &\qquad k+1 \le i < N, j = 2N-k;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_N(t), B_N(t)) &= al_N + a(\gamma_1(l_N, l_N, r_N) + \gamma_2(l_N, l_N, r_N)) + \\ &\quad 2a\bar{r}_N(1 - \bar{r}_N^{l_N})/r_N^2 + a/r_N^2, \quad i = j = N;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_N(t), B_j(t)) &= b(l_N \wedge l_j) + b(\gamma_1(l_N, l_j, r_N) + \\ &\quad \gamma_2(l_N, l_j, r_{2N-k})) + b\bar{r}_N(1 - \bar{r}_N^{l_j})/r_N^2, \\ &\quad i = N, N+1 \le j < 2N-k;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_N(t), B_{2N-k}(t)) &= b(l_N \wedge l_{2N-k}) + b(\gamma_1(l_N, l_{2N-k}, r_N) + \\ &\quad \gamma_2(l_N, l_{2N-k}, r_{2N-k})) + \\ &\quad b\bar{r}_N(1 - \bar{r}_N^{l_{2N-k}})/r_N^2 + \\ &\quad b\bar{r}_{2N-k}(1 - \bar{r}_{2N-k}^{l_N})/r_{2N-k}^2 + b/(r_N r_{2N-k}), \\ &\quad i = N, j = 2N-k;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) &= c(l_i \wedge l_j) + c(\gamma_1(l_i, l_j, r_{2N-k}) + \\ &\quad \gamma_2(l_i, l_j, r_{2N-k})), \quad N+1 \le i, j < 2N-k;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_{2N-k}(t)) &= c(l_i \wedge l_{2N-k}) + c(\gamma_1(l_i, l_{2N-k}, r_{2N-k}) + \\ &\quad \gamma_2(l_i, l_{2N-k}, r_{2N-k})) + \\ &\quad c\bar{r}_{2N-k} r_{2N-k}^2 (1 - \bar{r}_{2N-k}^{l_i})/r_{2N-k}^2, \\ &\qquad N+1 \le i < 2N-k, j = 2N-k;\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Cov}(B_i(t), B_j(t)) &= cl_{2N-k} + c(\gamma_1(l_{2N-k}, l_{2N-k}, r_{2N-k}) + \\ &\quad \gamma_2(l_{2N-k}, l_{2N-k}, r_{2N-k})) + \\ &\quad 2c\bar{r}_{2N-k}(1 - \bar{r}_{2N-k}^{l_i})/r_{2N-k}^2 + c/r_{2N-k}^2, \\ &\quad i = j = 2N-k.\end{aligned}$$

Proof: The proof of this theorem appears in Garg [9].

Hence, the steady-state average production quantity at each operation is equal to the total average demand in a period and the steady-state average inventory position at each operation is equal to its target level. The covariance of $\boldsymbol{Y}(t)$ under steady-state is independent of the lead times. In fact, only the steady-state covariances of the inventory positions of the two end points, $Y_N(t)$ and $Y_{2N-k}(t)$, are non-zero. The buffer stock levels at each stage are increasing functions of the lead times, and as we shall see in the next section, the target inventory positions are also increasing in the lead times.

At steady state, setting $l_i = l$ and $r_i = 1$, $\forall i$, gives some interesting results. The steady-state expectations of inventory positions, $\boldsymbol{Y}(t)$, and production quantities, $\boldsymbol{X}(t)$, are unchanged. The steady-state expectations of the buffer stock levels, $\boldsymbol{B}(t)$, are simplified considerably. In this case, $[r]_{dg} = \boldsymbol{I}$, $\boldsymbol{R} = 0$, $a = \sigma_1^2$, $b = \rho\sigma_1\sigma_2$, $c = \sigma_2^2$ and $\gamma_1(l,l,1) = \gamma_2(l,l,1) = 0$. The non-zero elements of $\boldsymbol{W}$, the steady-state covariance of $\boldsymbol{Y}(t)$, now become identical to the covariances of the demand. The modified steady-state covariances of $\boldsymbol{X}(t)$ and $\boldsymbol{B}(t)$ are:

$$\lim_{t\to\infty} \mathrm{Cov}(\boldsymbol{X}(t), \boldsymbol{X}(t)) = \boldsymbol{C}^T\boldsymbol{W}\boldsymbol{C}$$

and

$$\lim_{t\to\infty} \mathrm{Cov}(\boldsymbol{B}(t), \boldsymbol{B}(t)) = \lim_{t\to\infty} \mathrm{Cov}(\boldsymbol{Y}(t), \boldsymbol{Y}(t)) + l\mathrm{Cov}(\boldsymbol{X}(t), \boldsymbol{X}(t))$$

We can obtain the above equation also by observing that when $\boldsymbol{R} = 0$, as $t \to \infty$, $\boldsymbol{Y}(t)$ and $\boldsymbol{X}(t)$ become *iid* random variables.

2.4 Determination of the System Parameters

We now determine the target stock positions, T_i, and the restoration coefficients, r_i. As mentioned earlier, the target stock positions are analogous to the order-up-to-points in a base stock policy and are designed to cope with uncertainties in demands. In this divergent system, successor stages generate demands at each stage. Only the two end-points observe final customer demands. In order to satisfy demands at each stage in each period, we could set the target stock levels very high. However, this would result in excessive inventory holding costs. On the other hand, if we tried to reduce the inventory holding costs by setting the target stock positions low, the probability of our not being able to satisfy demands at each stage will be higher. Therefore, we would like to determine the target stock positions that will minimize the total in-process inventory holding costs while achieving the desired service levels or target fill rates.

The service level at each stage is expressed as the probability of satisfying the demands from its downstream stage(s). However, the service level at a stage is degraded because of uncertainties in supplies from its upstream site, and because of the inherent imbalance in inventories in divergent systems. Therefore, in order to attain the desired service level at a stage, the required service level at that stage should be higher than the desired level.

As in Tang [24] and in Lee and Tang [16], we have replaced the stochastic constraints with their deterministic equivalents. We have set the centered expected values to be at least twice their standard deviations. In this case, we can view the factor 2 as the *safety factor*. For example, if the centered random variables were normally distributed, a safety factor of 2 would provide a service level of over 95%. Therefore this criterion results in fill rates that are reasonably close to 100%.

The constraints of this stochastic program are

$$\begin{aligned}\lim_{t\to\infty} EB_i(t) \geq\ & \lim_{t\to\infty} EX_{i+1}(t+l_{i+1}) + 2\,[\mathrm{Cov}(B_i(t),B_i(t)) + \\ & \mathrm{Cov}(X_{i+1}(t+l_{i+1}),X_{i+1}(t+l_{i+1})) - \\ & 2\mathrm{Cov}(B_i(t),X_{i+1}(t+l_{i+1}))]^{1/2}, \\ & \quad \text{for } i \in \{1,\ldots,2N-k\} \setminus \{k,N,2N-k\};\end{aligned} \tag{12}$$

$$\begin{aligned}\lim_{t\to\infty} EB_k(t) \geq\ & \lim_{t\to\infty} EX_{k+1}(t+l_{k+1}) + EX_{N+1}(t+l_{N+1}) + \\ & 2\,[\mathrm{Cov}(B_k(t),B_k(t)) + \mathrm{Cov}(X_{k+1}(t+l_{k+1}),X_{k+1}(t+l_{k+1})) - \\ & 2\mathrm{Cov}(B_k(t),X_{k+1}(t+l_{k+1})) - \\ & 2\mathrm{Cov}(B_k(t),X_{N+1}(t+l_{N+1})) + \\ & \mathrm{Cov}(X_{N+1}(t+l_{N+1}),X_{N+1}(t+l_{N+1})) - \\ & 2\mathrm{Cov}(X_{k+1}(t+l_{k+1}),X_{N+1}(t+l_{N+1}))]^{1/2};\end{aligned} \tag{13}$$

$$\lim_{t\to\infty} EB_N(t) \geq \mu_1 + \lim_{t\to\infty} 2\left[\mathrm{Cov}(B_N(t),B_N(t)) + \sigma_1^2\right]^{1/2}; \tag{14}$$

$$\lim_{t\to\infty} EB_{2N-k}(t) \geq \mu_2 + \lim_{t\to\infty} 2\left[\mathrm{Cov}(B_{2N-k}(t), B_{2N-k}(t)) + \sigma_2^2\right]^{1/2}; \tag{15}$$

$$\lim_{t\to\infty} EX_i(t) \geq \lim_{t\to\infty} 2\left[\mathrm{Cov}(X_i(t), X_i(t))\right]^{1/2}, \quad i = 1, \ldots, 2N-k. \tag{16}$$

Based on the results of theorem 1, the $2N-k$ constraints in (16) can be simplified to

$$\mu_1 + \mu_2 \geq 2\sqrt{[a + 2b + c]}, \tag{17}$$

$$\mu_1 \geq 2\sqrt{a}, \tag{18}$$

$$\mu_2 \geq 2\sqrt{c}. \tag{19}$$

The expectation of the buffer stock level is a function of the target stock positions. Therefore, we can rewrite (12) through (15) as

$$\begin{aligned} T_i &\geq (l_i + 1)(\mu_1 + \mu_2) + 2\left[(a+b)F(l_i, r_N) + (b+c)F(l_i, r_{2N-k})\right]^{1/2}, \\ &\qquad i \in \{1, \ldots, k\}; \end{aligned} \tag{20}$$

$$T_i \geq (l_i + 1)\mu_1 + 2\left[aF(l_i, r_N)\right]^{1/2}, \quad i \in \{k+1, \ldots, N-1\}; \tag{21}$$

$$T_i \geq (l_i + 1)\mu_2 + 2\left[cF(l_i, r_{2N-k})\right]^{1/2}, \quad i \in \{N+1, \ldots, 2N-k-1\}; \tag{22}$$

$$T_N \geq (l_N + 1)\mu_1 + 2\left[a/r_N^2 + al_N + 2a\bar{r}_N l_N / r_N + \sigma_1^2\right]^{1/2}; \tag{23}$$

$$\begin{aligned} T_{2N-k} &\geq (l_{2N-k} + 1)\mu_2 + \\ &\quad 2\left[c/r_{2N-k}^2 + cl_{2N-k} + 2c\bar{r}_{2N-k} l_{2N-k}/r_{2N-k} + \sigma_2^2\right]^{1/2}. \end{aligned} \tag{24}$$

Notice that the covariances in the above-mentioned constraints are functions of r_N and r_{2N-k} only and are independent of the other restoration coefficients. Therefore, the decision variables in the chance-constrained program are T_i, r_N and r_{2N-k}. Our formulation does not incorporate the penalty cost of shortages in the objective function explicitly because for most real applications this penalty cost is hard to estimate. Managers typically use service rate metrics such as fill rates for their inventory policies.

Let h_i be the cost of holding one unit of a product at buffer i for one period. We can formulate this problem of minimizing the total inventory cost, problem (P), as

$$
\begin{aligned}
(P): \qquad & \min \sum_i h_i T_i \\
& \text{s.t.} \quad (17)\text{–}(24), \\
& r_N, r_{2N-k} \in (0,1].
\end{aligned}
$$

We will now focus on obtaining the optimal solution to the stochastic program (P).

Theorem 2 The optimal solution of problem (P) is given by the solutions to the following equations:

$$
\begin{aligned}
r_N &= \frac{2}{1+(\kappa\sigma_1/\mu_1)^2} \\
r_{2N-k} &= \frac{2}{1+(\kappa\sigma_2/\mu_2)^2}
\end{aligned}
$$

where κ is the safety factor used in the constraints of the stochastic program.

Proof: See Garg [9] for the proof of this theorem. ■

From theorem 2 we can see that the optimal values of r_N and r_{2N-k} are decreasing in the squared coefficient of variation of demands of the respective end-products and the safety factor, which is equal to 2 here. Therefore, a higher variability in end-product demands will require greater smoothing.

As mentioned earlier, asymptotic covariance stationarity of the system requires that $0 < r_i < 2$, $\forall i$. Interestingly enough, the optimal values of r_N and r_{2N-k} will always lie within this range. Note that the optimal value of r_N (or r_{2N-k}) will be greater than 1 if the squared coefficient of variation of the demand of product 1 (or product 2) is less than $1/\kappa^2$ (equal to 1/4 here), or the variability in the demand is quite low. Since the squared coefficient of variation of demands is greater than 1/4 in most realistic situations, the optimal value of the restoration coefficients will be less than one. Therefore, in the ensuing analyses we assume that $0 < r_i \leq 1$, $\forall i$.

Also note that functions a, b, and c are essentially the *effective* covariances of the demands. The absolute values of a, b and c are increasing in the restoration coefficients from lemma 2. As a result, the effective demand covariances at each stage are a, b and c.

Constraints (17)–(19) limit the effective variability of demands observed at each stage of the system. Therefore, if the squared coefficient of variation of the demands is high, the optimal value of the restoration coefficient will be lower so that the effective variability is reduced. The restoration coefficients also serve to reduce the probability of infeasibilities due to negative order quantities at a stage.

We would now like to obtain some qualitative insights into how the optimal target stock levels would compare to the order-up-to points of a base-stock inventory policy. In general, the target stock levels would be higher because in the absence of ordering costs, the base-stock policy has been shown to be optimal. Under our production control policy we would be following a true base stock policy if the optimal solution to the stochastic program (P) were r_N and r_{2N-k} equal to 1. Therefore, high variability in demands of the end-products will result in a greater difference between the optimal solution of the stochastic program, (P), and the result of a base-stock inventory policy. These and other properties of the optimal solution are summarized below:

Lemma 3 The solution to Problem (P') has the following properties:

(i) The optimal solution is not necessarily at $r_N = 1$ and $r_{2N-k} = 1$.

(ii) Target stock positions, T_i, are increasing in lead times, l_i.

(iii) Steady-state buffer stock levels, $\lim_{t\to\infty} EB_i(t)$, are increasing in lead times, l_i.

Proof: Results follow from theorem 2 and lemma 2. ■

3 Postponement Through Standardization

We can now apply the model for the divergent system from Section 2 to study postponement through standardization. Without any loss of generality, consider a system with two end-products, each requiring N stages, observing *iid* demands. In the divergent system that results, we assume that k is the point of product differentiation (strictly speaking, differentiation occurs after stage k). Stages 1 through k are common to both products, therefore, there is only one common operation at each of such stages. From stage $k+1$ onwards, there are two distinct operations, one for each product. We use the indices $k+i$ and $N+i$ to denote the two operations corresponding to stage $k+i$. Since each product is assumed to require N stages, k, the point of product differentiation, can vary between 0 and $N-1$. If there is no common stage for the two products, then $k = 0$. In order to retain the distinct identities of the two products, the maximum number of stages that one can standardize is $N-1$.

3.1 Total Relevant Cost Function

The effect of delaying the point of product differentiation can be evaluated on the basis of the total relevant cost. This total relevant cost consists of the inventory holding costs (which were determined in Section 2.4), the cost of standardizing components and the change in processing costs due to standardization. Let S_i be the equivalent net average cost per period of standardizing components and the operational processes at stage i. Assume that the unit processing costs for the two products are identical at each stage. Let p_i and q_i be the unit processing costs for stage i before and after standardization respectively. Let $\delta_i = q_i - p_i$. And let $H(k; r_N, r_{2N-k})$ denote the total inventory costs per period when operation k is the point of product differentiation. Therefore, $H(k; r_N, r_{2N-k})$ is the objective function value for the optimal solution to problem (P') of Section 2.4.

There are three decision variables in the total relevant cost minimization problem: k, r_N and r_{2N-k}. Let $Z(k; r_N, r_{2N-k})$ be the total relevant cost per period when operation k is the point of product differentiation. We can express $Z(k; r_N, r_{2N-k})$ as

$$\begin{aligned} Z(k; r_N, r_{2N-k}) &= \sum_{i=1}^{k} S_i + H(k; r_N, r_{2N-k}) + (\mu_1 + \mu_2) \sum_{i=1}^{k} q_i + \\ &\quad (\mu_1 + \mu_2) \sum_{i=k+1}^{N} p_i. \end{aligned} \tag{25}$$

Note that some components of $Z(k; r_N, r_{2N-k})$ are incurred as one-time fixed costs while the others are per period variable costs. However, these one-time costs can be converted to equivalent per period costs given the expected life of the products and the discounting factor. The problem of determining the optimal point of product differentiation, problem (Q), can be formulated as

$$
\begin{aligned}
(Q): \quad & \min_{k} \min_{r_N, r_{2N-k}} Z(k; r_N, r_{2N-k}) \\
\text{s.t.} \quad & r_N, r_{2N-k} \in (0,2) \\
& k \in \{0, \ldots, N-1\}.
\end{aligned}
$$

From theorem 2 we can obtain the optimal solution to problem (Q) once k is fixed, and this solution of r_N and r_{2N-k} is independent of k. Therefore, this problem can be solved by performing a simple search over k. For the sake of convenience, henceforth we will refer to function $Z(k; r_N, r_{2N-k})$ as $Z(k)$.

We first determine the marginal cost of deferring the point of product differentiation from stage k to stage $k+1$, $Z(k+1) - Z(k)$. Assuming $h_{k+i} = h_{N+i}$ and $l_{k+i} = l_{N+i}$ for $i = 1, \ldots, N-k$, this marginal cost can be expressed as

$$
Z(k+1) - Z(k) = S_{k+1} + (\mu_1 + \mu_2)\delta_{k+1} + h_{k+1}\theta(l_{k+1}, r_N, r_{2N-k}), \tag{26}
$$

where

$$
\begin{aligned}
\theta(l_{k+1}, r_N, r_{2N-k}) &= 2\left[(a+b)F(l_{k+1}, r_N) + (b+c)F(l_{k+1}, r_{2N-k})\right]^{1/2} - \\
& \quad 2\left[aF(l_{k+1}, r_N)\right]^{1/2} - 2\left[cF(l_{k+1}, r_{2N-k})\right]^{1/2}.
\end{aligned} \tag{27}
$$

If $h_{k+i} \neq h_{N+i}$ and $l_{k+i} \neq l_{N+i}$ for $i = 1, \ldots, N-k$, then the resulting expression is a simple modification of the equations (26) and (27). If we define h'_{k+i} and l'_{k+i} to be the unit inventory holding cost per period and the lead time at stage $k+i$ respectively after standardization, then we could set $h'_{k+i} = \max\{h_{k+i}, h_{N+i}\}$ and $l'_{k+i} = \max\{l_{k+i}, l_{N+i}\}$ for $i = 1, \ldots, N-k$ and derive similar expressions.

Note that deferring the point of product differentiation from stage k to $k+1$ affects only the inventory at buffer $(k+1)$. This result implicitly assumes that delaying differentiation does not affect the target stock levels at other stages. This difference in inventory is represented by $\theta(l_{k+1}, r_N, r_{2N-k})$ and is independent of k if $l_i = l, \forall i$. If $\theta(l_{k+1}, r_N, r_{2N-k})$ is negative then deferring the point of differentiation results in inventory savings; otherwise, deferring the point of differentiation results in higher inventories. In the case of $\theta(l_{k+1}, 1, 1)$ equation (27) simplifies to

$\theta(l_{k+1},1,1) = 2\sqrt{(l_{k+1}+1)(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)} - 2\sqrt{(l_{k+1}+1)\sigma_1^2} - 2\sqrt{(l_{k+1}+1)\sigma_2^2}$, or $\theta(l_{k+1},1,1) = 2\sqrt{l_{k+1}+1}\{\sqrt{Var(D_1+D_2)} - \sqrt{Var(D_1)} - \sqrt{Var(D_2)}\}$. We now develop some structural properties of $\theta(l_{k+1}, r_N, r_{2N-k})$.

Lemma 4 $\theta(l_{k+1}, r_N, r_{2N-k}) \le 0$ and $\theta(l_{k+1}, r_N, r_{2N-k})$ is non-increasing in the lead time, $l_{k+1}, \forall k$, if $\sigma_1/\mu_1 = \sigma_2/\mu_2$, or if $\rho \le 0$.

Proof: The proof is fairly straightforward for the case when $\rho \le 0$. When $\sigma_1/\mu_1 = \sigma_2/\mu_2$, from theorem 2 we have $r_N^* = r_{2N-k}^* = r$, hence,

$$\theta(l_{k+1}, r, r) = 2F(l_{k+1}, r)^{1/2}\left[\sqrt{a+2b+c} - \sqrt{a} - \sqrt{c}\right]$$

We prove the first result by noting that $b \le \sqrt{ac}$ and $F(l_{k+1}, r) > 0$. To obtain the second result it is sufficient to show that $F(l_{k+1}, r)$ is non-decreasing in l_{k+1}. This follows directly from lemma 2. ∎

This proposition implies that when $\sigma_1/\mu_1 = \sigma_2/\mu_2$ or $\rho \le 0$, delaying the point of product differentiation always results in inventory savings. In addition, these savings are greater as the lead time, l_{k+1}, increases.

3.2 Special Cases of Cost Structures

In general, the total relevant cost function, $Z(k)$, is non-linear and difficult to characterize. We consider some special cases of costs in which some structural results of the total relevant cost function can be readily obtained. However, these special cases do not limit the applicability of this analysis to different situations. The special cases of costs can be used in combination with one another to model systems with fairly general cost structures.

Let $G(k) = [Z(k+1) - Z(k)] - [Z(k) - Z(k-1)]$. From the definition of $G(k)$, $Z(k)$ is convex if $G(k) \ge 0$ for $1 \le k \le N-1$ and $Z(k)$ is concave if $G(k) \le 0$ for $1 \le k \le N-1$. From (26) we rewrite $G(k)$ as

$$G(k) = S_{k+1} - S_k + (\mu_1+\mu_2)(\delta_{k+1} - \delta_k) + \theta_{k+1}h_{k+1} - \theta_k h_k.$$

If $Z(k)$ is convex in k, then the optimal point of product differentiation, k^*, is given by the following expression:

$$k^* = \min\{N-1, \min\{k \geq 0 : Z(k+1) - Z(k) \geq 0\}\} \tag{28}$$

Case 1: $S_k = S$, $\delta_k = \delta$, $h_k = h$, $\sigma_1/\mu_1 = \sigma_2/\mu_2$, and $l_k \leq l_{k+1}$ (or $l_k \geq l_{k+1}$), $\forall k$.
In this case, the costs of standardizing the components and the processes at each stage are the same, and the unit inventory holding costs are also identical at each stage. This situation would occur in a production line in which the first $N-1$ stages are testing operations and no significant value is added to the products at these stages. The last stage could be an assembly operation where high value components are added. For this case, $G(k) = h(\theta(l_{k+1}, r, r) - \theta(l_k, r, r))$, $\forall k$. If the lead times are non-increasing progressively down the system, i.e., $l_k \geq l_{k+1}$, then $G(k) \geq 0$ (from Lemma 4), which implies that $Z(k)$ is convex in k. Therefore, the optimal point of product differentiation can be determined from equation (28).

On the other hand, if $l_k \leq l_{k+1}$ then $G(k) \leq 0$, $\forall k$, implying that $Z(k)$ is concave in k. In this case the optimal point of differentiation is one of the extreme points, $k^* = 0$ or $k^* = N-1$. We can resolve this dilemma by comparing the values of $Z(N-1)$ and $Z(0)$:

$$Z(N-1) - Z(0) = (N-1)\,[S + (\mu_1 + \mu_2)\delta] + h \sum_{k=1}^{N-1} \theta(l_k, r, r)$$

This equation results in two cases:

1. If $(N-1)\,[S + (\mu_1 + \mu_2)\delta] > -h \sum_{k=1}^{N-1} \theta(l_k, r, r)$, then $k^* = 0$.
2. If $(N-1)\,[S + (\mu_1 + \mu_2)\delta] \leq -h \sum_{k=1}^{N-1} \theta(l_k, r, r)$, then $k^* = N-1$.

The two conditions above imply that we delay the point of differentiation to stage $N-1$ only if the savings in the inventory holding costs are greater than the cost of delaying the point of product differentiation.

Case 2: $S_k = S$, $\delta_k = \delta$, $h_k = kh$, and $\sigma_1/\mu_1 = \sigma_2/\mu_2$.
In this case the inventory holding cost increases linearly in the number of operation,

while the cost of standardizing the components and processes and the difference in processing cost after standardization are identical at each stage. This case can occur when the same incremental value is added to the product at each stage. Here $G(k) = h((k+1)\theta_{k+1} - k\theta_k)$. If $l_k \le l_{k+1}$, then $G(k) \le 0,\ \forall k$, implying that $Z(k)$ is concave in k. Therefore, $k^* = 0$ or $N-1$.

If $l_k = l,\ \forall k,\ \theta_k = \theta$ and $G(k) = h\theta \le 0$. Therefore, the optimal point of differentiation is either $k^* = 0$ or $N-1$. Again, on comparing the values of $Z(N-1)$ and $Z(0)$, we get

$$\begin{aligned} Z(N-1) - Z(0) &= (N-1)\left[S + (\mu_1 + \mu_2)\delta + \theta h(N/2)\right] \\ &= (N-1)\left[S + (\mu_1 + \mu_2)\delta + \theta h_N/2\right] \end{aligned}$$

From this equation we get

1. If $S + (\mu_1 + \mu_2)\delta \ge -\theta h_N/2,\ k^* = 0$.
2. If $S + (\mu_1 + \mu_2)\delta < -\theta h_N/2,\ k^* = N-1$.

$S + (\mu_1 + \mu_2)\delta$ is the cost incurred in deferring the point of product differentiation by one stage while $-\theta h_N/2$ is the savings in inventory accruing from this decision. It is interesting to note that the inventory savings due to deferring the point of differentiation are measured in terms of half the finished products' inventory cost. Again, if the inventory savings accruing from deferring the point of differentiation are less than the investment required to standardize operations, we do not defer the point of product differentiation.

Case 3: $S_k = \pi_1 h_k,\ \delta_k = \pi_2 h_k,\ \sigma_1/\mu_1 = \sigma_2/\mu_2$, and $l_k = l,\ \forall k$
In essence, this case assumes that the cost of standardizing an operation and the difference in the processing costs due to standardization are proportional to the value of the product at that stage. Since $l_k = l$, we have $\theta(l_k, r_N, r_{2N-k}) = \theta,\ \forall k$. One can write the marginal cost (or benefit) of deferring the point of product differentiation as

$$Z(k) - Z(k-1) = h_k\left[\pi_1 + (\mu_1 + \mu_2)\pi_2 + \theta\right].$$

It follows from the above equation that if $\pi_1 + (\mu_1 + \mu_2)\pi_2 + \theta \ge 0$ then $Z(k)$ is an increasing function of k; otherwise it is a decreasing function of k. Therefore,

- If $\pi_1 + (\mu_1 + \mu_2)\pi_2 + \theta \geq 0$, the optimal point of product differentiation is $k^* = 0$.
- If $\pi_1 + (\mu_1 + \mu_2)\pi_2 + \theta < 0$, the optimal point of product differentiation is $k^* = N-1$.

4 Standardization and Process Sequencing

Postponement can be effected through standardization and through process re-sequencing. In this section we study the interactions between standardization and process sequencing decisions. In order to study the effect of standardization, we first need to recapitulate the sources of product differentiation: differences in components, and differences in processes. Manufacturing systems having several end-products can have three different types of standardizations at a stage:

- Component standardization (products at that stage may be differentiated),
- Process standardization (products at that stage may be differentiated),
- Product standardization.

Note that product standardization is a much stronger condition than the other two types of standardizations. For example, even if the components and the processes at a stage are standardized, products flowing through that stage may be differentiated. However, if the products at a stage are standardized, the components and the processes at that stage have to be standardized.

We consider a system with N operations (or processes). These operations can be performed in any sequence. Since there are N operations, there are also N stages (or positions) in the sequence of operations. We need to assign each operation to a stage in the sequence. The models are fairly general as there are no restrictions on the number of end-products. We now present the models for joint process sequencing and standardization problems.

4.1 Model for Process Sequencing and Standardization

Define s_{ij} as the cost of performing operation i at stage j. Let α_{ij} be the net savings resulting from standardizing components used in operation i at stage j. These savings can be due to the benefits of risk-pooling in component inventories and, the effect of quantity discounts in purchasing, minus the cost of redesigning the components and the products. Also define β_{ij} as the net savings from standardizing the processes that constitute operation i at stage j. These savings reflect the economies of scope minus the process redesign cost. Note that α_{ij} and β_{ij} can be negative if the savings due to standardization are less than the costs of standardizing the components and the processes respectively.

In addition, let us define the following indicator variables:

$$z_{ij} = \begin{cases} 1 & \text{if operation } i \text{ is performed at stage } j. \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if components for operation } i \text{ performed at stage } j \text{ are standardized.} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if processes for operation } i \text{ performed at stage } j \text{ are standardized.} \\ 0 & \text{otherwise} \end{cases}$$

The standardization and process sequencing optimization problem, (SPS), can be represented as follows:

$$(SPS): \qquad \min \sum_i \sum_j (s_{ij} z_{ij} - \alpha_{ij} y_{ij} - \beta_{ij} x_{ij}) \tag{29}$$

s.t.

$$\sum_i z_{ij} = 1, \quad \forall j \tag{30}$$

$$\sum_j z_{ij} = 1, \quad \forall i \tag{31}$$

$$z_{ij} \geq x_{ij}, \quad \forall i, j \tag{32}$$

$$z_{ij} \geq y_{ij}, \quad \forall i, j \tag{33}$$

$$x_{ij}, y_{ij}, z_{ij} \in \{0, 1\}, \quad \forall i, j.$$

Problem (SPS) has two sets of constraints. Note that equations (30) and (31) are the constraints of the assignment problem. Constraints (32) and (33) are the standardization constraints. Constraint (32) ensures that processes for operation i at stage j can only be standardized if operation i is performed at stage j. Similarly, (33) shows that components for operation i at stage j can only be standardized if operation i is performed at stage j. We now focus on the techniques for solving problem (SPS) by obtaining special properties of its constraint matrix.

Let $\boldsymbol{A}$ represent the assignment constraints matrix (equations (30) and (31)). One can write the two standardization constraints compactly as

$$-\boldsymbol{Iz} + \boldsymbol{Ix} \leq 0 \tag{34}$$

$$-\boldsymbol{Iz} + \boldsymbol{Iy} \leq 0 \tag{35}$$

The constraint matrix for (SPS) can be represented as

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{A} & 0 & 0 \\ -\boldsymbol{I} & 0 & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} & 0 \end{pmatrix}$$

Theorem 3 Matrix $\boldsymbol{S}$ for problem (SPS) is totally unimodular.

Proof: Total unimodularity of matrices is preserved under the following transformations (Seymour [22]):

(i) Permuting rows or columns,

(ii) Taking a transpose,

(iii) Multiplying a row or a column by -1,

(iv) Pivoting,

(v) Adding an all-zero row or column, or adding a row or column with one non-zero element, being ± 1,

(vi) Repeating a row or a column.

Note that $\boldsymbol{A}$, the constraint matrix for the assignment problem, is proven to be totally unimodular. We will now show that $\boldsymbol{S}$ results from a series of transformations of $\boldsymbol{A}$ that preserve total unimodularity. From the above, we have $(\boldsymbol{A} \ -\boldsymbol{I}\ )^T$ is totally unimodular, and so is $(\boldsymbol{A} \ -\boldsymbol{I} \ -\boldsymbol{I}\ )^T$. By repeated application of transformation (v) above matrix can be transformed into matrix $\boldsymbol{S}$. ■

Since the constraint matrix for problem (SPS) is totally unimodular, the linear programming solution for this problem will be integral.

4.2 A Model for Process Sequencing and Delayed Differentiation

We now develop a model to jointly optimize the operation sequence and the point of product differentiation. As mentioned earlier, delaying product differentiation requires the standardization of both components and processes. Therefore, we need to consider the net benefit due to these standardizations. Let the net benefit due to the two types of standardizations be denoted by $\overline{\alpha}_{ij} = \alpha_{ij} + \beta_{ij}$.

In addition to yielding savings due to standardizations, delaying product differentiation also reduces the inventories of the products themselves. From theorem 2 and equations (20)-(24), note that if the product is standardized at stage j, the net inventory holding cost due to performing operation i at stage j is

$$h_i \left[(\mu_1 + \mu_2)(l_i + 1) + 2\sqrt{(a+b)F(l_i, r_N) + (b+c)F(l_i, r_{2N-k})}\right],$$

and otherwise it is

$$h_i \left[(\mu_1 + \mu_2)(l_i + 1) + 2\sqrt{aF(l_i, r_N)} + 2\sqrt{cF(l_i, r_{2N-k})}\right].$$

Define

$$u_j = \begin{cases} 1 & \text{if the products are standardized at stage } j. \\ 0 & \text{otherwise} \end{cases}$$

The total cost, TC, for the problem can be expressed as

$$\begin{aligned} TC \;=\; & \sum_i \sum_j \Big\{ s_{ij} z_{ij} + (1-u_j) z_{ij} h_i \Big[2\sqrt{aF(l_i, r_N)} + \\ & 2\sqrt{cF(l_i, r_{2N-k})} + (\mu_1 + \mu_2)(l_i + 1) \Big] - \\ & u_j z_{ij} \big[\overline{\alpha}_{ij} - h_i \big[(\mu_1 + \mu_2)(l_i + 1) + \\ & 2\sqrt{(a+b)F(l_i, r_N) + (b+c)F(l_i, r_{2N-k})} \big] \big] \Big\} \end{aligned} \qquad (36)$$

After simplifying, we can rewrite equation (36) as

$$TC = \sum_i \sum_j \left[s'_{ij} - u_j \alpha'_{ij} \right] z_{ij}, \qquad (37)$$

where

$$\begin{aligned} \alpha'_{ij} \;=\; & \overline{\alpha}_{ij} + 2h_i \left[\sqrt{aF(l_i, r_N)} + \sqrt{cF(l_i, r_{2N-k})} \right] - \\ & 2h_i \left[(a+b)F(l_i, r_N) + (b+c)F(l_i, r_{2N-k}) \right]^{1/2} \end{aligned}$$

and s'_{ij} is given by

$$s'_{ij} = s_{ij} + h_i \left[2\sqrt{aF(l_i, r_N)} + 2\sqrt{cF(l_i, r_{2N-k})} + (\mu_1 + \mu_2)(l_i + 1) \right].$$

The joint postponement and process sequencing problem, problem (DD), can be formulated as

$$(DD): \qquad \min \; TC$$

s. t.

$$\sum_i z_{ij} = 1, \quad \forall j \qquad (38)$$

$$\sum_j z_{ij} = 1, \quad \forall j \qquad (39)$$

$$u_j \geq u_{j+1}, \quad j \in \{1, \ldots, N-1\} \tag{40}$$

$$u_j, z_{ij} \in \{0, 1\}, \quad \forall i, j$$

In this formulation, as before, equations (38) and (39) are the assignment problem constraints. Constraint (40) ensures that the products can be standardized at a stage j only if they are also standardized at all stages 1 through $j-1$ and $u_N = 0$ and $u_0 = 1$. Note that the matrix representing the constraints of problem (DD) is not totally unimodular. We will now focus on obtaining the solution to this problem.

Lemma 5 The optimal solution to the non-linear integer programming problem, (DD), for an N-stage system can be obtained by solving N linear assignment problems.

Proof: Note that the value of the last common stage can vary between 0 and $N-1$. From constraint (40) we can see that setting the last common stage will uniquely determine $\boldsymbol{u}$, the vector of u_js. Once $\boldsymbol{u}$ is determined and its values are substituted in (DD), the resulting problem is a linear assignment problem. ∎

5 Conclusions

In this paper we have applied the model for a divergent production-inventory system to derive qualitative insights into decisions regarding postponement. We show that postponement results in inventory savings. However, inventory savings are just a component of the total costs that need to be considered in making postponement decisions. We look at three different cases of cost structures to obtain the cost-effective postponement strategy in each case. These cases of cost structures can then be combined with each other to model supply chains with very general cost structures.

We have also developed models to study the joint effect of standardization and process sequencing. The resulting models are integer programming formulations. We show that a linear programming solution to one of the formulations will result in integer solutions. We also study the joint delayed differentiation and process sequence optimization problem for an N-stage system. The problem can be formulated as a non-linear integer program. We also show that the optimal solution to the integer program can be obtained by solving N linear programs.

Our models assume that the cost of re-sequencing processes is not sequence-dependent. This is an area for future research.

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