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ON RANDOM AND MEAN EXPONENTS FOR UNITARILY INVARIANT PROBABILITY MEASURES ON $\mathbb{GL}_n(\mathbb{C})$.

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Dedicated to Jacob Palis for his sixtieth birthday.

1. Introduction

Given a probability measure μ on the space of invertible $n \times n$ complex matrices satisfying a mild integrability condition, we have, by Oseledec's Theorem, n random exponents $r_1 \geq r_2 \geq \ldots \geq r_n \geq -\infty$ such that for almost every sequence $\ldots g_k \ldots g_1 \in \mathbb{GL}_n(\mathbb{C})$ the limit $\lim \frac{1}{k} \log \|g_k \ldots g_1 v\|$ exists for every $v \in \mathbb{C}^n \setminus \{0\}$ and equals one of the r_i, $i = 1 \ldots n$, see Gol'dsheid and Margulis [4] or Ruelle [8] or Oseledec [7]. The numbers $r_1, \ldots, r_n$ are called Lyapunov exponents. In our context we may call them random Lyapunov exponents or even just random exponents. If the measure is concentrated on a point A, these numbers $\lim \frac{1}{n} \log \|A^n v\|$ are $\log|\lambda_1|, \ldots,$ $\log|\lambda_n|$ where $\lambda_i(A) = \lambda_i$, $i = 1 \ldots n$, are the eigenvalues of A written with multiplicity and $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$.

The integrability condition for Oseledec's Theorem is

$$g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ is } \mu - \text{integrable}$$

where for a real valued function f, $f^+ = \max[0, f]$. Here we will assume more so that all our integrals are defined and finite, namely:

$$(*) \qquad g \in \mathbb{GL}_n(\mathbb{C}) \to \log^+(\|g\|) \text{ and } \log^+(\|g^{-1}\|) \text{ are } \mu\text{-integrable.}$$

We will prove:

Theorem 1. *If μ is a unitarily invariant measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k = 1, \ldots, n$,*

$$\int_{A \in \mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{k} \log|\lambda_i(A)| d\mu(A) \geq \sum_{i=1}^{k} r_i.$$

By unitary invariance we mean $\mu(U(X)) = \mu(X)$ for all unitary transformations $U \in \mathbb{U}_n(\mathbb{C})$ and all μ-measurable $X \in \mathbb{GL}_n(\mathbb{C})$.

Corollary 2.

$$\int_{A \in \mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{n} \log^+|\lambda_i(A)| d\mu(A) \geq \sum_{i=1}^{n} r_i{}^+.$$

Theorem 1 is not true for general measures on $\mathbb{GL}_n(\mathbb{C})$ or $\mathbb{GL}_n(\mathbb{R})$ even for $n = 2$. Consider

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and give probability $1/2$ to each. Then the left hand integral is zero but as is easily seen the right hand sum is positive. So, in this case the inequality goes the other way. We do not know a characterization of measures which make Theorem 1 valid. We would find such a characterization interesting.

The numbers $\sum_{i=1}^{k} r_i$ have a direct geometric interpretation. Let $\mathbb{G}_{n,k}(\mathbb{C})$ denote the Grassmannian manifold of k dimensional vector subspaces in $\mathbb{C}^n$, $A|G_{n,k}$ the restriction of A to the subspace $G_{n,k}$ and ν the natural unitarily invariant probability measure on $\mathbb{G}_{n,k}(\mathbb{C})$.

Theorem 3. *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then,*

$$\sum_{i=1}^{k} r_i = \int_{A \in \mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k} \in \mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

We may then restate Theorem 1 in the form we prove it.

Theorem 4. *If μ is a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ then, for $k = 1, \ldots, n$*

$$\int_{A \in \mathbb{GL}_n(\mathbb{C})} \sum_{i=1}^{k} \log|\lambda_i(A)| d\mu(A) \geq \int_{A \in \mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k} \in \mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

There is a considerable literature on random Lyapunov exponents and quite general criteria which guarantee that they are non-zero and even distinct. According to Bougerol and Lacroix in 1985 in [2] "The subject matter initiated by Bellman was fully developed by Furstenberg, Guivarc'h, Kesten, Le Page and Raugi." We refer to [2] for references prior to 1985 and to three others: Gol'dsheid and Margulis [4], Guivarc'h and Raugi [5] and Ledrappier [6].

Our interest in Theorem 1 and Theorem 4 was motivated by some questions in dynamical systems theory, see Burns, Pugh, Shub and Wilkinson [3]. Theorem 1 for $k = 1$, the orthogonal group and $\mathbb{GL}_n(\mathbb{R})$ was raised there.

We also get a version of Theorem 4 without the logarithms.

Theorem 5. *Let μ be a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$ and $1 \leq k \leq n$. Then*

$$\int_{A \in \mathbb{GL}_n(\mathbb{C})} \prod_{i=1}^{k} |\lambda_i(A)| d\mu(A) \geq \int_{A \in \mathbb{GL}_n(\mathbb{C})} \int_{G_{n,k} \in \mathbb{G}_{n,k}(\mathbb{C})} |\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}) d\mu(A).$$

There is a special case of Theorems 4 and 5 of that is good to keep in mind. Our proof relies it.

Let $A \in \mathbb{GL}_n(\mathbb{C})$ and μ be the Haar measure on $\mathbb{U}_n(\mathbb{C})$ (the unitary subgroup of $\mathbb{GL}_n(\mathbb{C})$) normalized to be a probability measure. In this case Theorem 5 becomes:

Theorem 6. *Let $A \in \mathbb{GL}_n(\mathbb{C})$. Then, for $1 \leq k \leq n$,*

$$\int_{U \in \mathbb{U}_n(\mathbb{C})} \sum_{i=1}^{k} \log |\lambda_i(UA)| d\mu(U) \geq \int_{G_{n,k} \in \mathbb{G}_{n,k}(\mathbb{C})} \log |\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k})$$

and

$$\int_{U \in \mathbb{U}_n(\mathbb{C})} \prod_{i=1}^{k} |\lambda_i(UA)| d\mu(U) \geq \int_{G_{n,k} \in \mathbb{G}_{n,k}(\mathbb{C})} |\operatorname{Det}(A|G_{n,k})| d\nu(G_{n,k}).$$

When $k = 1$, $|\lambda_1(UA)| = \rho(UA)$ is the spectral radius of UA. The Grassmannian manifold is identical to the complex projective space $\mathbb{P}_{n-1}(\mathbb{C})$. Integration on this manifold can be reduced to the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{R}^{2n}$ so that

Corollary 7. *Let $A \in \mathbb{GL}_n(\mathbb{C})$. Then*

$$\int_{U \in \mathbb{U}_n(\mathbb{C})} \log |\rho(UA)| d\mu(U) \geq \int_{x \in \mathbb{S}^{2n-1}} \log \|Ax\| d\nu(x)$$

and

$$\int_{U\in\mathbb{U}_n(\mathbb{C})} |\rho(UA)|d\mu(U) \geq \int_{x\in\mathbb{S}^{2n-1}} \|Ax\|d\nu(x).$$

We expect a similar result for orthogonally invariant probability measures on $\mathbb{GL}_n(\mathbb{R})$ but we have not proven it. Here we content ourselves with the case $n = 2$.

Theorem 8. *Let μ be a probability measure on $\mathbb{GL}_2(\mathbb{R})$ satisfying*

$$g \in \mathbb{GL}_2(\mathbb{R}) \to \log^+(\|g\|) \text{ and } \log^+(\|g^{-1}\|) \text{ are } \mu-\text{integrable}.$$

a. If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^+(\mathbb{R})$ then,

$$\int_{A\in\mathbb{GL}_2^+(\mathbb{R})} \log|\lambda_1(A)|d\mu(A) = \int_{A\in\mathbb{GL}_2^+(\mathbb{R})} \int_{x\in\mathbb{S}^1} \log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

b. If μ is a $\mathbb{SO}_2(\mathbb{R})$ invariant measure on $\mathbb{GL}_2^-(\mathbb{R})$, whose support is not contained in $\mathbb{RO}_2(\mathbb{R})$ i.e. in the set of scalar multiples of orthogonal matrices, then

$$\int_{A\in\mathbb{GL}_2^-(\mathbb{R})} \log|\lambda_1(A)|d\mu(A) > \int_{A\in\mathbb{GL}_2^-(\mathbb{R})} \int_{x\in\mathbb{S}^1} \log\|Ax\|d\mathbb{S}^1(x)d\mu(A).$$

Here $\mathbb{GL}_2^+(\mathbb{R})$ (resp. $\mathbb{GL}_2^-(\mathbb{R})$) is the set of invertible matrices with positive (resp. negative) determinant. Theorem 8 is proved in section 5.

2. A More General Theorem.

Theorem 4 is actually a special case of the much more general Theorem 11 below. Before we state Theorem 11 we need some preliminaries.

A flag F in $\mathbb{C}^n$ is a sequence of vector subspaces of $\mathbb{C}^n$: $F = (F_1, F_2, \ldots, F_n)$, with $F_i \subset F_{i+1}$ and Dim $F_i = i$. The space of flags is called the flag manifold and we denote it by $\mathbb{F}_n(\mathbb{C})$. Now it is easy to see that $\mathbb{F}_n(\mathbb{C})$ may be represented by $\mathbb{GL}_n(\mathbb{C})/\mathbb{R}_n(\mathbb{C})$ or by $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$, where $\mathbb{R}_n(\mathbb{C})$ is the subgroup of $\mathbb{GL}_n(\mathbb{C})$ of upper triangular matrices and $\mathbb{T}^n(\mathbb{C})$ is the subgroup of $\mathbb{GL}_n(\mathbb{C})$ consisting of diagonal matrices with complex numbers of modulus 1, so $\mathbb{T}^n(\mathbb{C}) = \mathbb{U}_n(\mathbb{C}) \cap \mathbb{R}_n(\mathbb{C})$. Regarding $\mathbb{F}_n(\mathbb{C})$ as $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$ we see that $\mathbb{F}_n(\mathbb{C})$ has a natural $\mathbb{U}_n(\mathbb{C})$-invariant probability measure.

An invertible linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ naturally induces a map $A_\sharp$ on flags by

$$A_\sharp(F_1, F_2, \ldots, F_n) = (AF_1, AF_2, \ldots, AF_n).$$

The flag manifold and the action of a linear map A on $\mathbb{F}_n(\mathbb{C})$ is closely related to the QR algorithm, see Shub and Vasquez [9] for a discussion of this. In particular if F is a fixed flag for A i.e. $A_\sharp F = F$, then A is upper triangular in a basis corresponding to the flag F, with the eigenvalues of A appearing on the diagonal in some order: $\lambda_1(A,F), \ldots, \lambda_n(A,F)$.

Let

$$\mathbb{G} = \{A \in \mathbb{GL}_n(\mathbb{C}) \ : \ |\lambda_1(A)| > |\lambda_2(A)| > \ldots > |\lambda_n(A)|\}.$$

Then, there is a unique flag F such that $A_\sharp(F) = F$ and such that $\lambda_i(A,F) = \lambda_i(A)$ for $i = 1, \ldots, n$. We call this flag the QR flag of A and let $QR : \mathbb{G} \to \mathbb{F}_n(\mathbb{C})$ be the map which associates to $A \in \mathbb{G}$ its QR flag. It follows from Shub-Vasquez [9] and the discussion of fixed point manifolds below that QR is a smooth mapping.

Now fix $A \in \mathbb{GL}_n(\mathbb{C})$, define $\mathbb{U}_n(\mathbb{C})A = \{UA \ : \ U \in \mathbb{U}_n(\mathbb{C})\}$ and consider $\mathbb{G}_A = \mathbb{G} \cap (\mathbb{U}_n(\mathbb{C})A)$. Assume that $\mathbb{G}_A \neq \emptyset$. If we restrict QR to $\mathbb{G}_A$ then $QR : \mathbb{G}_A \to \mathbb{F}_n(\mathbb{C})$ is in fact a locally trivial fibration whose fibers are the orbits of a $\mathbb{T}^n(\mathbb{C})$ action we now describe.

Let $D \in \mathbb{T}^n(\mathbb{C})$ and $U \in \mathbb{U}_n(\mathbb{C})$ and let $QR(UA) = U_1\mathbb{R}_n(\mathbb{C})$ where $U_1 \in \mathbb{U}_n(\mathbb{C})$. Let

$$\Phi_A : \mathbb{T}^n(\mathbb{C}) \times \mathbb{G}_A \to \mathbb{G}_A$$

be defined by $\Phi_A(D,UA) = U_1DU_1^{-1}UA$. In section 4 we establish

Proposition 1. *1) $\Phi_A(D,UA)$ is well defined.*
2) $QR(\Phi_A(D,UA)) = QR(UA)$.
3) $\Phi_A : \mathbb{T}^n(\mathbb{C}) \times \mathbb{G}_A \to \mathbb{G}_A$ is an action of $\mathbb{T}^n(\mathbb{C})$ on $\mathbb{G}_A$ whose orbits are the fibers of $QR : \mathbb{G}_A \to \mathbb{F}_n(\mathbb{C})$.
4) If $D = Diag(d_1, \ldots, d_n)$ then $\lambda_i(\Phi_A(D,UA)) = d_i\lambda_i(UA)$ and in particular $|\lambda_i|$ is constant on the fibers of $QR : \mathbb{G}_A \to \mathbb{F}_n(\mathbb{C})$ for $i = 1, \ldots, n$.

Let

$$\mathbb{V}_A = \{(U,F) \in \mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \ : \ (UA)_\sharp F = F\}.$$

We denote by Π_1 and Π_2 the restrictions to $\mathbb{V}_A$ of the projections $\mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{U}_n(\mathbb{C})$ and $\mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{F}_n(\mathbb{C})$. We define an action of $\mathbb{T}^n(\mathbb{C})$ on $\mathbb{V}_A$ denoted $\Psi_A : \mathbb{T}^n(\mathbb{C}) \times \mathbb{V}_A \to \mathbb{V}_A$ by

$$\Psi_A(D)(U, U_1\mathbb{T}^n(\mathbb{C})) = (U_1DU_1^{-1}U, U_1\mathbb{T}^n(\mathbb{C})).$$

Proposition 2. *1) Ψ_A is well defined and smooth.*
2) The orbits of Ψ_A are the fibers of $\Pi_2 : \mathbb{V}_A \to \mathbb{F}_n(\mathbb{C})$.

We consider the manifold

$$\mathbb{V} = \{(A,F) \in \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \ : \ A_\sharp F = F\}$$

and the restrictions to $\mathbb{V}$ of the two projections $\mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{GL}_n(\mathbb{C})$ and $\mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{F}_n(\mathbb{C})$ which we again denote by Π_1 and Π_2. By the Jordan Canonical Form Theorem the map Π_1 is surjective. Except on a set of positive codimension, ${\Pi_1}^{-1}(A)$ consists of $n!$ points corresponding to the permutations of the eigenspaces of $A \in \mathbb{GL}_n(\mathbb{C})$. The fibers of the map Π_2 are more complicated.

For $c \in \mathbb{C} \setminus \{0\}$ we write $c\mathbb{U}_n(\mathbb{C})$ to mean $\{cU \ : \ U \in \mathbb{U}_n(\mathbb{C})\}$.

Definition 9. Let $f : \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ be continuous.
1) f is $\mathbb{U}_n(\mathbb{C})$ or unitarily invariant if $f(UA, F) = f(A, F)$ for all $(A, F) \in \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C})$ and $U \in \mathbb{U}_n(\mathbb{C})$, and if $f|c\mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C})$ is constant for every $c \in \mathbb{C} \setminus \{0\}$.
2) For $B \in \mathbb{GL}_n(\mathbb{C})$ let $g(B) = \max_{(B,F)\in\mathbb{V}} f(B, F)$. We say that f is $\mathbb{T}^n(\mathbb{C})$ or torally invariant if $g(\Phi_A(D, B)) = g(B)$ for all $A \in \mathbb{G}$, $B \in \mathbb{G}_A$ and $D \in \mathbb{T}^n(\mathbb{C})$.

Examples of $\mathbb{U}_n(\mathbb{C})$ and $\mathbb{T}^n(\mathbb{C})$ invariant functions are
1) For $1 \le k \le n$ let $f_k(A, F) = |\operatorname{Det}(A|F_k)|$ where $F = (F_1, F_2, \dots, F_n) \in \mathbb{F}_n(\mathbb{C})$.
2) $\log f_k(A, F)$ where $f_k(A, F)$ is as in 1).

Remark 10. If $A_\sharp F = F$ then $|\operatorname{Det}(A|F_k)| = \prod_{i=1}^k |\lambda_i(A, F)|$.

Given a continuous $f : \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$, let $g : \mathbb{GL}_n(\mathbb{C}) \to \mathbb{R}$ be defined by $g(B) = \max_{(B,F)\in\mathbb{V}} f(B, F)$.

Theorem 11. *Let $f : \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ be continuous, unitarily and torally invariant. Let μ be a unitarily invariant probability measure on $\mathbb{GL}_n(\mathbb{C})$ satisfying $(*)$. Then*

$$\int_{A\in\mathbb{GL}_n(\mathbb{C})} g(A)d\mu(A) \ge \int_{A\in\mathbb{GL}_n(\mathbb{C})} \int_{F\in\mathbb{F}_n(\mathbb{C})} f(A, F)d\nu(F)d\mu(A).$$

It is now fairly simple to see how Theorem 11 implies Theorem 4. If $f_k(A, F) = \log|\operatorname{Det}(A|F_k)|$ then, by Remark 10, $g(A) = \sum_{i=1}^k \log|\lambda_i(A)|$ where $|\lambda_1(A)| \ge |\lambda_2(A)| \ge \dots \ge |\lambda_n(A)|$ are the absolute values of the eigenvalues of A. So the left hand integrals in Theorem 4 and 11 are the same. To see that the right hand integrals are the same consider the natural fibration $\Pi_k : \mathbb{F}_n(\mathbb{C}) \to \mathbb{G}_{n,k}(\mathbb{C})$ given by $\Pi_k(F_1, \dots, F_n) = F_k$. Then $|\operatorname{Det}(A|\Pi_k F)| = |\operatorname{Det}(A|F_k)|$ and it is easy to see that

$$\int_{F\in\mathbb{F}_n(\mathbb{C})} \log|\operatorname{Det}(A|F_k)|d\nu(F) = \int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})} \log|\operatorname{Det}(A|G_{n,k})|d\nu(G_{n,k}).$$

We will say more about this in section 4. So we are done.

We now turn to the proof of Theorem 11 which follows from the consideration of a special case.

Let $A \in \mathbb{GL}_n(\mathbb{C})$. We put Haar measure μ on $\mathbb{U}_n(\mathbb{C})$ normalized to be a probability measure. Thus the next proposition is a special case of Theorem 11.

Proposition 3. *Let $f : \mathbb{GL}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ be continuous, unitarily and torally invariant. Let*

$$\mathbb{V}_A = \{(U, F) \in \mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \ : \ (UA)_\sharp F = F\}$$

and $g(B) = \max_{(B,F) \in \mathbb{V}_A} f(B, F)$. Then

$$\int_{U \in \mathbb{U}_n(\mathbb{C})} g(UA) d\mu(U) \geq \int_{U \in \mathbb{U}_n(\mathbb{C})} \int_{F \in \mathbb{F}_n(\mathbb{C})} f(UA, F) d\nu(F) d\mu(U).$$

We now see that Proposition 3 implies Theorem 11. Disintegrate the measure μ of Theorem 11 along the orbits of $\mathbb{U}_n(\mathbb{C})$ obtaining $\mathbb{U}_n(\mathbb{C})$ invariant probability measures on each orbit. Identifying an orbit with $\mathbb{U}_n(\mathbb{C})$ we see that these measures are left invariant on $\mathbb{U}_n(\mathbb{C})$ hence they are Haar measures. Now Proposition 3 applies orbit by orbit. Integrating the inequality over the space of orbits proves Theorem 11.

Note that it is sufficient to prove Proposition 3 when A is not a constant times a unitary matrix, for otherwise $g(UA)$ and $f(UA, F)$ are both equal to the constant in the definition of unitary invariance. Thus the integrals are equal since they are equal to this constant. We will assume below that A is not a constant times a unitary matrix i.e. A is not conformal.

Note that in Proposition 3 the right hand integral does not depend on U since f is unitarily invariant. Thus it is not necessary to integrate over $\mathbb{U}_n(\mathbb{C})$, the first integral is constant.

Now we restate Proposition 3 in its simpler form.

Proposition 4. *Let $f : \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ be continuous and torally invariant, suppose A is not unitary or a scalar times a unitary. Let*

$$\mathbb{V}_A = \{(U, F) \in \mathbb{U}_n(\mathbb{C}) \times \mathbb{F}_n(\mathbb{C}) \ : \ (UA)_\sharp F = F\}.$$

Let $g(B) = \max_{(B,F) \in \mathbb{V}_A} f(F)$. Then

$$\int_{U \in \mathbb{U}_n(\mathbb{C})} g(U) d\mu(U) \geq \int_{F \in \mathbb{F}_n(\mathbb{C})} f(F) d\nu(F).$$

Now we outline the proof of Proposition 4. We use the diagram

$$\begin{array}{ccccc} & & \mathbb{V}_A & & \\ & \Pi_1 \swarrow & & \searrow \Pi_2 & \\ \mathbb{U}_n(\mathbb{C}) & & & & \mathbb{F}_n(\mathbb{C}) \end{array}$$

to transfer the right hand integral over $\mathbb{F}_n(\mathbb{C})$ to an integral over $\mathbb{U}_n(\mathbb{C})$. First we identify a subset of $\mathbb{U}_n(\mathbb{C})$ over which we will integrate.

Let $\mathbb{G}_1$ be the open subset of $\mathbb{U}_n(\mathbb{C})$ consisting of those U such that the eigenvalues of UA are of distinct modulus. In this case we write them as

$$\lambda_i = \lambda_i(UA), \quad 1 \le i \le n,$$

where $|\lambda_1| > \ldots > |\lambda_n|$.

Proposition 5. *$\mathbb{G}_1$ is an open set of full measure in $\mathbb{U}_n(\mathbb{C})$, i.e. $\mu(\mathbb{G}_1) = 1$.*

Lemma 1. *Let $f : \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ be continuous and torally invariant. Let $g(B) = \max_{(B,F)\in\mathbb{V}_A} f(F)$. Then*

$$\int_{F\in\mathbb{F}_n(\mathbb{C})} f(F)d\nu(F) = \int_{U\in\mathbb{G}_1} \sum_{(U,F)\in\mathbb{V}_A} f(F) \prod_{j<i} \left|1 - \frac{\lambda_i(UA,F)}{\lambda_j(UA,F)}\right|^{-2} d\mu(U) \le$$

$$\int_{U\in\mathbb{G}_1} g(U) \sum_{\sigma\in\Sigma_n} \prod_{j<i} \left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}\right|^{-2} d\mu(U)$$

with Σ_n the group of permutations over the set $\{1, 2, \ldots, n\}$.

Proposition 5 and Lemma 1 are proved in section 4. Proposition 3 and 4 follow from Proposition 5, Lemma 1 and from the next proposition.

Proposition 6.

$$\int_{U\in\mathbb{G}_1} g(U) \sum_{\sigma\in\Sigma_n} \prod_{j<i} \left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}\right|^{-2} d\mu(U) = \int_{U\in\mathbb{G}_1} g(U)d\mu(U).$$

We will prove Proposition 6 in Section 4 by decomposing the two integrals along the fibers of the QR fibration on which $g(U)$ is constant.

Proposition 7. *The normal Jacobian of the QR fibration is $\prod_{j<i} \left|1 - \frac{\lambda_i}{\lambda_j}\right|^{-2}$ where $\lambda_i = \lambda_i(UA)$ are the eigenvalues of UA whith $|\lambda_1| > \ldots > |\lambda_n|$. Hence*

$$\int_{U\in\mathbb{G}_1} g(U) \sum_{\sigma\in\Sigma_n} \prod_{j<i} \left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}\right|^{-2} d\mu(U) =$$

$$\int_{F\in\mathbb{F}_n(\mathbb{C})} g(U) \int_{U\in QR^{-1}(F)} \sum_{\sigma\in\Sigma_n} \prod_{j<i} \left|\frac{1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}}{1 - \frac{\lambda_i}{\lambda_j}}\right|^{-2} d\mu(QR^{-1}(F))(U)d\nu(F)$$

and

$$\int_{U\in\mathbb{G}_1} g(U)d\mu(U) = \int_{F\in\mathbb{F}_n(\mathbb{C})} g(U) \int_{U\in QR^{-1}(F)} \prod_{j<i} \left|1 - \frac{\lambda_i}{\lambda_j}\right|^{2} d\mu(QR^{-1}(F))(U)d\nu(F).$$

Proposition 7 is proved in Section 4. Finally in Section 4 we prove

Proposition 8.

$$\int_{U\in QR^{-1}(F)} \prod_{j<i} \left|1-\frac{\lambda_i}{\lambda_j}\right|^2 d\mu(QR^{-1}(F))(U) = \int_{U\in QR^{-1}(F)} \sum_{\sigma\in\Sigma_n} \prod_{j<i} \left|\frac{1-\frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}}{1-\frac{\lambda_i}{\lambda_j}}\right|^{-2} d\mu(QR^{-1}(F))(U).$$

Now Proposition 7 and Proposition 8 prove Proposition 6 and we are done. To summarize it remains to prove Theorem 3, Proposition 1, Proposition 2, Proposition 5, Lemma 1, Proposition 7 and Proposition 8.

3. Manifolds of fixed points

The manifolds $\mathbb{V}$ and $\mathbb{V}_A$ are manifolds of fixed points. In this section we discuss integration formulas for manifolds of fixed points and prove Lemma 1 and Proposition 7. We begin by recalling the co-area formula.

3.1. **The Co-area Formula.** Let $\mathbb{X}$ and $\mathbb{Y}$ be real Riemannian manifolds. We denote by $d\mathbb{X}$ and $d\mathbb{Y}$ the associated volume forms. Suppose $F:\mathbb{X}\to\mathbb{Y}$ is a smooth surjective map and suppose that the derivative $DF(x): T_x\mathbb{X}\to T_{f(x)}\mathbb{Y}$ is surjective for almost all $x\in\mathbb{X}$. The horizontal space H_x of $T_x\mathbb{X}$ is defined as the orthogonal complement to Ker $DF(x)$. The horizontal derivative of F at x is the restriction of $DF(x)$ to H_x. The normal Jacobian $NJ(F(x))$ is the absolute value of the determinant of the horizontal derivative defined almost everywhere on X:

$$NJ(F(x)) = |\,\mathrm{Det}\ (DF(x)|_{H_x})\,|.$$

The map F defines a fibration of $\mathbb{X}$ with base $\mathbb{Y}$ and fibers $F^{-1}(y)$, $y\in\mathbb{Y}$. Integration over X with respect to this fibration generalizes Fubini's formula:

Theorem 12. *(Co-area Formula) Let $F:\mathbb{X}\to\mathbb{Y}$ be a smooth map of real Riemannian manifolds satisfying the preceeding surjectivity conditions. Then, for any integrable $f:\mathbb{X}\to\mathbb{R}$*

$$\int_{x\in\mathbb{X}} f(x)d\mathbb{X}(x) = \int_{y\in\mathbb{Y}}\int_{x\in F^{-1}(y)} \frac{f(x)}{NJ(F(x))} dF^{-1}(y)(x)d\mathbb{Y}(y).$$

Remark 13. In the co-area formula, $d\mathbb{X}$ and $d\mathbb{Y}$ are the volume forms associated with the Riemannian structures over $\mathbb{X}$ and $\mathbb{Y}$, $dF^{-1}(y)$ is the volume form on $F^{-1}(y)$ equipped with the induced metric.

Remark 14. The co-area formula also extends to complex Riemannian manifolds. In that case the normal jacobian is equal to

$$NJ(F(x)) = |\,\mathrm{Det}\ (DF(x)|_{H_x})\,|^2.$$

This follows immediately from the fact that if $A : \mathbb{C}^n \to \mathbb{C}^n$ is a complex linear map and $A_{\mathbb{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the real map it defines, then

$$|\,\mathrm{Det}\ A_{\mathbb{R}}| = |\,\mathrm{Det}\ A|^2.$$

Remark 15. When $DF(x) : T_x\mathbb{X} \to T_{f(x)}\mathbb{Y}$ is onto, the normal Jacobian is equal to

$$NJ(F(x)) = (\mathrm{Det}\ DF(x)DF(x)^*)^{1/2}$$

so that

$$\int_{x\in\mathbb{X}} f(x)d\mathbb{X}(x) = \int_{y\in\mathbb{Y}}\int_{x\in F^{-1}(y)} \frac{f(x)}{(\mathrm{Det}\ DF(x)DF(x)^*)^{1/2}} dF^{-1}(y)(x)d\mathbb{Y}(y)$$

and in the complex case (see Remark 14)

$$\int_{x\in\mathbb{X}} f(x)d\mathbb{X}(x) = \int_{y\in\mathbb{Y}}\int_{x\in F^{-1}(y)} \frac{f(x)}{\mathrm{Det}\ DF(x)DF(x)^*} dF^{-1}(y)(x)d\mathbb{Y}(y).$$

Remark 16. The co-area formula also extends to the case of maps $F : \mathbb{X} \to \mathbb{Y}$ between algebraic varieties by considering the restriction of F to the smooth part of X.

3.2. **Manifolds of Fixed Points.** Let $\mathcal{F}$ and $\mathcal{M}$ be compact Riemannian manifolds and a smooth map $\Phi : \mathcal{F} \times \mathcal{M} \to \mathcal{M}$ be given. Let

$$\Psi : \mathcal{F} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$$

be defined by $\Psi(f, m) = (\Phi(f, m), m)$. Suppose Ψ is transversal to

$$\Delta = \{(m, m)\ :\ m \in \mathcal{M}\} \subset \mathcal{M} \times \mathcal{M}.$$

Then

$$\mathcal{V} = \Psi^{-1}(\Delta) = \{(f, m) \in \mathcal{F} \times \mathcal{M}\ :\ \Phi(f, m) = m\}$$

is a submanifold in $\mathcal{F} \times \mathcal{M}$. We denote by $\Pi_{\mathcal{F}}$ and $\Pi_{\mathcal{M}}$ the restrictions to $\mathcal{V}$ of the projections $\mathcal{F} \times \mathcal{M} \to \mathcal{F}$ and $\mathcal{F} \times \mathcal{M} \to \mathcal{M}$. By Sard's Theorem, almost all $f \in \mathcal{F}$ are regular values of $\Pi_{\mathcal{F}} : \mathcal{V} \to \mathcal{F}$. For these $f \in \mathcal{F}$ the corresponding fixed points $m \in \mathcal{M}$, i.e. $(f, m) \in \mathcal{V}$, are isolated in $\mathcal{M}$. Since $\mathcal{M}$ is compact these fixed points are finite.

Theorem 17. *Let $\mathcal{F}_{\pitchfork}$ denote the set of $f \in \mathcal{F}$ which are regular values of $\Pi_{\mathcal{F}}$. Let $G : \mathcal{M} \to \mathbb{R}$ be a continuous function. Then*

$$\int_{m\in\mathcal{M}} G(m)d\mathcal{M}(m) = \int_{f\in\mathcal{F}_{\pitchfork}} \sum_{m\in\Pi_{\mathcal{F}}^{-1}(f)} G(m)\frac{NJ(\Pi_{\mathcal{M}}(f, m))}{\mathrm{Vol}\,\Pi_{\mathcal{M}}^{-1}(m)NJ(\Pi_{\mathcal{F}}(f, m))} d\mathcal{F}(f).$$

Remark 18. The integral is taken over the set $\mathcal{F}_{\pitchfork}$ of regular values of $\Pi_{\mathcal{F}}$. We note that $f \in \mathcal{F}_{\pitchfork}$ if and only if for all $m \in \mathcal{M}$, such that $(f, m) \in \mathcal{V}$, $id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f, m)$ is invertible.

Proof. We apply the co-area formula to the function $G(m)NJ(\Pi_{\mathcal{M}}(f,m))/\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)$ defined over $\mathcal{V}$ with respect to the projection $\Pi_{\mathcal{M}}$. This gives

$$\int_{(f,m)\in\mathcal{V}} \frac{G(m)NJ(\Pi_{\mathcal{M}}(f,m))}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)} d\mathcal{V}(f,m) =$$

$$\int_{m\in\mathcal{M}}\int_{(f,m)\in\Pi_{\mathcal{M}}^{-1}(m)} \frac{G(m)NJ(\Pi_{\mathcal{M}}(f,m))}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)NJ(\Pi_{\mathcal{M}}(f,m))} d\Pi_{\mathcal{M}}^{-1}(m)(f,m)d\mathcal{M}(m) = \int_{m\in\mathcal{M}} G(m)d\mathcal{M}(m).$$

We now apply the same formula to the same function with respect to the the projection $\Pi_{\mathcal{F}}$. We notice that the fiber $\Pi_{\mathcal{F}}^{-1}(f)$ consists in a finite number of fixed points so that:

$$\int_{(f,m)\in\mathcal{V}} \frac{G(m)NJ(\Pi_{\mathcal{M}}(f,m))}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)} d\mathcal{V}(f,m) = \int_{f\in\mathcal{F}_{\pitchfork}} \sum_{m\in\Pi_{\mathcal{F}}^{-1}(f)} G(m)\frac{NJ(\Pi_{\mathcal{M}}(f,m))}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)NJ(\Pi_{\mathcal{F}}(f,m))} d\mathcal{F}(f)$$

and we are done.

Now we compute the normal Jacobians in terms of the partial derivatives of $\Phi : \mathcal{F}\times\mathcal{M} \to \mathcal{M}$. The Riemannian structure we put on $\mathcal{V}$ is the restriction of the product structure on $\mathcal{F}\times\mathcal{M}$.

Lemma 2. *Let $f \in \mathcal{F}_{\pitchfork}$ and $(f,m) \in \mathcal{V}$. Then the tangent space of $\mathcal{V}$ at (f,m) is*

$$T_{(f,m)}\mathcal{V} = \{(\dot f, \dot m) \in T_f\mathcal{F}\times T_m\mathcal{M} \ : \ \dot m = (id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f,m))^{-1}D_{\mathcal{F}}\Phi(f,m)\dot f\}.$$

Proof. This is a consequence of Remark 18.

If we put together Lemma 2, Theorem 17, and Blum-Cucker-Shub-Smale [1] Lemma 3, page 242, we have:

Theorem 19. *Let $G : \mathcal{M} \to \mathbb{R}$ be a continuous function. Then, for real manifolds*

$$\int_{m\in\mathcal{M}} G(m)d\mathcal{M}(m) =$$

$$\int_{f\in\mathcal{F}_{\pitchfork}} \sum_{m\in\Pi_{\mathcal{F}}^{-1}(f)} G(m)\frac{|\operatorname{Det}(D_{\mathcal{F}}\Phi(f,m)D_{\mathcal{F}}\Phi(f,m)^*)|^{1/2}}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)|\operatorname{Det}(id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f,m))|} d\mathcal{F}(f).$$

For complex manifolds this formula becomes

$$\int_{m\in\mathcal{M}} G(m)d\mathcal{M}(m) =$$

$$\int_{f\in\mathcal{F}_{\pitchfork}} \sum_{m\in\Pi_{\mathcal{F}}^{-1}(f)} G(m)\frac{|\operatorname{Det}(D_{\mathcal{F}}\Phi(f,m)D_{\mathcal{F}}\Phi(f,m)^*)|}{\operatorname{Vol}\Pi_{\mathcal{M}}^{-1}(m)|\operatorname{Det}(id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f,m))|^2} d\mathcal{F}(f).$$

Similarly we may also evaluate integrals defined on $\mathcal{F}$ using the fibration over $\mathcal{M}$. Suppose that $S : \mathcal{F}_{\pitchfork} \to \mathcal{V}$ is a smooth section of $\mathcal{V}$ defined on $\mathcal{F}_{\pitchfork}$ or on an open set of $\mathcal{F}_{\pitchfork}$ i.e. $\Pi_{\mathcal{F}}S = id_{\mathcal{F}_{\pitchfork}}$.

Theorem 20. *Let* $H : \mathcal{F}_{\pitchfork} \to \mathbb{R}$ *be an integrable function defined on* $\mathcal{F}_{\pitchfork}$ *or on an open set in* $\mathcal{F}_{\pitchfork}$*. Then, for real manifolds*

$$\int_{f\in\mathcal{F}_{\pitchfork}} H(f)d\mathcal{F}(f) =$$

$$\int_{m\in\mathcal{M}}\int_{(\Pi_{\mathcal{M}}S)^{-1}(m)} H(f)\frac{|\operatorname{Det}(id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f,m))|}{|\operatorname{Det}(D_{\mathcal{F}}\Phi(f,m)D_{\mathcal{F}}\Phi(f,m)^*)|^{1/2}}d\mathcal{F}(f)$$

and for complex manifolds

$$\int_{f\in\mathcal{F}_{\pitchfork}} H(f)d\mathcal{F}(f) =$$

$$\int_{m\in\mathcal{M}}\int_{(\Pi_{\mathcal{M}}S)^{-1}(m)} H(f)\frac{|\operatorname{Det}(id_{T_m\mathcal{M}} - D_{\mathcal{M}}\Phi(f,m))|^2}{|\operatorname{Det}(D_{\mathcal{F}}\Phi(f,m)D_{\mathcal{F}}\Phi(f,m)^*)|}d\mathcal{F}(f).$$

4. Proofs of Theorem 3, Propositions 1, 2, 5, Lemma 1 and of Propositions 7 and 8.

4.1. **Proof of Theorem 3.** If not explicitely stated this Theorem is inherent in the works of Furstenberg, Guivarc'h, Raugi, Gol'dsheid, Margulis and possibly other sources. See also Bougerol-Lacroix. We sketch a proof.

We consider two auxilliary spaces and maps:
1) $\prod_{i=1}^{\infty}\mathbb{GL}_n(\mathbb{C})$ equipped with the product measure $\hat{\mu}$, and $\sigma : \prod_{i=1}^{\infty}\mathbb{GL}_n(\mathbb{C}) \hookleftarrow$ the one sided shift:

$$\sigma(\dots g_p \dots g_1) = (\dots g_p \dots g_2).$$

2) $\prod_{i=1}^{\infty}\mathbb{GL}_n(\mathbb{C})\times\mathbb{G}_{n,k}(\mathbb{C})$ with the measure $\hat{\mu}\times\nu$ and the map $\tau : \prod_{i=1}^{\infty}\mathbb{GL}_n(\mathbb{C})\times\mathbb{G}_{n,k}(\mathbb{C}) \hookleftarrow$ defined by

$$\tau((\dots g_p \dots g_1), G_{n,k}) = (\sigma(\dots g_p \dots g_2), g_1(G_{n,k})).$$

$\hat{\mu}$ is invariant and ergodic for σ and $\hat{\mu}\times\nu$ is invariant for τ (here we use the unitary invariance of μ). It follows from Birkoff's Ergodic Theorem and the invariance of the measure $\hat{\mu}\times\nu$ for the map τ that $\lim \frac{1}{p}\log|\operatorname{Det}(g_p\dots g_1|G_{n,k})|$ exists a.e. in $\prod_{i=1}^{\infty}\mathbb{GL}_n(\mathbb{C})\times\mathbb{G}_{n,k}(\mathbb{C})$, and the integral of $\lim\frac{1}{p}\log|\operatorname{Det}(g_p\dots g_1|G_{n,k})|$ equals $\int_{A\in\mathbb{GL}_n(\mathbb{C})}\int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})}\log|\operatorname{Det}(A|G_{n,k})|d\nu(G_{n,k})d\mu(A)$. Now by Oseledec's theorem for almost all $\hat{g} = (\dots g_p\dots g_1)$ the limit $\lim\frac{1}{p}\log|\operatorname{Det}(g_p\dots g_1|G_{n,k})|$ exists for almost all $G_{n,k}$ and equals $\displaystyle\sum_{i=1}^{k} r_i$. So

$$\sum_{i=1}^{k} r_i = \int_{A\in\mathbb{GL}_n(\mathbb{C})}\int_{G_{n,k}\in\mathbb{G}_{n,k}(\mathbb{C})}\log|\operatorname{Det}(A|G_{n,k})|d\nu(G_{n,k})d\mu(A).$$

4.2. **Proofs of Propositions 1 and 2.** We now turn, in section 3, to the case that $\mathcal{F} = \mathbb{U}_n(\mathbb{C})$, $\mathcal{M} = \mathbb{F}_n(\mathbb{C})$, $\mathcal{V} = \mathbb{V}_A$ and $\Phi(U,F) = (UA)_\sharp(F)$.

Lemma 3. *Suppose* $(UA)_\sharp(U_1\mathbb{T}^n(\mathbb{C})) = U_1\mathbb{T}^n(\mathbb{C})$. *Then, for any* $V \in \mathbb{U}_n(\mathbb{C})$ *one has*

$$(VA)_\sharp(U_1\mathbb{T}^n(\mathbb{C})) = U_1\mathbb{T}^n(\mathbb{C})$$

if and only if there exists $D \in \mathbb{T}^n(\mathbb{C})$ *such that* $U_1DU_1^{-1}U = V$.

Proof. If $(VA)_\sharp(U_1\mathbb{T}^n(\mathbb{C})) = U_1\mathbb{T}^n(\mathbb{C})$ then $U_1\mathbb{R}_n(\mathbb{C}) = VAU_1\mathbb{R}_n(\mathbb{C}) = VU^{-1}UAU_1\mathbb{R}_n(\mathbb{C}) = VU^{-1}U_1\mathbb{R}_n(\mathbb{C})$. So $U_1^{-1}UV^{-1}U_1\mathbb{R}_n(\mathbb{C}) = \mathbb{R}_n(\mathbb{C})$ and $U_1^{-1}UV^{-1}U_1$ is in $\mathbb{R}_n(\mathbb{C}) \cap \mathbb{U}_n(\mathbb{C}) = \mathbb{T}^n(\mathbb{C})$. So there is a $D \in \mathbb{T}^n(\mathbb{C})$ with $UV^{-1} = U_1DU_1^{-1}$ and $V = U_1D^{-1}U_1^{-1}U$.

On the other hand for, $D \in \mathbb{T}^n(\mathbb{C})$, $U_1DU_1^{-1}UAU_1\mathbb{R}_n(\mathbb{C}) = U_1DU_1^{-1}U_1\mathbb{R}_n(\mathbb{C}) = U_1D\mathbb{R}_n(\mathbb{C}) = U_1\mathbb{R}_n(\mathbb{C})$. So we are done.

Proof of Proposition 1. 1) If $QR(UA) = U_1\mathbb{R}_n(\mathbb{C}) = U_1'\mathbb{R}_n(\mathbb{C})$ then $U_1' = U_1D'$ for some $D' \in \mathbb{T}^n(\mathbb{C})$. Thus $U_1'DU_1'^{-1}UA = U_1D'DD'^{-1}U_1^{-1}UA = U_1DU_1^{-1}UA$. From $QR(UA) = U_1\mathbb{R}_n(\mathbb{C})$ we get $(UA)_\sharp U_1\mathbb{R}_n(\mathbb{C}) = U_1\mathbb{R}_n(\mathbb{C})$ so that $UA = U_1RU_1^{-1}$ for some $R \in \mathbb{R}_n(\mathbb{C})$. This gives $\Phi_A(D,UA) = U_1DU_1^{-1}UA = U_1DU_1^{-1}U_1RU_1^{-1} = U_1DRU_1^{-1}$. Thus the eigenvalues of $\Phi_A(D,UA)$ have distinct modulus and Φ_A is well defined.
2) Using $UA = U_1RU_1^{-1}$ we get $\Phi_A(D,UA)U_1 = U_1DU_1^{-1}UAU_1 = U_1DR$ so that

$$QR(\Phi_A(D,UA)) = QR(UA) = U_1\mathbb{R}_n(\mathbb{C}).$$

3) This assertion is exactly Lemma 3.
4) $\lambda_i(\Phi_A(D,UA)) = d_i\lambda_i(UA)$ is proved in 1). and $|\lambda_i|$ constant on the fibers of QR described in 3) and we are done.

Proof of Proposition 2. Similar to the proof of Proposition 1. it also uses Lemma 3.

4.3. **Proof of Lemma 1.** Lemma 3 has an immediate consequence:

Lemma 4. *a) The volume of the fibers* $\Pi_2^{-1}(F)$, *for* $F \in \mathbb{F}_n(\mathbb{C})$, *with* $\Pi_2 : \mathbb{V}_A \to \mathbb{F}_n(\mathbb{C})$, *is constant and equal to* $\operatorname{Vol}\mathbb{T}^n(\mathbb{C})$.
b) The volume of the fibers $QR^{-1}(F)$, *for* $F \in \mathbb{F}_n(\mathbb{C})$, *is constant and equals* $\operatorname{Vol}\mathbb{T}^n(\mathbb{C})$.

Next we turn our attention to the term $|\operatorname{Det} D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)^*|$. If we fix a flag F then $D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F) = D_{\mathbb{U}_n(\mathbb{C})}\Phi_F(U)$ where $\Phi_F(U) = UU_1\mathbb{T}^n(\mathbb{C})$ and U_1 defined by $A_\sharp F = U_1\mathbb{T}^n(\mathbb{C})$. Next we prove that the normal Jacobian of $\Phi_F(U)$ is constant.

Proposition 9. *Let* $\mathbb{U}_n(\mathbb{C})$ *act on* $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$ *by* $\Phi_F(U) = UU_1\mathbb{T}^n(\mathbb{C})$*. Then the normal jacobian of* $\Phi_F(U)$ *is independent of* F*,* U_1 *and* U *and equals* $\mathrm{Vol}\,\mathbb{T}^n(\mathbb{C})$*.*

Proof. First consider the case $U_1 = I_n$. Then $\Phi_F(U) = U\mathbb{T}^n(\mathbb{C})$ is the projection from $\mathbb{U}_n(\mathbb{C})$ to $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$. Before normalizing the Riemannian metric on $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$ to make the volume 1, the normal to the fiber is mapped isometrically to the tangent space of $\mathbb{U}_n(\mathbb{C})/\mathbb{T}^n(\mathbb{C})$. Now $R_{U_1} : \mathbb{U}_n(\mathbb{C}) \to \mathbb{U}_n(\mathbb{C})$ defined by $R_{U_1}(U) = UU_1$ is an isometry of $\mathbb{U}_n(\mathbb{C})$ and the fibers of Φ_F are the reciprocal images by R_{U_1} of the fibers of Φ_{I_n}. So the normal jacobians are constant. After normalization, the normal jacobians must equal $\mathrm{Vol}\,\mathbb{T}^n(\mathbb{C})$ to make $\mathrm{Vol}\,\mathbb{U}_n(\mathbb{C})$ equal 1.

Corollary 21. $|\,\mathrm{Det}\ D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)^*| = \mathrm{Vol}\,\mathbb{T}^n(\mathbb{C})$ *for any* $F \in \mathbb{F}_n(\mathbb{C})$ *and* $U \in \mathbb{U}_n(\mathbb{C})$*.*

Proof. By Remark 15 $|\,\mathrm{Det}\ D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)D_{\mathbb{U}_n(\mathbb{C})}\Phi(U,F)^*|$ is equal to the normalized Jacobian of $\Phi_F(U)$ and we apply Proposition 9.

Finally we have from Shub-Vasquez [9]

Proposition 10. $|\,\mathrm{Det}\,(id - D_{\mathbb{F}_n(\mathbb{C})}\Phi(U,F)| = \prod_{j<i}\left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}\right|$ *where* $\lambda_{\sigma(i)} = \lambda_i(UA,F)$ *and* $|\lambda_1| > \ldots > |\lambda_n|$*.*

Making the substitutions in Theorem 19 given by Corollary 21 and Proposition 10 we have

Theorem 22. *Let* $f : \mathbb{F}_n(\mathbb{C}) \to \mathbb{R}$ *be continuous. Then*

$$\int_{F\in\mathbb{F}_n(\mathbb{C})} f(F)d\nu(F) = \int_{U\in\mathbb{G}_1} \sum_{(U,F)\in\Pi^{-1}_{\mathbb{U}_n(\mathbb{C})}} f(F)\prod_{j<i}\left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(i)}}\right|^{-2} d\mu(U).$$

This proves Lemma 1.

4.4. **Proof of Proposition 7.** Similarly substituting in Theorem 20 gives

Theorem 23. *Let* $g : \mathbb{G}_1 \to \mathbb{R}$ *be integrable. Then*

$$\int_{U\in\mathbb{G}_1} g(U)d\mu(U) = \int_{F\in\mathbb{F}_n(\mathbb{C})}\int_{(U,F)\in\Pi^{-1}_{\mathbb{F}_n(\mathbb{C})}(F)} g(U)\prod_{j<i}\left|1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(i)}}\right|^2 d\Pi^{-1}_{\mathbb{F}_n(\mathbb{C})}(F)(U)d\nu(F).$$

This theorem proves Proposition 7.

4.5. **Proof of Proposition 8.** Since the fibers $QR^{-1}(F)$ for a given $F \in \mathbb{G}_1$ are isometric to $\mathbb{T}^n(\mathbb{C})$ we have to prove the equality

$$\int_{\mathbb{T}^n(\mathbb{C})} \prod_{j<i} \left| 1 - \frac{\lambda_i}{\lambda_j} \right|^2 d\mu(\mathbb{T}^n(\mathbb{C})) = \int_{\mathbb{T}^n(\mathbb{C})} \sum_{\sigma \in \Sigma_n} \prod_{j<i} \left| \frac{1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}}{1 - \frac{\lambda_i}{\lambda_j}} \right|^{-2} d\mu(\mathbb{T}^n(\mathbb{C})).$$

Let us denote the Van der Monde determinant

$$V(\lambda_1, \ldots, \lambda_n) = \begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j<i} (\lambda_i - \lambda_i).$$

The first integral is equal to

$$\int_{\mathbb{T}^n(\mathbb{C})} \prod_{j<i} \left| 1 - \frac{\lambda_i}{\lambda_j} \right|^2 d\mu(\mathbb{T}^n(\mathbb{C})) = \int_{\mathbb{T}^n(\mathbb{C})} \frac{|V(\lambda_1, \ldots, \lambda_n)|^2}{\prod_{j<i} |\lambda_j|^2} d\mu(\mathbb{T}^n(\mathbb{C})).$$

The Van der Monde is equal to

$$V(\lambda_1, \ldots, \lambda_n) = \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \lambda_1^{\sigma(1)-1} \ldots \lambda_n^{\sigma(n)-1}.$$

Here the sum is taken for any permutation σ in the symmetric group and $\epsilon(\sigma) = \pm 1$ denotes is signature. The square of the absolute value of this Van der Monde is

$$|V(\lambda_1, \ldots, \lambda_n)|^2 = \sum_{\sigma, \tau \in \Sigma_n} \epsilon(\sigma)\epsilon(\tau) \lambda_1^{\sigma(1)-1} \bar{\lambda}_1^{\tau(1)-1} \ldots \lambda_n^{\sigma(n)-1} \bar{\lambda}_n^{\tau(n)-1}.$$

Now we integrate these products over a product of circles:

$$\int_{0<\theta_k<2\pi} \lambda_k^{\sigma(k)-1} \bar{\lambda}_k^{\tau(k)-1} d\theta_k = |\lambda_k|^{\sigma(k)+\tau(k)-2} \int_{0<\theta_k<2\pi} \exp(i\theta_k(\sigma(k) - \tau(k))) d\theta_k.$$

Since $d\theta_k$ is a probability measure, this last integral is equal to 1 when $\sigma(k) = \tau(k)$ and 0 otherwise. For this reason

$$\int_{\mathbb{T}^n(\mathbb{C})} \prod_{j<i} \left| 1 - \frac{\lambda_i}{\lambda_j} \right|^2 d\mu(\mathbb{T}^n(\mathbb{C})) = \sum_{\sigma \in \Sigma_n} \frac{|\lambda_1|^{2\sigma(1)-2} \ldots |\lambda_n|^{2\sigma(n)-2}}{\prod_{j<i} |\lambda_j|^2}.$$

The second integral is equal to

$$\int_{\mathbb{T}^n(\mathbb{C})} \sum_{\sigma \in \Sigma_n} \prod_{j<i} \left| \frac{1 - \frac{\lambda_{\sigma(i)}}{\lambda_{\sigma(j)}}}{1 - \frac{\lambda_i}{\lambda_j}} \right|^{-2} d\mu(\mathbb{T}^n(\mathbb{C})) =$$

$$\int_{\mathbb{T}^n(\mathbb{C})} \sum_{\sigma \in \Sigma_n} \frac{|V(\lambda_1, \ldots, \lambda_n)|^2}{|V(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})|^2} \prod_{j<i} \left| \frac{\lambda_{\sigma(j)}}{\lambda_j} \right|^2 d\mu(\mathbb{T}^n(\mathbb{C})) = \sum_{\sigma \in \Sigma_n} \prod_{j<i} \left| \frac{\lambda_{\sigma(j)}}{\lambda_j} \right|^2.$$

The first and second integral are equal if and only if

$$\sum_{\sigma\in\Sigma_n}|\lambda_1|^{2\sigma(1)-2}\dots|\lambda_n|^{2\sigma(n)-2}=\sum_{\sigma\in\Sigma_n}\prod_{j<i}|\lambda_{\sigma(j)}|^2$$

or, in other terms, if and only if

$$\sum_{\sigma\in\Sigma_n}|\lambda_1|^{2\sigma(1)-2}\dots|\lambda_n|^{2\sigma(n)-2}=\sum_{\sigma\in\Sigma_n}|\lambda_{\sigma(1)}|^{2(n-1)}|\lambda_{\sigma(2)}|^{2(n-2)}\dots|\lambda_{\sigma(n-1)}|^2.$$

This last inequality is obvious.

4.6. **Proof of Proposition 5.** $\mathbb{G}_1$ is clearly open and semi-algebraic in $\mathbb{U}_n(\mathbb{C})$. For this reason, "full measure in $\mathbb{U}_n(\mathbb{C})$" is equivalent to "dense in $\mathbb{U}_n(\mathbb{C})$". We shall prove now this last property.

Consider $\mathbb{V}_{1,A}\subset\mathbb{U}_n(\mathbb{C})\times\mathbb{U}_n(\mathbb{C})$ defined by $(U_1,U_2)\in\mathbb{V}_{1,A}$ when $(U_2^*U_1AU_2)_{i,j}=0$ for $i>j$, that is the flag defined by U_2 is fixed by $(U_1A)_\#$. $\mathbb{V}_{1,A}$ is a connected smooth real algebraic manifold. It is a locally trivial bundle over $\mathbb{V}_A$ with fiber $\mathbb{T}^n(\mathbb{C})$. Since the map $(U_1,U_2)\to U_2^*U_1AU_2$ taking $\mathbb{U}_n(\mathbb{C})\times\mathbb{U}_n(\mathbb{C})$ into $\mathbb{GL}_n(\mathbb{C})$ is transversal to the upper triangular matrices, which can be seen by varying U_1 alone, it follows that $\mathbb{V}_{1,A}$ is also a smooth variety. So a polynomial which vanishes on an open set in $\mathbb{V}_{1,A}$ vanishes identically. It will suffice to prove that the set of $(U_1,U_2)\in\mathbb{V}_{1,A}$ such that U_1A has distinct eigenvalue modules is dense in $\mathbb{V}_{1,A}$. Now the eigenvalues of U_1A are the diagonal elements of $U_2^*U_1AU_2$. The set of $(U_1,U_2)\in\mathbb{V}_{1,A}$ where there are equal modulus eigenvalues on the diagonal is given by the equations

$$(\mathcal{P}_{i,k})\qquad (U_2^*U_1AU_2)_{i,i}\overline{(U_2^*U_1AU_2)_{i,i}}=(U_2^*U_1AU_2)_{k,k}\overline{(U_2^*U_1AU_2)_{k,k}}.$$

So, if we show for each (i,k) that there are (U_1,U_2) such that the equality fails, then the variety defined by $\mathcal{P}_{i,k}$ is nowhere dense and the finite union of nowhere dense sets is nowhere dense. Let $A=V_1DV_2$ be a singular decomposition of A: V_1 and V_2 are in $\mathbb{U}_n(\mathbb{C})$ and $D=\mathrm{Diag}(d_1,\dots,d_n)$ with $0<d_1\le\dots\le d_n$. We know by the hypothesis that there are at least two distinct d_i. This gives two unitary matrices U_1 and U_2 such that

$$U_2^*U_1AU_2=\mathrm{Diag}(d_1,\dots,d_n)$$

with some pair (d_{i_1},d_{i_2}) of different moduli. By composing U_2 with a permutation matrix P, $P^*U_2^*U_1AU_2P$ permutes d_{i_1}, d_{i_2} to any two positions we wish, so we are done.

5. Proof of Theorem 8.

We may decompose the measure μ along $\mathbb{SO}_2(\mathbb{R})$ orbits. Then we are reduced to comparing the integrals

$$\int_{\mathbb{SO}_2(\mathbb{R})} \log|\lambda_1(R_\theta A)|d\mu(\theta) = \int_{\mathbb{S}^1} \log\|A(\theta)\|d\theta$$

for Det $A > 0$ and

$$\int_{\mathbb{SO}_2(\mathbb{R})} \log|\lambda_1(R_\theta A)|d\mu(\theta) > \int_{\mathbb{S}^1} \log\|A(\theta)\|d\theta$$

for Det $A < 0$ unless A is a constant times a reflection in which case equality holds.

Without loss of generality we may assume that $|\,\mathrm{Det}\ A| = 1$ and hence that $\lambda_1(R_\theta A)\lambda_2(R_\theta A) = \pm 1$ for all θ as Det $A = \pm 1$. Now we consider

$$\mathbb{V}_A = \{(R_\theta, x) \in \mathbb{SO}_2(\mathbb{R}) \times \mathbb{S}^1 \ : \ (R_\theta A)x = x\}$$

and the two projections $\Pi_{\mathbb{SO}_2(\mathbb{R})} : \mathbb{V}_A \to \mathbb{SO}_2(\mathbb{R})$ and $\Pi_{\mathbb{S}^1} : \mathbb{V}_A \to \mathbb{S}^1$. Then

$$\int_{\mathbb{S}^1} \log\|A(\theta)\|d\theta =$$

$$\int_{\mathbb{SO}_2(\mathbb{R})} \log|\lambda_1(R_\theta A)| \left|1 - \frac{\lambda_2(R_\theta A)}{\lambda_1(R_\theta A)}\right|^{-1} + \log|\lambda_2(R_\theta A)| \left|1 - \frac{\lambda_1(R_\theta A)}{\lambda_2(R_\theta A)}\right|^{-1} d\mu(\theta) =$$

$$\int_{\mathbb{SO}_2(\mathbb{R})} \log|\lambda_1(R_\theta A)| \left(\left|1 - \frac{\lambda_2(R_\theta A)}{\lambda_1(R_\theta A)}\right|^{-1} - \left|1 - \frac{\lambda_1(R_\theta A)}{\lambda_2(R_\theta A)}\right|^{-1}\right) d\mu(\theta).$$

Now for $\lambda_1\lambda_2 = 1$

$$\left|1 - \frac{\lambda_2}{\lambda_1}\right|^{-1} - \left|1 - \frac{\lambda_1}{\lambda_2}\right|^{-1} = \frac{1}{1 - \frac{\lambda_2}{\lambda_1}} - \frac{1}{\frac{\lambda_1}{\lambda_2} - 1} = 1$$

while for $\lambda_1\lambda_2 = -1$

$$\left|1 - \frac{\lambda_2}{\lambda_1}\right|^{-1} - \left|1 - \frac{\lambda_1}{\lambda_2}\right|^{-1} = \frac{1}{1 - \frac{\lambda_2}{\lambda_1}} - \frac{1}{1 - \frac{\lambda_1}{\lambda_2}} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} < 1.$$

This proves Theorem 8 except for the possibility that Det $A = -1$ and $\log\|A(\theta)\|$ is identically zero, i.e. A is a reflection.

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