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Mathematics

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Matroid Reinforcement

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MATROID REINFORCEMENT

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Abstract. For a matroid M, Edmonds proved that its ground set contains k disjoint bases if and only if $|A| \geq k(r(E) - r(\bar{A}))$ for every subset A of the ground set E. Here r is the rank function of M. We study the system of inequalities $x(A) \geq k(r(E) - r(\bar{A}))$, $0 \leq x \leq u$. We show that if u is integer valued then this defines a polyhedron with integer extreme points. We also show that this is a TDI system. We give a simple combinatorial algorithm for solving the associated optimization problem. Related results have been obtained by Frank & Tardos with the use of generalized polymatroids.

1. Introduction

Consider a graph $G = (V, E)$, for a family of disjoint vertex-sets $\{S_1, \ldots, S_p\}$ let $\delta(S_1, \ldots, S_p)$ be the set of edges with endpoints in different sets of this family. It has been proved in [Tutte, 1961] and [Nash-Williams, 1961] that G contains k edge-disjoint spanning trees if and only if

$$|\delta(S_1, \ldots, S_p)| \geq k(p-1),$$

for every partition $\{S_1, \ldots, S_p\}$ of V.

Let $M = (E, r)$ be a matroid where E is its ground set and r is its rank function. The theorem above was generalized in [Edmonds, 1965] to prove that M contains k disjoint bases if and only if

$$|A| \geq k\big(r(E) - r(\bar{A})\big),$$

for every subset $A \subseteq E$. Here $\bar{A} = E \setminus A$.

The following *matroid reinforcement* problem was introduced in [Cunningham, 1985b] for graphic matroids. For every element e let $x(e)$ be the number of parallel copies to be made of e, notice that $x(e) = 0$ means that e is not taken at all. Let $u(e)$ be an upper bound for $x(e)$. Let $d(e)$ be the cost of making a copy of e. The linear program below finds a minimum cost set of elements containing k disjoint bases.

$$\text{minimize } dx \tag{1}$$

subject to

$$x(S) \geq k\Big(r(E) - r(\bar{S})\Big), \text{ for all } S \subseteq E, \tag{2}$$

$$0 \leq x(e) \leq u(e). \tag{3}$$

In this note we study the linear program above. We show that if u is integer valued then (2)-(3) defines a polyhedron with integer vertices. We also show that if d is integer then the linear program above has a dual optimal solution that is integer. This means that (2)-(3) is totally dual integral (TDI). We give a simple combinatorial algorithm to solve this linear program. The case of graphic matroids was solved in [Cunningham, 1985b], later a faster algorithm was given in [Barahona, 2002].

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In [Frank and Tardos, 1988] similar results were shown for the slightly different version below:

$$\begin{aligned}&\text{minimize } dx\\ &\text{subject to}\\ &x(S) \geq k\Big(r(E) - r(\bar{S})\Big) - |A|, \text{ for all } S \subseteq E,\\ &x(e) \geq 0.\end{aligned}$$

Here every element e is taken, and $x(e)$ represents the number of *extra* copies of it. They used the theory of Generalized Polymatroids.

Now we introduce some notation. If x is a function that associates with every element $e \in E$ a value $x(e)$, we use $x(S)$ to denote $\sum_{e \in S} x(e)$, for $S \subseteq E$.

A property of the rank function r that will be used in the next section is the *submodular inequality*, namely for any two subsets A and B of E,

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

This note is organized as follows. In Section 2 we study a problem that will be used as a subroutine. In Section 3 we give an algorithm for solving (1)-(3).

2. A key subroutine

The problem below will be used in the next section:

$$\text{minimize } u(A) - k\Big(r(E) - r(\bar{A})\Big) \tag{4}$$

The minimization in (4) is over all sets $A \subseteq E$. We can divide by k and since $r(E)$ is a constant we obtain

$$\text{minimize } w(A) + r(\bar{A}). \tag{5}$$

Following [Cunningham, 1985b] we treat (5) by associating a variable $y(e)$ to each element e and solving

$$\text{maximize } y(E) \tag{6}$$

$$\text{subject to}$$

$$y(S) \leq r(S), \text{ for all } S \subseteq E, \tag{7}$$

$$y \leq w. \tag{8}$$

First notice that for any $A \subseteq E$, $y(E) = y(A) + y(\bar{A}) \leq w(A) + r(\bar{A})$. Next we need to find a set A such that the equality holds.

Suppose we apply the greedy algorithm to solve (6)-(8). Starting from a feasible vector $\bar{y}$, we raise each component $\bar{y}(e)$ until either $\bar{y}(e) = w(e)$, or e is in a set S such $\bar{y}(S) = r(S)$. Such a set is called tight. Consider two tight sets A and B, their union is also tight, because

$$\begin{aligned}&r(A) + r(B) = \bar{y}(A) + \bar{y}(B) = \bar{y}(A \cup B) + \bar{y}(A \cap B) \leq\\ &r(A \cup B) + r(A \cap B) \leq r(A) + r(B).\end{aligned}$$

Let $\bar{A}$ be the union of all tight sets, this is also tight. Then $\bar{y}(E) = \bar{y}(A) + \bar{y}(\bar{A}) = w(A) + r(\bar{A})$. So A solves (5).

Here the key operation is to find for a given element e a set S minimizing $r(S)-\bar{y}(S)$, with $e \in S$. This gives the amount by which $\bar{y}(e)$ can be increased. For the case of graphic matroids this reduces to a minimum cut problem, see [Cunningham, 1985a], [Picard and Queyranne, 1982], [Padberg and Wolsey, 1983]. For the general case a strongly polynomial algorithm was given in [Cunningham, 1984], where it is assumed the existence of an oracle that tests independence in M.

The next lemma shows a property of the solution of (5), needed in the next section.

Lemma 1. *Let A be the solution obtained for* (5)*. Let $e \in A$, suppose we increase $w(e)$ and let A' be the new solution obtained. Then either $A' = A$ or $A' \subset A$.*

Proof. We can continue applying the greedy algorithm starting with the vector $\bar{y}$ obtained before increasing $w(e)$. Let y' be the new vector obtained. Then either $y'(e) = w(e)$ or e is in a tight set. In the first case $A' = A$, and in the second case $A' \subset A$. □

3. Reinforcement

Consider the pair of dual linear programs

$$\text{minimize } dx \tag{9}$$

subject to

$$x(S) \geq k\Big(r(E) - r(\bar{S})\Big), \text{ for all } S \subseteq E, \tag{10}$$

$$0 \leq x(e) \leq u(e). \tag{11}$$

$$\max \sum \gamma(S)\, k\Big(r(E) - r(\bar{S})\Big) - \sum u(e)\beta(e) \tag{12}$$

$$\sum_{\{S:\, e \in S\}} \gamma(S) \leq d(e) + \beta(e), \quad \text{for all } e, \tag{13}$$

$$\gamma \geq 0, \quad \beta \geq 0. \tag{14}$$

We are going to present a dual algorithm, where constraints (13) and (14) are always satisfied, and the value of (12) always increases. A primal vector is being constructed so that complementary slackness and primal feasibility are reached at the end.

We start with $\bar{\gamma} = 0$, $\bar{\beta} = 0$, $\bar{x} = 0$. We are going to choose a set $S \subseteq E$ and increase $\bar{\gamma}(S)$ by ϵ. We have to ensure that inequalities (13) are satisfied. So let H be the set of elements such that (13) are satisfied as equation. We have to increase by ϵ all values $\bar{\beta}(e)$, for $e \in S \cap H$. Then the objective function changes by

$$\epsilon\Big(k\big(r(E) - r(\bar{S})\big) - u(S \cap H)\Big).$$

Thus we choose S by solving

$$\text{minimize } u(A \cap H) - k\Big(r(E) - r(\bar{A})\Big). \tag{15}$$

The minimization in (15) is over $A \subseteq E$. This is done as in Section 2. We should set $w(e) = 0$ for all $e \notin H$. Notice that $A = \emptyset$ gives the value zero, so the minimum in (15) is always nonpositive. If the minimum in (15) is negative we use the largest value of ϵ so that a new inequality (13) becomes tight. This is

$$\bar{\epsilon} = \min\Big\{\bar{d}(e) = d(e) - \sum_{\{T\,:\,e\in T\}} \bar{\gamma}(T) \mid e \in S \setminus H\Big\}. \tag{16}$$

If this minimum is taken over the empty set we say that $\bar{\epsilon} = \infty$. In this case the dual problem is unbounded and the primal problem is infeasible.

Now assume that an edge f gives the minimum in (16). We change $(\bar{\gamma}, \bar{\beta})$, add f to H and solve (15) again. Let S' be the new solution of (15). If $S' = S$ then $\bar{\beta}(f)$ should increase, and to satisfy complementary slackness we set

$$\bar{x}(f) = u(f). \tag{17}$$

If $S' \subset S$, $f \notin S'$, then $\bar{\beta}(f)$ remains equal to zero. In this case $\bar{x}(f)$ can take a value less than $u(f)$. We set

$$\bar{x}(f) = k\bar{y}(f), \tag{18}$$

where $\bar{y}(f)$ is the value obtained when applying the greedy algorithm in (6)- (8). We shall see below that at the end $\bar{x}$ will be feasible and complementary slackness will be satisfied. We increase $\bar{\gamma}(S')$ and $\bar{\beta}(e)$ for $e \in S' \cap H$.

We continue until either one of the following two cases arise.

- Case 1: $\bar{\epsilon} = \infty$, this implies that the dual problem is unbounded.
- Case 2: $S' = \emptyset$, at this point an optimal solution has been obtained, as shown in the Lemma below.

Lemma 2. *Suppose that the solution obtained for* (4) *is $A = \emptyset$. Let $\bar{y}$ be the solution obtained for* (6)-(8)*. If we set $\bar{x} = k\bar{y}$, then $\bar{x}$ is feasible for* (9)-(11) *and complementary slackness is satisfied.*

Proof. Notice that the minimum in (4) is zero. If we set $u = \bar{x}$ and solve (4) again the solution remains the same. So all inequalities (10) are satisfied.

The algorithm produces a nested sequence of sets $E = S_1 \supset S_2 \supset \dots S_k = \emptyset$ with $\bar{x}(\bar{S}_i) = kr(\bar{S}_i)$. We have that $\bar{\gamma}_S$ can be positive only for sets in this sequence.

We have $\bar{x}(E) = kr(E)$ and $\bar{x}(\bar{S}_i) + \bar{x}(S_i) = \bar{x}(E) = kr(E)$. Thus $\bar{x}(S_i) = k(r(E) - r(\bar{S}_i)$.

We have already seen that whenever $\bar{\beta}(e) > 0$ then $\bar{x}(e) = u(e)$, thus $\bar{x}$ and $(\bar{\gamma}, \bar{\beta})$ satisfy complementary slackness. □

The formal description of the algorithm is below.

Algorithm A

- **Step 0**. Start with $\bar{\gamma} = 0$, $\bar{\beta} = 0$, $\bar{x} = 0$, $S = E$, $H = \emptyset$.
- **Step 1**. Compute $\bar{\epsilon}$ as in (16). If $\bar{\epsilon} = \infty$ stop, the problem is infeasible.
  Otherwise update $\bar{\gamma}_S \leftarrow \bar{\gamma}_S + \bar{\epsilon}$,
  $\bar{\beta}_e \leftarrow \bar{\beta}_e + \bar{\epsilon}$ for all $e \in H \cap S$.
- **Step 2**. Let f be an element giving the minimum in (16), add f to H. Solve problem (15) to obtain S'.

- **Step 3**. If $S' = S$ update $\bar{x}$ as in (17). Otherwise update $\bar{x}$ as in (18). If $S' = \emptyset$ stop. Otherwise set $S \leftarrow S'$ and go to Step 1.

Since at each iteration one element is added to H, this algorithm takes at most $|E|$ iterations. As seen in Section 2, at each iteration the key operation is to find for a given element e, a set $S \subseteq E$ that minimizes $r(S) - y(S)$, with $e \in S$.

When $k = 1$ and $u = \infty$, Algorithm A reduces to the greedy algorithm for matroids.

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