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From Dynamics to Computation and Back?

Michael Shub
IBM T. J. Watson Research Center
P. O. Box 218
Yorktown Heights, NY 10598

Research Division
Almaden - Austin - Beijing - Haifa - T. J. Watson - Tokyo - Zurich

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Michael Shub

Dedicated to Steve Smale on his seventieth birthday.

Felicitations

It is a pleasure for me to be celebrating Steve's seventieth birthday. Twenty years ago I sent him a note congratulating him on his fiftieth birthday and wishing him another half century as productive as the first. Twenty years later I can say, so far so good. Real computation and complexity and now learning theory are added to the tremendous influence he has had on twentieth (now twenty first)century mathematics. Steve is an impossible act for his students to follow and there is no end in sight. Last week I was in Paris where I visited the Monet museum. Monet was doing his best work at eighty-five. I expect that Steve will be doing the same.

The title of my talk echoes the title of the last Smalefest for Steve's sixtieth birthday "From Topolology to Computation". I met Steve in 1962. He had by this time finished his immersion theorem-turning the sphere inside out, the generalized Poincaré conjecture and the H-cobordism theorem. He had found a horseshoe on the beaches of Rio and had begun his modern restructuring of the geometric theory of dynamical systems, focusing on the global stable and unstable manifolds and their intersections. The period 1958 to 1962 had been incredibly creative for Steve. The number of Steve's remarkable accomplishments still boggles my imagination. And they are not of a whole, one following from the other, but rather disparate independent inventions. By 1962 Steve had already left finite dimensional differential topology. So I missed this wonderful part of his career. But luckily for the subject and for me he had not left dynamical systems. So that is where my story begins. My work in dynamics has been tremendously influenced by Steve. Now after years of collaboration with Steve, Felipe and Lenore on computation, I find that some of the techniques that Steve and I used in our sequence of papers on Bezout's theorem and complexity may be useful again back in dynamics.

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What I am reporting on today is joint work with Keith Burns, Charles Pugh, Amie Wilkinson, and Jean-Pierre Dedieu see [**BuPuShWi**],[**DeSh**]. Much of what I am saying is taken from these two papers without specific attribution.The material may be found in much expanded form in these two references.

1. Introduction

In his 1967 survey paper on dynamical systems [**Sm2**] Steve asked for stable properties which hold for most dynamical systems and which in some sense describe the orbit structure of the system. The concepts under study at the time were structural stability and Ω stability, which roughly require that at least on the most dynamically interesting sets the orbit structure of the dynamical system be locally constant under perturbation of the system up to continuous change of coordinates. By work beginning with Steve's work on the horseshoe, Anosov's [**An**] structural stability theorem and Steve's Ω stability theorem [**Sm2**], uniform hyperbolicity of the dynamics is known to imply Ω or structural stability. A remarkable feature of these new results, which set them apart from previous work on structural stability by Andronov-Pontryagin and Peixoto, is the complexity of the dynamics encompassed. The horseshoe, strange attractors and globally hyperbolic dynamics are chaotic. They exhibit exponentially sensitive dependence on initial conditions. Thus, while in some sense the future history of a particular orbit may be too difficult to predict, the ensemble of orbits in these stable systems is topologically rigid in its behaviour.

One of the major achievements of the uniformly hyperbolic theory of dynamical systems is the work of Anosov, Sinai, Ruelle and Bowen on the ergodic theory of uniformly hyperbolic systems. Anosov proved that smooth volume preserving globally hyperbolic systems are ergodic. Sinai, Ruelle and Bowen extended this work to specifically constructed invariant measures for general uniformly hyperbolic systems now called SRB measures. The ergodicity of these measures asserts that although particular histories are difficult to compute the statistics of these histories, the probability that a point is in a given region at a given time, is captured by the measure.

Steve's program is however not accomplished, since structurally stable, Ω stable and uniformly hyperbolic systems are not dense in the space of dynamical sysytems [**Sm1, AbSm**]. Much of the work in dynamical systems in recent years has been an attempt to extend the results of the uniformly hyperbolic theory to more general systems. One theme is to relax the notion of uniform hyperbolicity to non-uniform or partial hyperbolicity and then to conclude the existence of measures sharing ergodic properties of the SRB measures. The Proceedings of the Seattle AMS Summer Symposium on Smooth Ergodic Theory will surely contain much along these lines. In particular, you can find the survey of recent progress on ergodicity of partially hyperbolic systems [**BuPuShWi**] included there. There is much more available concerning the quadratic family, Henon and Lorenz attractors and more, but I will not try to reference that work here. It is my feeling that much of the work proving the presence of non-uniform hyperbolicity or non-zero Lyapunov exponents (which is the same) is too particular to low dimensions to be able to apply in general.

This paper reports on a result in the theory of random matrices which is an analogue in linear algebra of a mechanism we may hope to use to find non-zero Lyapunov exponents for general dynamical systems.

2. Rich Families

In rich enough families individual members may inherit family properties.

This sentence which a truism in ordinary language sometimes also applies in mathematics. The first theorem I learned from Steve in his 1962 course on infinite dimensional topology, the Abraham transversality theorem is an example. Let us recall that a smooth map $F : M \to N$ between differentiable manifolds is transversal to the submanifold W of N if $TF(x)(T_xM)$ contains a vector space complement to $T_{F(x)}W$ in $T_{F(x)}N$ for every $x \in M$ such that $F(x) \in W$.

We give a simple finite dimensional version of the Abraham transversality theorem which is valid in infinite dimensions [**AbRo**]. Let $\mathcal{P}$ be a finite dimensional smooth manifold which we will think of as a space of parameters for a space of maps . Suppose $\Phi : \mathcal{P} \times M \to N$ is a smooth map. For $p \in \mathcal{P}$ let $\Phi_p = \Phi(p, -)$ which is a smooth mapping from M to N. Suppose that Φ transversal to W. Then $V = \Phi^{-1}(W)$ is a smooth submanifold of $\mathcal{P} \times M$. Let $\Pi_1 : V \to \mathcal{P}$ be the projection of $\mathcal{P} \times M$ onto the first factor restricted to V. The following proposition is then an exercise in counting dimensions of vector spaces.

Proposition 2.1. *With Φ,M,$\mathcal{P}$, N,W and V as above and $p \in \mathcal{P}$; Φ_p is transversal to W on M if and only if p is a regular value of Π_1.*

Now by Sard's Theorem it follows that almost every $p \in \mathcal{P}$ is a regular value of Π_1. So we have proven a version of Abraham's theorem.

Theorem 2.2. *If $\Phi : \mathcal{P} \times M \to N$ is a smooth map transversal to W then Φ_p is transversal to W for almost every $p \in \mathcal{P}$.*

Thus almost every member of the family Φ_p inherits the transversality property from the transversality of the whole family. The richness of the family is expressed by the transversality of the mapping Φ.

Here is another, more dynamical, example of our truism in ergodic theory [**PugSh1**].

Theorem 2.3. *If $\Phi : \mathbb{R}^n \times X \to X$ is an ergodic action of $\mathbb{R}^n$ on a probability space X then for almost every $r \in \mathbb{R}^n$, $\Phi_r : X \to X$ is ergodic.*

The ergodicity of the family is inherited by almost all elements. Further examples of our truism in ergodic theory are provided by the Mautner phenomenon [**Mo**],[**BreMo**]. Both Theorem 2.3 and the Mautner phenomenon are proven via representation theory. The richness in the family comes from the Lie group structure and the ergodicity of the group.

We would like to have a notion of *richness of a family of dynamical systems* and *Lyapunov exponent of the family* so as to be able *to conclude that most or at least many of the elements of the family have some non-zero exponents when the family does.* For the notion of Lyapunov exponent of the family we shall use the exponents of random products of elements of the family with respect to a probability measure on the space of systems.

We begin in the next section with linear maps where we use as a notion of richness the unitary invariance of the probability distribution on the space of matrices.

3. Unitarily invariant measures on $\mathbb{GL}_n(\mathbb{C})$

Let L_i be a sequence of linear maps mapping finite dimensional normed vector spaces V_i to V_{i+1} for $i \in \mathbb{N}$. Let $v \in V_0 \backslash \{0\}$. If the limit $\lim \frac{1}{k} \log \|L_{k-1} \dots L_0(v)\|$ exists it is called a Lyapunov exponent of the sequence. It is easy to see that if two vectors have the same exponent then so does every vector in the space spanned by them. It follows that there are at most $dim(V_0)$ exponents. We denote them λ_j where $j \leq k \leq dim(V_0)$. We order the λ_i so that $\lambda_i \geq \lambda_{i+1}$ Thus it makes sense to talk about the Lyapunov exponents of a diffeomorphism f of a compact manifold M at a point $m \in M$, $\lambda_j(f, m)$ by choosing the sequence L_i equal to $Tf(f^i(m))$.

Given a probability measure μ on $\mathbb{GL}_n(\mathbb{C})$ the space of invertible $n \times n$ complex matrices we may form infinite sequences of elements chosen at random from μ by taking the product measure on $\mathbb{GL}_n(\mathbb{C})^{\mathbb{N}}$. Thus we may also talk about the Lyapunov exponents of sequences or almost all sequences in $\mathbb{GL}_n(\mathbb{C})^{\mathbb{N}}$.

Oseledec's Theorem applies in our two contexts.

For diffeomorphisms f Oseldec's theorem says that for any f invariant measure ν, for ν almost all $m \in M$, f has $dim(M)$ Lyapunov exponents at m, $\lambda_j(f, m)$ for $1 \leq j \leq dim(M)$.

For measures μ on $\mathbb{GL}_n(\mathbb{C})$ satisfying a mild integrability condition, we have n Lyapunov exponents $r_1 \geq r_2 \geq \dots \geq r_n \geq -\infty$ such that for almost every sequence $\dots g_k \dots g_1 \in \mathbb{GL}_n(\mathbb{C})$ the limit $\lim \frac{1}{k} \log \|g_k \dots g_1 v\|$ exists for every $v \in \mathbb{C}^n \backslash \{0\}$ and equals one of the r_i, $i = 1 \dots n$, see Gol'dsheid and Margulis [**GoMa**] or Ruelle [**Ru**] or Oseledec [**Os**]. We may call the numbers $r_1, \dots, r_n$ random Lyapunov exponents or even just random exponents. If the measure is concentrated on a point A, these numbers $\lim \frac{1}{n} \log \|A^n v\|$ are $\log |\lambda_1|, \dots, \log |\lambda_n|$ where $\lambda_i(A) = \lambda_i$, $i = 1 \dots n$, are the eigenvalues of A written with multiplicity and $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$.

The integrability condition for Oseledec's Theorem is

$$g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ is } \mu - \text{integrable}$$

where for a real valued function f, $f^+ = \max[0, f]$. Here we will assume more so that all our integrals are defined and finite, namely:

$$(*) \qquad g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ and } \log^+(\|g^{-1}\|) \text{ are } \mu\text{−integrable.}$$

In [**DeSh**]we prove:

THEOREM 3.1. *If μ is a unitarily invariant measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k = 1, \dots, n$,*

$$\int_{A \in \mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{k} \log |\lambda_i(A)| d\mu(A) \geq \sum_{i=1}^{k} r_i.$$

By unitary invariance we mean $\mu(U(X)) = \mu(X)$ for all unitary transformations $U \in \mathbb{U}_n(\mathbb{C})$ and all μ-measurable $X \in \mathbb{GL}_n(\mathbb{C})$.

Thus non-zero Lyapunov exponents for the family i.e. the random exponents implies that at least some of the individual linear maps have non-zero exponents i.e eigenvalues. The notion of richness here is unitary invariance of the measure. For complex matrices we have achieved part of our goal. Later we will suggest a way in which these results may be extended to dynamical systems.

Corollary 3.2.
$$\int_{A\in\mathbb{GL}_n(\mathbb{C})}\sum_{i=1}^{n}\log^{+}|\lambda_i(A)|d\mu(A)\geq\sum_{i=1}^{n}{r_i}^{+}.$$

Theorem 3.1 is not true for general measures on $\mathbb{GL}_n(\mathbb{C})$ or $\mathbb{GL}_n(\mathbb{R})$ even for $n=2$. Consider
$$A_1=\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad A_2=\begin{pmatrix}1&1\\0&1\end{pmatrix},$$
and give probability 1/2 to each. The left hand integral is zero but as is easily seen the right hand sum is positive. So, in this case the inequality goes the other way. We do not know a characterization of measures which make Theorem 3.1 valid.

In order to prove 3.1 we first identify the the right hand summation in terms of an integral. Let $\mathbb{G}_{n,k}(\mathbb{C})$ denote the Grassmannian manifold of k dimensional vector subspaces in $\mathbb{C}^n$. If $A\in\mathbb{GL}_n(\mathbb{C})$ and $G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})$, $A|G_{n,k}$ the restriction of A to the subspace $G_{n,k}$. Let ν be the natural unitarily invariant probability measure on $\mathbb{G}_{n,k}(\mathbb{C})$. The next theorem is a fairly standard fact.

Theorem 3.3. *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then,*
$$\sum_{i=1}^{k}r_i=\int_{A\in\mathbb{GL}_n(\mathbb{C})}\int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})}\log|\det(A|G_{n,k})|d\nu(G_{n,k})d\mu(A).$$

We may then restate Theorem 3.1.

Theorem 3.4. *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k=1,\dots,n$*
$$\int_{A\in\mathbb{GL}_n(\mathbb{C})}\sum_{i=1}^{k}\log|\lambda_i(A)|d\mu(A)\geq\int_{A\in\mathbb{GL}_n(\mathbb{C})}\int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})}\log|\det(A|G_{n,k})|d\nu(G_{n,k})d\mu(A).$$

Theorems 3.4 reduces to a special case.

Let $A\in\mathbb{GL}_n(\mathbb{C})$ and μ be the Haar measure on $\mathbb{U}_n(\mathbb{C})$ (the unitary subgroup of $\mathbb{GL}_n(\mathbb{C})$) normalized to be a probability measure. In this case Theorem 3.4 becomes:

Theorem 3.5. *Let $A\in\mathbb{GL}_n(\mathbb{C})$. Then, for $1\leq k\leq n$,*
$$\int_{U\in\mathbb{U}_n(\mathbb{C})}\sum_{i=1}^{k}\log|\lambda_i(UA)|d\mu(U)\geq\int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})}\log|\det(A|G_{n,k})|d\nu(G_{n,k})$$

We expect similar results for orthogonally invariant probability measures on $\mathbb{GL}_n(\mathbb{R})$ but we have not proven it except in dimension 2.

Theorem 3.6. *Let μ be a probability measure on $\mathbb{GL}_2(\mathbb{R})$ satisfying*
$$g\in\mathbb{GL}_2(\mathbb{R})\to\log^{+}(\|g\|)\text{ and }\log^{+}(\|g^{-1}\|)\text{ are }\mu\text{−integrable.}$$
a. If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^{+}(\mathbb{R})$ then,
$$\int_{A\in\mathbb{GL}_2^{+}(\mathbb{R})}\log|\lambda_1(A)|d\mu(A)=\int_{A\in\mathbb{GL}_2^{+}(\mathbb{R})}\int_{x\in\mathbb{S}^1}\log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

b. If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^-(\mathbb{R})$, whose support is not contained in $\mathbb{RO}_2(\mathbb{R})$ i.e. in the set of scalar multiples of orthogonal matrices, then

$$\int_{A\in\mathbb{GL}_2^-(\mathbb{R})} \log|\lambda_1(A)|d\mu(A) > \int_{A\in\mathbb{GL}_2^-(\mathbb{R})}\int_{x\in\mathbb{S}^1} \log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

Here $\mathbb{GL}_2^+(\mathbb{R})$ (resp. $\mathbb{GL}_2^-(\mathbb{R})$) is the set of invertible matrices with positive (resp. negative) determinant.

4. Proofs and the Complexity of Bezout's Theorem

In our series of papers on complexity and Bezout's theorem, Steve and I concentrated on the manifold of solutions $V = \{(P,z) \in \mathbb{P}(H_{(D)}) \times \mathbb{P}(\mathbb{C}^n) | P(z) = 0\}$ and the two projections

$$\begin{array}{ccc} & V & \\ \overset{\Pi_1}{\swarrow} & & \overset{\Pi_2}{\searrow} \\ \mathbb{P}(H_{(D)}) & & \mathbb{P}(\mathbb{C}^n) \end{array}$$

in order to transfer integrals over $\mathbb{P}(H_{(D)})$ to integrals over $\mathbb{P}(\mathbb{C}^n)$. See [**BlCuShSm**]

Here $(D) = (d_1, \cdots, d_{n-1})$ and $H_{(D)}$ is the vector space of homogeneous polynomials systems $P = (P_1, \dots, P_{n-1})$ where each P_i is a homogeneous polynomial of degree d_i in n complex variables. For a vector space V, $\mathbb{P}(V)$ denotes the projective space of V.

Our proof of 3.5 relies heavily on this technique, but with respect to a manifold of fixed points.

A flag F in $\mathbb{C}^n$ is a sequence of vector subspaces of $\mathbb{C}^n$: $F = (F_1, F_2, \dots, F_n)$, with $F_i \subset F_{i+1}$ and $\dim F_i = i$. The space of flags is called the flag manifold and we denote it by $\mathbb{F}_n(\mathbb{C})$. An invertible linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ naturally induces a map $A_\sharp$ on flags by

$$A_\sharp(F_1, F_2, \dots, F_n) = (AF_1, AF_2, \dots, AF_n).$$

The flag manifold and the action of a linear map A on $\mathbb{F}_n(\mathbb{C})$ is closely related to the QR algorithm, see [**ShVa**] for a discussion of this. In particular if F is a fixed flag for A i.e. $A_\sharp F = F$, then A is upper triangular in a basis corresponding to the flag F, with the eigenvalues of A appearing on the diagonal in some order: $\lambda_1(A,F), \dots, \lambda_n(A,F)$.

Let

$$\mathbb{V}_A = \{(U,F) \in \mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \ : \ (UA)_\sharp F = F\}.$$

We denote by Π_1 and Π_2 the restrictions to $\mathbb{V}_A$ of the projections $\mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{U}_n(\mathbb{C})$ and $\mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{F}_n(\mathbb{C})$. $\mathbb{V}_A$ is a manifold of fixed points. We use the diagram

$$\begin{array}{ccc} & \mathbb{V}_A & \\ \overset{\Pi_1}{\swarrow} & & \overset{\Pi_2}{\searrow} \\ \mathbb{U}_n(\mathbb{C}) & & \mathbb{F}_n(\mathbb{C}) \end{array}$$

in order to transfer the right hand integral in 3.5 over $\mathbb{F}_n(\mathbb{C})$ to an integral over $\mathbb{U}_n(\mathbb{C})$.

5. A dynamical systems analogue

Is there a notion of richness for a family $\mathcal{P}$ of diffeomorphisms of a compact manifold M which would allow us to conclude that at least some members of the family have non-zero exponents?

We introduce now a notion of richness of $\mathcal{P}$ which might, in some situations, be sufficient to deduce properties of the exponents of elements of $\mathcal{P}$ from those of the random exponents. This notion was suggested to us by some preliminary numerical experiments and by the results in the setting of random matrix products in section 3.

We focus on the problem for $M = S^n$, the n-sphere. Let μ be Lebesgue measure on S^n normalized to be a probability measure, and let m be Liouville measure on $T_1(S^n)$, the unit tangent bundle of S^n, similarly normalized to be a probability measure. The orthogonal group $O(n+1)$ acts by isometries on the n-sphere and so induces an action on the space of μ-preserving diffeomorphisms by

$$f \mapsto O \circ f, \qquad \text{for } O \in O(n+1).$$

Let ν be a probability measure supported on $\mathcal{P} \subset \mathrm{Diff}^r_\mu(S^n)$. We say that ν is *orthogonally invariant* if ν is preserved by every element of $O(n+1)$ under the action described above.

For example, let

$$\mathbb{F}_n(\mathbb{C}) = O(n+1)f = \{O \circ f \mid O \in O(n+1)\},$$

for a fixed $f \in \mathrm{Diff}^r_\mu(S^n)$ Defining ν by transporting Haar measure on $O(n+1)$ to $\mathcal{P}$, we obtain an orthogonally-invariant measure. Because $O(n+1)$ acts transitively on $T_1(S^n)$, a random product of elements of $\mathcal{P}$ will pick up the behavior of f in almost all tangent directions — the family is reasonably rich in that sense.

Let ν be an orthogonally invariant measure on $\mathcal{P}$. The largest random Lyapunov exponent for $\mathcal{P}$, which we will denote by $R(\nu)$, can be expressed as an integral:

$$R(\nu) = \int R(\nu, x)\, d\mu = \int_{\mathrm{Diff}^r_\mu(S^n)} \int_{T_1(S^n)} \ln \|Df(x)v\|\, dm\, d\nu.$$

We define the *mean largest Lyapunov exponent* to be

$$\Lambda(\nu) = \int_{\mathrm{Diff}^r_\mu(S^n)} \int_{S^n} \lambda_1(f, x)\, d\mu\, d\nu$$

where $\lambda_1(f, x)$ is the largest Lyapunov exponent of f at x.

Question 5.1. Is there a positive constant $C(n)$ — perhaps 1 — depending on n alone such that $\Lambda(\nu) \geq C(n)R(\nu)$?

If the answer to Question 5.1 were affirmative, then a positive measure set of elements of $\mathcal{P}$ would have areas of positive exponents, (assuming a mild nondegeneracy condition on ν). We add here that this type of question has been asked before and has been the subject of a lot of research. What is new is the notion of richness which allows us to express the relation between exponents as an inequality of integrals.

The question is already interesting for S^2. Express S^2 as the sphere of radius $1/2$ centered at $(1/2, 0)$ in $\mathbf{R} \times \mathbf{C}$, so that the coordinates $(r, z) \in S^2$ satisfy the equation

$$|r - 1/2|^2 + |z|^2 = 1/4.$$

In these coordinates define a twist map $f_\epsilon : S^2 \to S^2$, for $\epsilon > 0$, by

$$f_\epsilon(r, z) = (r, \exp(2\pi i r \epsilon) z).$$

Let $\mathcal{P}$ be the orbit $O(3)f$ and let ν be the push forward of Haar measure on $O(3)$. A very small and inconclusive numerical experiment seemed to indicate that for ϵ close to 0 the inequality may hold with $C(n) = 1$. It seemed the constant may decrease as the twist increases speed.

Michel Herman thinks Question 5.1 has a negative answer, precisely for the twist map example f_ϵ, for ϵ very small due to references cited in section 6 of [**BuPuShWi**]. Perhaps more and better experiments would shed some light on the question. Whether or not Herman is correct, it would be interesting to know if other lower bound estimates are available with an appropriate concept of richness of the family.

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IBM T. J. Watson Research Center, P. O. Box 218, Yorktown Heights, NY 10598

E-mail address: mshub@us.ibm.com