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On a Question in Linear Programming and Its Application in Decentralized Allocation

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Abstract

For a linear programming problem $z^* = min\{cx : Ax \geq b, p \leq x \leq q\}$, consider the following question: Given $\delta > 0$, describe the set, S, of column vectors, with their corresponding costs, such that any one of these columns when introduced in the constraint matrix A guarantees a decrease (over that of z^*) in the optimum solution of the new linear program by at least δ. This and related questions arise while designing an iterative allocation scheme in which a single buyer wants to acquire a set of items by soliciting bids from multiple suppliers via a competitive bidding process.

In this note, we characterize the set S and and show that the separation problem over S can be solved in polynomial time. This is then used to solve a related optimization problem in each iteration of the bidding process.

1 Introduction

There has been a lot of recent interest in the problem of decentralized resource allocation and its applications in electronic commerce. A general setting for decentralized allocation is one where there are multiple agents with a utility function for the different resources and the allocation problem is to distribute the resource in an optimal way. A key difference from classical optimization is that the utility functions of the agents are private information and are not explicitly known to the decision maker.

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For example, an emerging application is in negotiations for electronic procurement where a buyer is looking to buy several items from a pool of suppliers. However, the cost functions of the suppliers are private information and unknown to the buyer.

An approach that is often used for this setting is to design an interactive mechanism where based on a "market signal" such as price for each item, the agents can propose bids based on a decentralized private cost model. The key requirements for such a design to be practical are: (i) convergence to an "equilibrium solution" in a finite number of steps, and (ii) the "equilibrium solution" is optimal for each of the agents, given the market signal. One approach for implementing such mechanisms is the use of primal-dual approaches where the resource allocation problem is formulated as a linear program and the dual prices are used as market signals [2, 3, 10, 1, 4, 8]. Each agent can then use the dual price vector to propose a profit maximizing bid, for the next round, based on her private cost model. Here, the assumption is that the agents attempt to maximize their profits in each round. This assumption is referred to as the *myopic best response* [5]. In a procurement setting with a single buyer and multiple suppliers, the buyer uses a linear program to allocate her demand by choosing a set of cost minimizing bids and then use the dual price variables to signal the suppliers. In order to guarantee convergence a large enough price decrement is used on all non-zero dual prices in each iteration.

In this paper we explore an alternate design where, the market signal provided to each supplier is based on the current cost of procurement for the buyer. Each supplier is then required to submit new bid proposals that reduce the procurement cost (assuming other suppliers keep their bids unchanged) by some large enough decrement δ. We show that, for each supplier, generating a profit maximizing bid that decreases the procurement cost for the buyer by at least δ can be done in polynomial time. This implies that in designs where the bids are not common knowledge, each supplier and the buyer can engage in an "algorithmic conversation" to identify such proposals in a polynomial number of steps. In addition, we show that such a mechanism converges to an "equilibrium solution" where all suppliers are at their profit maximizing solution given the cost and the required decrement δ. At the heart of this design lies a fundamental sensitivity analysis problem of linear programming - given a linear program and its optimal solution, identify the set of new columns such that any one of these columns when introduced in the linear program reduces the optimum solution by at least δ.

The paper is organized as follows: In Section 2, we introduce a fundamental question in linear programming and answer it in Section 2.1. This result is then used in Section 2.2 to solve an optimization problem central to the allocation problem. A variant of the problem in Section 2.1 is described in Section 2.3. The iterative scheme for the allocation problem is described in detail in Section 3 followed by the properties at termination in Section 3.1. We conclude in Section 4.

2 A Question in Linear Programming

Consider the linear programming problem

$$P: \min\{cx : Ax \geq b, x \in \mathbb{R}^n_+\}$$

Let $FP = \{x \in \mathbb{R}^n_+ : Ax \geq b\}$. We assume that all data is rational, FP is non-empty and P has an optimum solution. Let z^* be the optimum solution to P. Now, consider the following problem:

Question 1: Given $\delta > 0$, characterize the set $S \subseteq \mathbb{R}^{n+1}$ of vectors $(\bar{c}, \bar{a})$ where $\bar{c} \in \mathbb{R}, \bar{a} \in \mathbb{R}^n$ such that

$$\min\{[c\ \bar{c}]w : [A\ \bar{a}]w \geq b, w \in \mathbb{R}^{n+1}_+\} \leq z^* - \delta$$

The question asks for a characterization of the set of column vectors, with their corresponding costs, such that any one of these columns when introduced in the constraint matrix A of P guarantees a decrease in the objective function by at least δ.

2.1 Characterization of the set S

It is easy to see that S is a closed cone minus the origin: if $(\bar{c}, \bar{a}) \in S$ then $\lambda(\bar{c}, \bar{a}) \in S\ \forall \lambda > 0$. Let

$$\tilde{P}: \min\{[c\ \bar{c}]w : [A\ \bar{a}]w \geq b, w \in \mathbb{R}^{n+1}_+\}$$

Consider the dual of $\tilde{P}$:

$$\tilde{D}: \max\{yb : yA \leq c, y\bar{a} \leq \bar{c}, y \in \mathbb{R}^m_+\}$$

and let $FD = \{y : yA \leq c, y\bar{a} \leq \bar{c}, y \in \mathbb{R}^m_+\}$. Note that $(\bar{c}, \bar{a}) \in S$ if and only if the inequality $yb \leq z^* - \delta$ is a valid inequality [9] for FD. This then implies that $\exists\ (u, v) \geq 0,\ v \neq 0$ where $u \in \mathbb{Q}^n, v \in \mathbb{Q}$ such that

$$\begin{aligned} \sum_{j=1}^{n} u_j a_{ij} + v\bar{a}_i &\geq b_i \ i = 1, ..., m \\ \sum_{j=1}^{n} u_j c_j + v\bar{c} &\leq z^* - \delta \end{aligned} \tag{1}$$

Let $g_i = v\bar{a}_i,\ i = 1, ..., m;\ g_{m+1} = v\bar{c}$ and let Q be the polyhedron of the feasible points corresponding to the following system of inequalities:

$$\begin{aligned} \sum_{j=1}^{n} u_j a_{ij} + g_i &\geq b_i \ i = 1, ..., m \\ \sum_{j=1}^{n} u_j c_j + g_{m+1} &\leq z^* - \delta \end{aligned} \tag{2}$$

Theorem 2.1 $S = \{\lambda h : h \in proj_g(Q), \lambda > 0\}$

Proof: If $g^* \in \text{proj}_g(Q)$, then $\exists\ u \geq 0$ such that (2) is satisfied and hence $(\bar{c}, \bar{a}) = g^*$ satisfies (1). Thus, $g^* \in S$ and hence $\lambda g^* \in S\ \forall\ \lambda > 0$. If $s = (\bar{c}, \bar{a}) \in S$, there exists $(u, v) \geq 0, v > 0$ satisfying (1). Then, $g = vs \in \text{proj}_g(Q)$. ∎

Corollary 2.2 $S \cup \{\bar{0}\}$ *is a polyhedral convex cone.*

2.2 The Separation Problem on S

We consider the separation problem on the set $S \in \mathbb{R}^{n+1}$ obtained as a solution to Question 1.

<u>**Separation Problem:**</u> Given a vector $s_0 = (\bar{c}^*, \bar{a}^*) \in \mathbb{Q}^{n+1}$, decide whether $s_0 \in S$ or not, and, in the latter case, find a vector $t \in \mathbb{Q}^{n+1}$ such that $ts < ts_0\ \forall\ s \in S$.

For $s_0 = (\bar{a}^*, \bar{c}^*) \in \mathbb{Q}^{n+1}$, membership in S can be checked by solving the linear program $\min\{[c\ \bar{c}^*]w : [A\ \bar{a}^*]w \geq b, w \in \mathbb{R}^{n+1}_+\}$. Otherwise, it is easy to see that the system

$$\begin{aligned}\sum_{j=1}^{n} u_j a_{ij} &\geq b_i - \bar{a}_i{}^* \; i = 1, ..., m \\ \sum_{j=1}^{n} u_j c_j &\leq z^* - \delta - \bar{c}^*\end{aligned}$$

is infeasible. Farkas' lemma [11] provides us with multipliers $t_i \geq 0, i = 1, ..., m+1$, $t_i \in \mathbb{Q}$ such that

1. $\sum_{i=1}^{m}(-a_{ij})t_i + c_j t_{m+1} \geq 0, j = 1, ..., n$
2. $\sum_{i=1}^{m}(-\bar{a}_i{}^*)t_i + \bar{c}_j{}^* t_{m+1} \geq 0$
3. $\sum_{i=1}^{m} -b_i t_i + (z^* - \delta)t_{m+1} < 0$

Let $t = (-t_1, ..., -t_m, t_{m+1})$. Then $ts_0 \geq 0$. If $s \in S$, then $ts < 0$ for otherwise (1) would be infeasible. Thus, $ts < ts_0 \; \forall \; s \in S$.

2.3 Related Problems

The description of S can easily be extended for arbitrary linear programs. For $0 \leq p < q$, consider the linear program :

$$P' : \; \min\{cx : Ax \geq b, p \leq x \leq q\}$$

Let z^* be the optimum solution to P'.

Question 1$'$: Given $\delta > 0$, characterize the set $S \subseteq \mathbb{R}^{n+1}$ of vectors $(\bar{c}, \bar{a})$ where $\bar{c} \in \mathbb{R}, \bar{a} \in \mathbb{R}^n$ such that

$$\min\{[c \; \bar{c}]w : [A \; \bar{a}]w \geq b, p \leq w \leq q\} \leq z^* - \delta$$

if and only if $(\bar{c}, \bar{a}) \in S$.

Let $Q' = \{(u, g) : (u, g) \text{ satisfies } (2), p \leq u \leq q\}$. Then, $S = \{\lambda h : h \in \text{proj}_g(Q'), \frac{1}{q} \leq \lambda \leq \frac{1}{p}\}$. Of particular interest is the case when $p = 0, q = 1$. In this case, S is a closed, convex, polyhedral set.

A similar analysis can be used to investigate *replacing* a given column (instead of adding a new column).

Question 2: Given $\delta > 0$ and $j \in \{1, 2, ..., n\}$, characterize the set $S_j \subseteq \mathbb{R}^{n+1}$ of vectors $(\bar{c}, \bar{a})$ where $\bar{c} \in \mathbb{R}, \bar{a} \in \mathbb{R}^n$ such that

$$\min\{[c^{\bar{j}}\ \bar{c}]w : [A^{\bar{j}}\ \bar{a}]w \geq b, p \leq w \leq q\} \leq z^* - \delta$$

if and only if $(\bar{c}, \bar{a}) \in S_j$, where $c^{\bar{j}}$ and $A^{\bar{j}}$ are the cost vector and constraint matrix, respectively, obtained after removing variable x_j from P'.

The separation problems on the solution sets of Question $1'$ and Question 2 can be solved similarly.

3 An Allocation Problem

A buyer is soliciting bids to buy m items $I_1, ..., I_m$. The quantity required for item I_i is $d_i \in \mathbb{Q}_+$. Supplier j, $j = 1, ..., n$ proposes a bid (c_j, a_j) where $a_j \in \mathbb{Q}^n_+$ is the vector indicating the number of units of each item supplier j will supply and $c_j \in \mathbb{Q}_+$ is the total price charged to supply all the items in a_j. Let $d = (d_1, d_2, ..., d_m) \in \mathbb{Q}^m_+$.

We make the following assumptions in an iterative allocation scheme described below.

(a) Any fractional allocation of her bid vector is acceptable to each supplier. If a bid (c, a) gets a fractional allocation, say f, where $0 < f < 1$, then the supplier promises to supply the item vector fa for a total price of fc.

(b) For each supplier, the production cost for each item is proportional to the number of items. The utility function of each supplier is her profit obtained as the difference between the price she charges for the allocation and her production cost for that allocation.

(c) All bids are bounded above by the demand vector d. That is, for any bid (c, a), $a \leq d$.

Fix $\delta > 0$. The iteration scheme can be summarized as follows: Given the bids by the suppliers, the buyer solves an allocation problem with the objective of minimizing her procurement cost. Each supplier's allocation is privately reported to her. The buyer and each supplier then engage in a private (algorithmic) conversation to come up with a bid which is (a) at least as profitable to the supplier (if fully awarded) as the profit resulting from current allocation and (b) guaranteed to decrease the buyer's cost by at least δ. If such a bid is available, the supplier proposes it as her new bid. Otherwise, the previous bid is retained. The scheme now proceeds to the next iteration. If none of the suppliers propose a new bid, the allocation process stops.

We now describe the process in detail.

Step 1: *(initialization)* Let (c_j^0, a_j^0), $j = 1, 2, ..., n$ be the initial set of bids submitted by the suppliers. Let $A^0 = [a_1^0 \; a_2^0 ... \; a_n^0] \in \mathbb{Q}^{m \times n}$ and $c^0 = (c_1^0, c_2^0, ..., c_n^0) \in \mathbb{Q}^m$. Fix $\delta > 0$ and set $r = 0$.

Step 2: *(buyer's problem)* The buyer solves an allocation problem to minimize her cost:

$$P^r : \; \min\{c^r x : A^r x \geq d; 0 \leq x \leq 1\}$$

Let z^r and x^r be the optimum cost and the corresponding solution vector, respectively, for this problem. The buyer now executes the following steps:

(a) Reports the allocation x_j^r privately to each supplier j, $j = 1, ..., n$.

(b) For each j, computes the set S_j^r of column vectors, with their corresponding cost, such that replacing the bid of supplier j, in P^r, by any vector in S_j^r guarantees a decrease (over that of z^r) in the optimum solution to the allocation problem by at least δ (Question 2 in Section 2.3).

Step 3 *(algorithmic interaction between buyer and suppliers)* The profit of supplier j from the allocation x_j^r is $v_j^r = (c_j^r - a_j^r p_j) x_j^r$, where $p_j \in \mathbb{Q}_+^m$ is the vector of unit production costs for supplier j.

Let $S_0 = \{(\bar{c}, \bar{a}) : \bar{a} \leq d\} \subseteq \Re^{n+1}$. The optimization problem for the supplier j is

$$L_j^r : \; \max\{\bar{c} - p_j \bar{a} - v_j^r : (\bar{c}, \bar{a}) \in S_j^r \cap S_0\}$$

The buyer and supplier j, $j = 1, ..., n$, cooperate privately to solve the optimization problem L_j^r. Note that L_j^r is solvable in polynomial time (Section 2.3).

If it exists, let (c_j^*, a_j^*) be the optimum solution vector for L_j^r. Since the extreme points and extreme rays of $S_j^r \cap S_0$ are rational vectors, $(c_j^*, a_j^*) \in \mathbb{Q})$. If the optimum value to L_j^r is non-negative, supplier j updates her bid: $(c_j^{r+1}, a_j^{r+1}) = (c_j^*, a_j^*)$. Otherwise, supplier j retains her previous bid $(c_j^{r+1}, a_j^{r+1}) = (c_j^r, a_j^r)$.

Step 4 (*exit criterion*) If $(c_j^{r+1}, a_j^{r+1}) = (c_j^r, a_j^r)\ \forall\ j$, **STOP**; otherwise $r = r + 1$ and return to Step 2.

3.1 Properties at Termination

Since the cost of the buyer reduces by at least δ in each iteration, the allocation process terminates in a finite number of iterations. It is easy to see that the allocation problem satisfies the following properties at termination.

- Given the bids by the suppliers, the buyer minimizes her cost.
- For each supplier j, given the bids of all other suppliers $k \neq j$, $\nexists$ any bid $(\bar{c}, \bar{a})$, which if she were to propose would satisfy the following two conditions simultaneously: (i) gives more profit and (ii) reduces the buyers cost by at least δ.

Given an optimum basic solution to a linear program, there exists a finite, non-trivial perturbation of its cost function such that the optimum basic solution to the perturbed problem stays the same. Using this well-known property of linear programs [11], an interesting *ex-post* δ-optimality property can be proved when the iterative scheme terminates:

- At termination, let the buyer's total cost be c^*. Then $\exists\ \delta > 0$ such that the following property is true for *each* supplier: Given the bids of all other suppliers $k \neq j$, $\nexists$ any bid $(\bar{a}, \bar{c})$ for supplier j which keeps the buyer's cost at c^* and increases her profit by more than δ.

4 Conclusions

The main purpose of this note is to introduce a fundamental question in linear programming which asks for a description of the set of column vectors, with their corresponding costs, such that any one of these vectors when introduced in the constraint matrix guarantees a fixed decrease in the objective function. A variant in which a column is replaced instead of being added is also considered. We characterize the sets and show that the separation problem over these sets can be solved in polynomial time.

The solution to these problems is then used to design an iterative allocation scheme in which a single buyer wants to acquire a set of items by soliciting bids from multiple suppliers via a competitive bidding process.

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