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Stochastic Optimization for Lake Eutrophication Management

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STOCHASTIC OPTIMIZATION FOR LAKE EUTROPHICATION MANAGEMENT

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1 INTRODUCTION

Man-made (or artificial) eutrophication has been considered as one of the most serious water quality problems of lakes during the last 25-40 years. Increasing discharges of domestic and industrial waste water and the intensive use of crop fertilizers — all leading to growing nutrient loads of the recipients — can be mentioned among the major causes of this undesirable phenomenon. The typical symptoms of eutrophication are among others sudden algal blooms, water coloration, floating water plants and debris, excreation of toxic substances causing taste and odor problems of drinking water and fish kills. These symptoms can easily result in limitations of water use for domestic, agricultural, industrial or recreational purposes.

One of the major features of artificial eutrophication is that although the consequences appear within the lake, the cause — the gradual increase of nutrients (various phosphorous and nitrogen compounds) reaching the lake — and most of the possible control measures lie in the region. Consequently, eutrophication management requires analysis of complex interactions between the water body and its surrounding region. In the lake, different biological, chemical and hydrophysical processes — all being time and space dependent, furthermore non-linear — are important, while in the region one must take into account human activities generating nutrient, residuals and control measures determining that portion of the emission which reaches the water body.

Eutrophication management requires a sound understanding of all these processes and activities, which belong to diverse disciplines. Additionally, various uncertainties and stochastic features of the problem have to be also taken into account — the estimation of loads from infrequent observations and the dependence of water quality on hydrologic and meteorologic factors. The fact that

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we are dealing with a stochastic environment is especially important for shallow lakes, primarily due to the absence of thermal stratification that predicates a much more definite response to randomness as would be the case for deep lakes.

The paper presents an approach that combines descriptive, simulation and management optimization models. The roles of each of these models is discussed in Section 2. The derivation of the aggregated lake and planning type nutrient load models to be used in the management model is the subject of Section 3. Alternative management models are formulated in Section 4. Two of them were implemented: a 'true' stochastic model (which uses as the starting point of the iterative solution procedure the corresponding deterministic model) and a linear programming approach capturing stochastic features of the problem through a linearized expectation-variance model. A brief survey of the solution approach to the stochastic problem is presented in Section 5. In Section 6 the methods are applied to Lake Balaton and the results compared.

This chapter is based on the original technical report Somlyódy and Wets (1985) parts of which appeared in Somlyody and Wets (1988).

2 The Approach

The approach to eutrophication and eutrophication management is based on the idea of decomposition and aggregation; Somlyódy (1982 and 1983b). The first step is to *decompose* the problem into smaller, tractable units forming a hierarchy of issues (and models), such as biological and chemical processes in the lake, sediment-water interaction, water circulation and mass exchange, nutrient loads, watershed processes and possible control measures, as well as the influence of uncontrollable meteorological factors, etc. This step is followed by *aggregation*, the aim of which is to preserve and integrate only the issues that are essential for the higher level of the analysis, ruling out unnecessary details. The procedures followed for the derivation of the eutrophication management optimization model (EMOM) presented in this paper may be found in Somlyódy (1983a) and Somlyódy and van Straten (1985).

The major assumptions for the application of EMOM to Lake Balaton are:

1. The lake is shallow with vertically uniform water quality.
2. The lake can be subdivided into sequentially connected basins.
3. The lake is phosphorus (P) limited, like most water bodies, and thus nutrients other than P are not involved in the analysis.
4. A single water quality indicator, the maximum annual chlorophyll-a concentration $(\mathrm{Chl}-\mathrm{a})_{\max}$, is used for defining trophic state and the goals of management.
5. A linear relationship holds between $(\mathrm{Chl}-\mathrm{a})_{\max}$ and the annual average P load.

6. The management horizon is short-term (a few years). Longer-term renewal processes between the lake and its sediment layer and the staging of investments are out of the scope of the present effort.

7. Only certain types of P sources and associated control alternatives are taken into account.

3 Formulation of the Stochastic Model

Based on the assumptions made in the approach and the insights gained from the study on Lake Balaton, the short term response of water quality to load reduction — taking into account macroscopic effects of biological and biochemical processes, interbasin mass exchanges, and the influence of stochastic factors — can be written as

$$\mathbf{Y} = E\{\mathbf{Y}_0\} + \mathbf{w} - (D + d\mathbf{w}) \triangle \mathbf{L} \tag{1}$$

where the elements of the m-vector $\mathbf{Y}$ represent the water quality in the m basins as measured by $(\mathrm{Chl}-\mathrm{a})_{\max}$, the m-vector $\mathbf{Y}_0$ is the uncontrolled nominal state, and the symbol $E\{\cdot\}$ denotes the expectation operator. The m-vector $\triangle\mathbf{L}$ expresses the reduction in P load due to controls

$$\triangle\mathbf{L} = E\{\mathbf{L}_0\} - \mathbf{L} \tag{2}$$

where the elements of $\mathbf{L}$ are the annual mean volumetric biologically available P load in each basin $i = 1, \ldots, m$. Biologically available P refers to the fraction of P that can be taken up by algae and thus contribute to short term trophic status of the water body. The random vector $\mathbf{w}$ represents the impact of non-controllable meteorological factors in each basin. (Stochastic variables and parameters are represented in bold-face.)

The elements of the matrix D are the reciprocals of lumped reaction rates. The main diagonal comprises primarily the effect of biological and biochemical processes in the basins, and the off-diagonal elements refer to interbasin exchange due to hydrological flow and mixing. The meaning of the slopes d_i is similar to that of the diagonal elements of D. The term $d_i\mathbf{w}_i \triangle \mathbf{L}_i$ expresses a change in the random component of the water quality indicator in basin i due to the impact of weather. Of course, the effect of the random fluctuations $\mathbf{w}_i$ caused by meteorology decreases if the loads diminish.

Next we model the P loads. The units of $\mathbf{L}_{0i}$ and $\mathbf{L}_i$ are measured in [mg/m^{3}d]. The absolute annual mean load $\mathbf{L}^a$ is the daily flow [mg/d] averaged over an entire year, thus

$$\mathbf{L}_i = \mathbf{L}_i^a / V_i \tag{3}$$

where V_i is the volume of basin i. To model the term (2) we must first analyze the contributions to the absolute annual mean load $\mathbf{L}_i^a$ and then divide by the volume. We consider three P sources as indicated in Figure 1:

1. direct point-source sewage L_S,

2. indirect point-source sewage flowing into a tributary of the lake L_{SN},

3. miscellaneous load from various point and non-point sources.

The sewage loads L_S and L_{SN} are known, deterministic, and biologically available. The uncertain biologically available (reactive) portion of the tributary load is

$$\mathbf{L}_D + \delta(\mathbf{L}_T - \mathbf{L}_D) \tag{4}$$

where $\mathbf{L}_D$ is the dissolved reactive P load and $\delta(\mathbf{L}_T - \mathbf{L}_D)$ is the proportion of the undissolved P load (the difference between the total P load and the dissolved P load) that becomes available through biological or biochemical processes. The availability ratio δ is about 0.2.

The basic control options for the P loads are:

1. precipitation by sewage treatment plants for both the direct and tributary point-source sewage loads, with corresponding control variables x_S and x_{SN}, respectively, considered as continuous variables.

2. pre-reservoir systems, established on tributaries before the water enters the lake, consisting of two segments: one that removes particulate P through sedimentation, and one that removes dissolved P through benthic eutrophication in reed basins, sorption, etc, with corresponding control variables x_P and x_D, respectively. The pre-reservoir control variables should normally be considered as binary $\{0, 1\}$ variables but for numerical tractability are modeled as continuous.

The control variables are modeled as removal coefficients with

$$0 \leq r^- \leq x \leq r^+ \leq 1. \tag{5}$$

Consider a simple situation for a single basin, as in Figure 1, with one tributary with a single point-source sewage inflow and one point-source direct sewage inflow. The original uncontrolled load $\mathbf{L}$ is:

$$\mathbf{L}_0^a = \mathbf{L}_D + \delta(\mathbf{L}_T - \mathbf{L}_D) + L_S + L_{SN} + \mathbf{L}_{NC} \tag{6}$$

where $\mathbf{L}_{NC}$ is the portion of the load that is beyond the controls considered. The controlled load of the basin is

$$\mathbf{L}^a = (1-x_D)\,[\mathbf{L}_D - (1-r_t)x_{SN}L_{SN}] + \delta(1-x_P)(\mathbf{L}_T - \mathbf{L}_D) + (1-x_S)L_S + \mathbf{L}_{NC} \tag{7}$$

where r_t is the retention coefficient defining that portion of P from the tributary sewage discharge that is retained in the tributary and does not reach the lake. There is an obvious interaction between the impact of sewage treatment plants on a given tributary x_{SN} and the impact of the pre-reservoir segment x_D on the same tributary, which is discussed below. With equations (6) and (7) we obtain

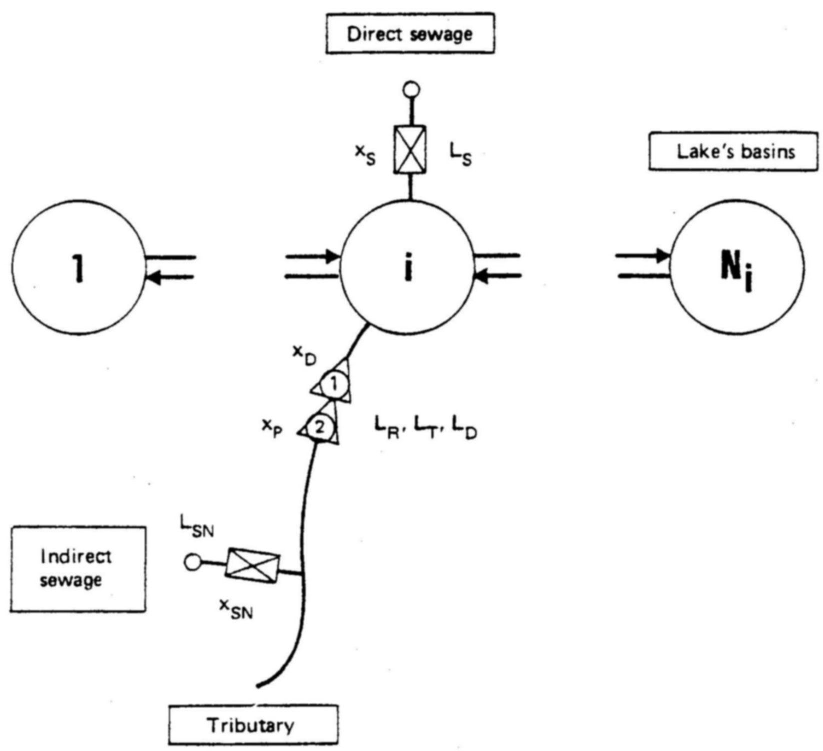

Figure 1: Simple illustration of Lake Eutrophication and Nutrient Load Modeling

an expression for the amount of dissolved P that is removed from the lake and its tributaries due to controls:

$$\begin{aligned}\triangle \mathbf{L}^a := E\left\{\mathbf{L}_0^a\right\} - \mathbf{L}^a = \quad & x_D\left[E\left\{\mathbf{L}_D\right\} - (1-r_t)x_{SN}L_{SN}\right] \\ & +(x_D - 1)\left[\mathbf{L}_D - E\left\{\mathbf{L}_D\right\}\right] \\ & +\delta\left[(x_P - 1)(\mathbf{L}_T - \mathbf{L}_D) + E\left\{\mathbf{L}_T - \mathbf{L}_D\right\}\right] \\ & +(1-r_t)x_{SN}L_{SN} + x_S L_S\end{aligned} \tag{8}$$

The terms in (8) have been rearranged for interpretive purposes.

- The first and fourth terms express the reduction in the *expectation* of the tributary's dissolved P load.
- The second term represents the effects of the reed basin on the *fluctuations* of the tributary's dissolved P load.
- The third term gives the modification in the particulate P load of the tributary.
- Finally, the fifth term expresses the control of the direct sewage load on the lake.

If we set all the control variables to zero in equation (8) we obtain the fluctuations in the original uncontrolled load, the expectation of which is zero.

It is apparent from equation (8) that the tributary load can be controlled by P precipitation at sewage treatment plans and/or by sedimentation and benthic removal in pre-reservoirs. The former influence only the expectation, whereas the latter influence linearly both the expectation and the variance.

Equation (8) is nonlinear in the control variables because of the product term $x_D \cdot x_{SN}$, which may cause difficulties in the optimization scheme. There are many ways to treat this issue. In the present case, the surface dependent character of benthic P removal in the second segment of the pre-reservoir system offers a possibility. Generally, for a reed reservoir one cannot estimate more than the P removal per unit of surface area, independent of the inflow concentration. This suggests that the effect of x_D can be approximated in terms of the original uncontrolled load, and the term involving $x_D \cdot x_{SN}$ can be dropped from (8). The price of this elimination of nonlinearity is twofold.

1. An upper limit should be specified for the impact of x_D stating that no more nutrient can be removed than that which reaches the lake via the tributary. In expectation, this constraint reads

   $$x_D E\left\{\mathbf{L}_D\right\} \leq E\left\{\mathbf{L}_D\right\} + (1-r_t)x_{SN}L_{SN} \tag{9}$$

   This relation should be applied to all realizations of $\mathbf{L}_D$, but this would introduce a stochastic constraint that will be difficult to manage in the optimization.

2. A new variable $x_U \geq x_D$ should replace x_D in the second term of (8) to account for the fact that the impact of the reservoir system on the *fluctuations* $\mathbf{L}_D - E\left\{\mathbf{L}_D\right\}$ is not restricted by the condition (9).

The general situation, when the i-th basin is fed by N_1 direct sewage discharges and N_2 tributaries each with M_m indirect sewage discharges, becomes

$$\begin{array}{rl}\triangle\mathbf{L}_i^a = & \sum_{m=1}^{N_2}\Big(x_D^m E\left\{\mathbf{L}_D^m\right\} \\ & +(x_U^m-1)\left[\mathbf{L}_D^m - E\left\{\mathbf{L}_D^m\right\}\right] \\ & +\delta\left[(x_P^m-1)(\mathbf{L}_T^m-\mathbf{L}_D^m)+E\left\{\mathbf{L}_T^m-\mathbf{L}_D^m\right\}\right] \\ & +\sum_{l=1}^{M_m}(1-r_t^{ml})x_{SN}^{ml}L_{SN}^{ml}\Big) \\ & +\sum_{n=1}^{N_1}x_S^n L_S^n\end{array} \tag{10}$$

and the equivalent version of (9) becomes

$$x_D^m E\left\{\mathbf{L}_D^m\right\} \leq E\left\{\mathbf{L}_D^m\right\} + \sum_{l=1}^{M_m}(1-r_t^{ml})x_{SN}^{ml}L_{SN}^{ml} \tag{11}$$

for each $m = 1, \ldots, N_2$.

Observations and careful analysis of the load (point versus non-point source contributions) and watershed are required for the derivation of the uncertain load components $\mathbf{L}_D$ and $\mathbf{L}_T$. Unfortunately, insufficient observations, short historical data, and our lack of understanding make the problem quite difficult; see Haith (1982) and Beck (1982). In general they have positive lower bounds and can be characterized by strongly skewed distributions. Very often they can be expressed as simple functions of annual mean streamflow rates $\mathbf{Q}$, which generally have much longer historical records than those for P loads. We return to this issue of the derivation of the load distributions in Section 6.

The final element of the model concerns the budget constraint. The cost of implementing the control options will be a combination of fixed costs for construction, and cost functions capturing the exponential growth in capital outlay and operating expense for increased P removal rates. For tractability in the optimization, the costs are modeled by piecewise linear functions increasing in the size of the control variable. To select among management alternatives of different investment costs (IC) and operational, maintenance and repair costs (OC), the total annual cost (TAC) term is used

$$\mathrm{TAC} = \sum_j(\mathrm{OC}_j + \alpha_j \mathrm{IC}_j) \tag{12}$$

where α_j is the capital recovery factor for project j. In the planning model the TAC is limited by annual budgetary constraints

$$\mathrm{TAC} \leq \beta \tag{13}$$

where β is the annual allocation of budget to the lake treatment plan, or expressing this in terms of the control variables

$$\sum_j c_j(x_j) \leq \beta. \tag{14}$$

A standard technique represents (14) as a linear constraint involving variables corresponding to each linear piece of $c_j(\cdot)$.

4 Formulation of the Eutrophication Management Optimization Model

There are a number of variants available in the building of the management optimization model that allow us to capture the stochastic features of the water quality management problem. For convenience of presentation, we substitute, regroup terms, and reindex the control variables in the model equations (1) through (10) to obtain an affine relation for the water quality indicators $(\mathbf{y}_i)_{i=1}^m$ of the type

$$\mathbf{y} = \mathbf{T}x - \mathbf{h} \tag{15}$$

where $\mathbf{h}_i$ incorporates all the noncontrollable factors that affect the water quality $\mathbf{y}_i$ in basin i, and the random coefficients associated to the $x-$variables in (10) determine the entries of the random matrix $\mathbf{T}$ through the transformation $(D + d\mathbf{w}) \triangle \mathbf{L}$. The decision variables have been reindexed as an $n-$vector $(x_1, \ldots, x_n)$, each x_j corresponding to a specific control project affecting the load in some basin i. $\mathbf{T}$ is thus a $m \times n-$matrix and $\mathbf{h}$ is an $m-$vector. We also write

$$y(x, \omega) = T(\omega)x - h(\omega) \tag{16}$$

for the preceding equation. The notation $y(x, \omega)$ stresses the dependence of the water quality indicators $y_i(x, \omega)_{i=1}^m$ on the decision variables x and on the existing (stochastic) environmental conditions ω that determine the entries of T and h.

The distribution function $G_y(x, \cdot)$ of the random vector $y(x, \cdot)$ also depends on the choice of the control measures. We could view our objective as finding x' that satisfies the constraints and such that for every other feasible x

$$G_y(x', \cdot) \geq G_y(x, \cdot), \tag{17}$$

i.e., such that for all $z \in R^m$

$$\text{prob}\left\{y(x', \cdot) < z\right\} \geq \text{prob}\left\{y(x, \cdot) < z\right\}.$$

If such an x' existed, it would, of course, be the 'absolute' optimal solution, since it guarantees the best water quality whatever be the actual realization of the random environment. There always exists such a solution if there are no budgetary limitations: simply build all possible projects to their physical upper bounds! It is precisely because there are budgetary limitations that we are led to choose a restricted number of treatment plants and/or pre-reservoirs. Unless the problem is very unusual there will be no choice of investment program that will dominate all other feasible programs in terms of the preference ordering suggested by (17).

We are thus forced to examine somewhat more carefully the objectives we want to achieve. We could, somewhat unreasonably, see the goal as bringing the water quality indicator to a near zero level in all basins. This would ignore the individual characteristics of each basin as well as the user-oriented criteria —

such as, for example, recreational versus agricultural. A more sensible approach is to choose the control measures to achieve certain desirable trophic states basin by basin. Let

$$\gamma_i, \quad i = 1, \ldots, m$$

be water quality goals expressed in terms of the indicator, $(\mathrm{Chl} - \mathrm{a})_{\max}$, each γ_i corresponding to the particular use of basin i. The sensitivity of the solution to these fixed levels γ_i would have to be a part of the overall analysis of the system. We are thus interested in the quantities

$$[y_i(x, w) - \gamma_i]_+ \qquad \text{for} \quad i = 1, \ldots, m,$$

that measure the deviations between realized water quality and the fixed goals γ_i, where $[z]_+$ denotes the nonnegative part of z

$$[z]_+ = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{if } z \geq 0. \end{cases}$$

The vector

$$[y_i(x, \cdot) - \gamma_i]_+, \qquad i = 1, \ldots, m,$$

is random with distribution function $G(x, \cdot)$ defined on R^m. The problem is to choose among all feasible control measures a program x' that generates the 'best' distribution $G(x', \cdot)$ by which one could again mean

$$G(x', z) \geq G(x, z)$$

for all $z \in R^m$.

Such an x' exists only in very unusual circumstances. We must find a way to compare the distribution functions that takes into account their particular characteristics but leads to a measure that can be expressed in terms of a scalar functional.

4.1 Reliability Criteria

A first possibility would be to introduce a pure *reliability criterion*, i.e., to fix in consultation with the decision maker certain reliability coefficients to guide in the choice of an investment program. More specifically, we would fix $0 < \alpha \leq 1$, so that among all feasible x we should restrict ourselves to those satisfying

$$\mathrm{prob}\,\{y(x, \cdot) < \gamma\} \geq \alpha. \tag{18}$$

Or preferably, if we take into account the fact that each basin should be dealt with separately, we would fix the reliability coefficients $(\alpha_i)_{i=1}^m$ and impose the constraints

$$\mathrm{prob}\,\{y_i(x, \cdot) < \gamma_i\} \geq \alpha_i, \ i = 1, \ldots, m. \tag{19}$$

where the scalars α or $(\alpha_i)_{i=1}^m$ would be chosen sufficiently large so that we would observe the unacceptable concentration level only on rare occasions. In terms of the distribution function G these constraints become

$$G(x, 0) \geq \alpha, \tag{20}$$

for (18), and

$$G_i(x, 0) \geq \alpha_i \qquad \text{for } i = 1, \ldots, m, \tag{21}$$

for (19) where the $G_i(x, \cdot)$ are the marginal distributions of the random variables $[y_i(x, \cdot) - \gamma_i]_+$. These are *probabilistic (or chance) constraints*. One refers to (18) as a *joint* probabilistic constraint. These model simple accept/reject criteria: namely, if $G(x, \cdot)$ is either larger than or equal to α, or for each $i = 1, \ldots, m$, $G_i(x, \cdot)$ is larger than or equal to α_i, then the investment program x is acceptable. This means that we 'compare' the possible distributions $\{G(x, \cdot), x \text{ feasible }\}$ at the single point α.

Assuming we opt for the more natural separable version of the probabilistic constraints (19), we would rely on the following model for the policy analysis

$$\begin{array}{lll} \text{find } x \in \mathrm{R}^n \text{ such that} & & \\ \quad r_j^- \leq x_j \leq r_j^+, & j = 1, \ldots, n, & \\ \quad \sum_{j=1}^n \alpha_{ij} x_j \leq b_i, & & \\ \quad \text{prob}\left\{\sum_{j=1}^n t_{ij}(\omega)\ x_j - h_i(\omega) < \gamma_i\right\} \geq \alpha_i, & i = 1, \ldots, m, & \\ \text{and } z = \sum_{j=1}^n c_j(x_j) \text{ is minimized} & & \end{array} \tag{22}$$

where as before the vector r^- and r^+ are upper and lower bounds on x, the inequalities $\sum_{j=1}^n \alpha_{ij} x_j \leq b_i$ describe the technological constraints and for every j

$$c_j : R \to R_+$$

is the cost function associated to project j, see (14). The overall objective would thus be to find the smallest possible budget that would guarantee meeting the present goals γ_i at least a portion α_i of the time.

We do not pursue this approach because it does not allow us to distinguish between situations where we almost meet the goals γ_i and those that generate 'catastrophic' situations, i.e., when some of the values of the $(y_i(x, \omega))_{i=1}^m$ would exceed by far $(\gamma_i)_{i=1}^m$. For a eutrophication model this is a serious shortcoming.

Let us also point out that probabilistic contraints involving affine functions with random coefficients are difficult to manage. We have only very limited knowledge about such constraints, and then only if the random coefficients $((t_{ij}(\cdot))_{j=1}^n, h_i(\cdot))$ are jointly normally distributed (cf. Section 1 of Wets (1983b) for a survey and relevant references). Since in environmental problems the coefficients are generally not normally distributed random variables we could not even use the few results that are available, except possibly by replacing the probabilistic constraints by approximations using Chebyshev's inequality, as suggested by Sinha, cf. Proposition 1.26 in Wets (1983b).

4.2 Recourse Formulation

A second possibility is to recognize that one should distinguish between situations that barely violate the desired water quality or levels $(\gamma_i)_{i=1}^m$ and those that deviate substantially from these norms. This suggests a formulation of our objective in terms of a penalization that would take into account the observed values of $[y_i(x,\omega) - \gamma_i]_+$ for $i = 1, \ldots, m$. We expect such a function

$$\Psi : \mathrm{R}^m \to \mathrm{R}$$

to have the following properties

(i) Ψ is nonnegative.

(ii) $\Psi(z) = 0$ if $z_i \leq 0$ for $i = 1, \ldots, m$,

(iii) Ψ is separable, i.e., $\Psi(z) = \sum_{i=1}^m \Psi_i(z_i)$.

This last property comes from the fact that the objectives for each basin are or may be different and there are essentially no 'joint rewards' to be accrued from having given concentration levels in neighboring basins, the interconnections between the basins being already modeled through the Equation (1). A more sophisticated model would still work with separate penalty functions $(\Psi_i(z_i))_{i=1}^m$ but instead of simply summing these penalties would treat them as multiple objectives. A solution to such a problem would eventually assign specific weights to each basis, making it equivalent to an optimization problem with single objective function. We assume that these weighting factors have been made available to or have been discovered by the model builder and have been incorporated in the functions Ψ_i themselves; note however that the methodology developed here would apply equally well to a multiple objective version of the model. In addition, to (i)-(iii) we would expect the following properties for $i = 1, \ldots, m$,

(iv) Ψ_i is differentiable, with derivative Ψ_i',

(v) Ψ_i' is monotone increasing, i.e., Ψ_i is convex,

(vi) $\Psi_i'(z_i) > 0$ whenever $z_i > 0$,

- relatively small if z_i is 'close' to 0,
- leveling off when z_i is much 'larger' than 0,

A couple of possibilities, both with $\Psi_i(z_i) = 0$ if $z_i \leq 0$, are

$$\Psi_i(z_i) = \beta_i z_i^2 \text{ if } z_i \geq 0,$$

with $\beta_i > 0$,

$$\Psi_i(z_i) = \beta_i(e^{z_i} - z_i - 1) \text{ if } z_i \geq 0,$$

also with $\beta_i > 0$.

A wide variety of functions have the desired properties, since what is at stake here is the creation of a (negative) utility function that measures the socio-economic consequences of the deterioration of the environment. The following class of functions provided a flexible tool for the analysis of these factors. Let $\Theta : \mathrm{R} \to \mathrm{R}_+$ be defined by

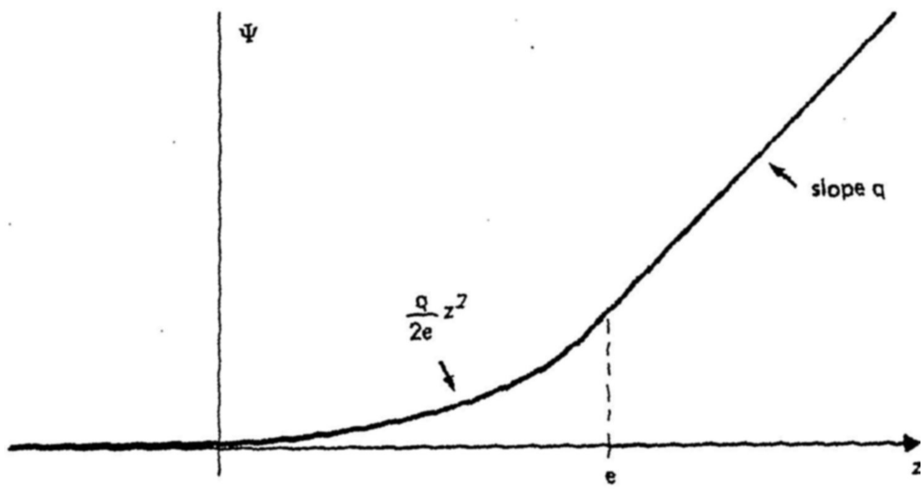

Figure 2: Criteria functions.

$$\Theta(\tau) := \begin{cases} 0 & \text{if } \tau \leq 0 \\ 1/2\tau^2 & \text{if } 0 \leq \tau \leq 1 \\ \tau - 1/2 & \text{if } \tau \geq 1. \end{cases} \tag{23}$$

This is a *piecewise linear-quadratic function*. The functions $(\Psi_i)_{i=1}^m$ are defined through

$$\Psi_i(z_i) = q_i e_i \Theta(e_i^{-1} z_i) \qquad \text{for} = 1, \ldots, m, \tag{24}$$

where q_i and e_i are positive quantities that allow us to scale each function Ψ_i in terms of slopes and the range of its quadratic component. By varying the parameters e_i and q_i we are able to model a wide range of preference relationships and study the stability of the solution under perturbation of these scaling parameters.

The objective is thus to find a program that in the average minimizes the penalties, or negative utilities, associated with exceeding the desired concentration levels. This leads us to the following formulation of the water quality management problem

$$\begin{array}{ll} \text{find } x \in R^n \text{ such that} & \\ r_j^- \leq x_j \leq r_j^+, & j = 1, \ldots, n, \\ \sum_{j=1}^n \alpha_{ij} x_j \leq b_i, & i = 1, \ldots, m, \\ \sum_{j=1}^n c_j(x_j) \leq \beta, & \\ \sum_{j=1}^n t_{ij}(\omega) x_j - y_i(\omega) = h_i(\omega), & i = 1, \ldots, m, \\ \text{and } z = E\left\{\sum_{i=1}^m q_i e_i \Theta\ (e_i^{-1}[y_i(\omega) - \gamma_i])\right\} \text{ is minimized} & \end{array} \tag{25}$$

where β is the available budget. This type of stochastic optimization problem goes under the name of *stochastic program with recourse*: a decision x

(the investment program) must be chosen before we can observe the outcome of the random events (the environment modeled here by the random quantities $t_{ij}(\omega), h_i(\omega)$) at which time a recourse decision is selected so as to make up whatever discrepancies there may be; the variables y_i are just measuring the difference between $\sum_j t_{ij}x_j$ and h_i. One refers to (25) as a program with *simple recourse* in that the recourse decision is uniquely determined by the first-stage decision x and the values taken on by the random variables.

It is very important to note that no attempt has been made at combining budgetary considerations and the penalty functions that measure the deviations from the desired concentration levels in a single objective function, although there are financial considerations that may affect the choice of the coefficients q_i and e_i of the penalty terms. In our approach we handle these two criteria separately. We rely on a (discrete) parametric analysis of the solution of (25) as a function of β, the available budget. An essentially equivalent approach would have been to formulate (25) as a multi-objective program with one objective corresponding to the penalizations terms and the other to the cost function.

In terms of the distribution functions $\{G(x,\cdot), x \text{ feasible}\}$ the entire 'tail' of the distributions enters into the comparison not just the value of $G(x,\cdot)$ at 0, as was the case in model (22) with probabilistic contraints. The objective function is

$$z = \sum_{i=1}^{m} q_i e_i \int_0^{\infty} \Theta(e_i^{-1} s) dG_i(x,s).$$

4.3 Expected Value Model

A third possibility is to essentially ignore the stochastic aspects of the eutrophication model and replace the random variables that appear in the formulation of the water quality management problem by fixed quantities. This would lead us to the following *deterministic optimization problem*:

$$\begin{array}{ll} \text{find } x \in R^n \text{ such that} & \\ r_j^- \le x_j \le r_j^+, & j = 1,\ldots,n, \\ \sum_{j=1}^n \alpha_{ij} x_j \le b_i, & i = 1,\ldots,m, \\ \sum_{j=1}^n c_j(x_j) \le \beta, & \\ \sum_{j=1}^n \hat{t}_{ij} x_j - y_i = \hat{h}_i, & i = 1,\ldots,m, \\ \text{and } z = \sum_{i=1}^m q_i e_i \Theta\left(e_i^{-1}[y_i - \gamma_i]\right) \text{ is minimized} & \end{array} \tag{26}$$

The choice of the parameters $\hat{t}_{ij}$ and $\hat{h}_i$ is left to the model builder. One possibility is to choose

$$\begin{aligned} \hat{t}_{ij} = \bar{t}_{ij} &= E\left[t_{ij}(\omega)\right], \\ \hat{h}_i = \bar{h}_i &= E\left[h_i(\omega)\right], \end{aligned}$$

i.e., replace the random quantities by their expectations. Without accepting the solution of (26), we could always use it as part of an initialization scheme for solving the stochastic optimization problem (25), and this is actually how the algorithm proceeds, see Section 5.

4.4 Optimal Reliability Levels

A fourth model in which reliability considerations again occupy a central role, but in which the shapes of the distribution functions $\{G_i(x,\cdot)\}_{i=1}^m$ play a much more important role than just their values at one point, allows for variable concentration levels. Again let $(\alpha_i)_{i=1}^m$ be scalars that correspond to desired reliability level. The objective is to find an investment program x such that

$$\text{prob}\{y_i(x,\cdot) < v_i\} \geq \alpha_i \quad \text{for } i=1,\ldots,m \tag{27}$$

but now the $(v_i)_{i=1}^m$ are also decision variables that we would like to choose as low as possible. There are a variety of ways measuring 'as low as possible', for example by minimizing

$$\sum_{i=1}^m q_i[v_i - \gamma_i]_+ \tag{28}$$

where the q_i are nonnegative scalars that assign different importance to meeting the desired water quality goals $(\gamma_i)_{i=1}^m$ in the various basins, or by minimizing

$$\max\nolimits_i \ v_i \tag{29}$$

i.e., by bringing the overall concentration level as far down as possible (at least a certain portion of the time determined by the $\alpha_i's$), or by minimizing as in model (25) the function

$$\sum_{i=1}^m q_i e_i \Theta\left(e_i^{-1}\ (v_i - \gamma_i)\right) \tag{30}$$

which penalizes the deviations from γ_i in a nonlinear manner, cf. Figure 2, or still to handle the minimization of the $(v_i)_{i=1}^m$ as a multiple objective optimization problem, each coordinate of v corresponding to an objective that we seek to minimize.

We formulate our optimization problem in terms of the objective (28) but any of the other variants could or should be considered. The optimization problem again involves probabilistic constraints but its structure now resembles much more the stochastic program with recourse (25) than it does the first model (22) involving probabilistic constraints. We obtain

$$\begin{array}{ll} \text{find } x \in R^n \text{ such that} & \\ \quad r_j^- \leq x_j \leq r_j^+, & j=1,\ldots,n, \\ \quad \sum_{j=1}^n \alpha_{ij}x_j \leq b_i, & i=1,\ldots,m, \\ \quad \sum_{j=1}^n c_j(x_j \leq \beta & \\ \quad \text{prob}\left\{\sum_{j=1}^n t_{ij}(\omega)\ x_j - h_i(\omega) - v_i < 0\right\} \geq \alpha_i, & i=1,\ldots,m, \\ \text{and } z = \sum_{i=1}^m q_i[v_i-\gamma_i]_+ \text{ is minimized.} & \end{array} \tag{31}$$

At this point it may be worthwhile to observe that (28) is just a limit case of (30). Recall that the range over which $q_i\ e_i\Theta(e_i^{-1}(\cdot - \gamma_i))$ is quadratic is $[0, e_i]$, cf. Figure 2. If we shrink this interval to 1 point, we are left with the piecewise linear function $q_i\,[\cdot - \gamma_i]_+$.

As for our earlier models, we should study the solution as a parametric function of β the available budget. However, solving (31) presents all the technical challenges mentioned in connection with the first model (22) involving probabilistic constraints. The presence of the $(v_i)_{i=1}^m$ has in no way simplified the problem, and in fact we do not know of any direct method for solving (31). One possibility is to find an approximation of (31) that could be handled by available linear or nonlinear programming techniques. We return to this in the next section.

4.5 Expectation-Variance Formulation

A fifth possibility is to deploy a model as in Somlyódy (1983a) that is based on expectation-variance considerations for the water quality indicators. The justification of the model relies on the validity of certain approximations, and one should be prepared to accept the solution with some circumspection. In the Lake Balaton Case Study the results for this expectation-variance model were confirmed by those obtained for the more accurate stochastic programming model (25)as shown in Section 6.

As a starting point for the construction of this model, consider the recourse objective function:

$$\sum_{i=1}^{m} q_i E\left\{(y_i(x,\cdot) - \gamma_i)_+^2\right\} \tag{32}$$

The objective being quadratic in the area of interest, and the distribution functions $G_i(x,\cdot)$ of the $y_i(x,\cdot)$ not being too far from normal, one should be able to recapture the essence of its effect on the decision process by considering just expectations and variances. This observation and the 'soft' character of the management problem (which means that there is a large degree of flexibility in the choice of the objective) suggest that we could substitute

$$\sum_{i=1}^{m} q_i \left(E\left\{y_i(x,\cdot) - \bar{y}_{0i}\right\} + \Theta\sigma\left\{y_i(x,\cdot) - \bar{y}_{0i}\right\}\right) \tag{33}$$

for (32) where Θ is a positive scalar (usually between 1 and 2.5), $\bar{y}_{0i} = E\left\{y_{0i}\right\}$ is the expected nominal state of basin i, and $\sigma\left\{\cdot\right\}$ denotes the standard deviation operator,

$$\sigma\left\{y_i(x,\cdot) - \bar{y}_{0i}\right\} = E\left\{\left(y_i(x,\cdot) - E\left\{y_i(x,\cdot)\right\}\right)^2\right\}^{1/2}. \tag{34}$$

Since for each $i = 1, \ldots, m$, the y_i are affine (linear plus a constant term) with respect to x, the expression for

$$E\left\{y_i(x,\cdot) - \bar{y}_{0i}\right\} = \sum_{j=1}^{n} \mu_{ij} x_j + \mu_{i0}$$

as a function of x is easy to obtain from equations (1) and (10). The μ_{ij} are the expectations of the coefficients of the x, and the μ_{i0} the expectation of the constant term. Unfortunately the same does not hold for the standard deviation $\sigma\{y_i(x,\cdot)-\bar{y}_{0i}\}$. Equations (1) and (10) suggest that

$$\sigma\{y_i(x,\cdot)-\bar{y}_{0i}\} \sim \left(\sum_j \sigma_{ij}^2 x_j^2\right)^{1/2} \tag{35}$$

where σ_{ij} is the part of the standard deviation that can be influenced by the decision variable x_j; for example, the standard deviation of the tributary load $L(\omega)_D$. Cross terms are for all practical purposes irrelevant in this situation since the total load in basin i is essentially the result of a sum of the loads generated by various sources that are independently controlled. This justifies using

$$\sum_{i=1}^{m} q_i \left[\left(\sum_{j=1}^{n} \mu_{ij}x_j\right) + \Theta\left(\sum_{j=1}^{n}\sigma_{ij}^2 x_j^2\right)^{1/2}\right] \tag{36}$$

instead of (33) as an objective for the optimization problem. This function is convex and differentiable on R_+^n except at $x=0$, and conceivably one could use a nonlinear programming package to solve the optimization problem:

$$\begin{array}{ll} \text{find } x \in \mathrm{R}^n \text{ such that} & \\ r_j^- \le x_j \le r_j^+, & j=1,\dots,n, \\ \sum_{j=1}^n \alpha_{ij}x_j \le b_i, & i=1,\dots,m, \\ \sum_{j=1}^n c_j(x_j) \le \beta & \\ \text{and } z = \sum_{i=1}^m q_i\left[\sum_{j=1}^n \mu_{ij}x_j + \Theta\left(\sum_{j=1}^n \sigma_{ij}^2 x_j^2\right)^{1/2}\right] \text{ is minimized.} & \end{array} \tag{37}$$

We can go one step further in simplifying the problem to be solved, namely, by replacing the term

$$\left(\sum_{j=1}^{n}\sigma_{ij}^2 x_j^2\right)^{1/2}$$

in the objective, by the linear (inner) approximation

$$\sum_{j=1}^{n}\sigma_{ij}x_j.$$

On each axis of R_+^n, no error is introduced by relying on this linear approximation, otherwise we are over-estimating the effect a certain combination of the x_j's will have on the variance of the concentration levels. Thus, at a given budget level we shall have a tendency to start projects that affect more strongly the variance if we use the linear approximation, and this is actually what we observed in practice (see Section 6). Assuming the cost functions c_j are piecewise linear, we have to solve the *linear* program

$$\begin{array}{ll} \text{find } x \in R^n \text{ such that} & \\ r_j^- \leq x_j \leq r_j^+, & j = 1, \ldots, n, \\ \sum_{j=1}^n \alpha_{ij} x_j \leq b_i, & i = 1, \ldots, m, \\ \sum_{j=1}^n c_j(x_j) \leq \beta & \\ \text{and } z = \sum_{i=1}^m q_i \sum_{j=1}^n (\mu_{ij} + \Theta\sigma_{ij}) x_j \text{ is minimized.} & \end{array} \tag{38}$$

We refer to this problem as the *linearized expectation-variance model*; see also Somlyódy (1983a) and Somlyódy and van Straten (1985).

For the sake of illustration, let us consider the $i-th$ basin of Figure 1 and suppose that there is no mass exchange with neighboring basins. To obtain a linear form in the x_j, we proceed as indicated in equation (9). To derive the remaining term in the objective of (38) we only need to consider the controllable portion of the variance of $\mathbf{y}_i(x) - \bar{y}_{0i}$. We rewrite (1) as

$$\mathbf{y}_i(x) - \bar{y}_{0i} = \mathbf{w}_i - (d_{ii} + d_i \mathbf{w}_i)\Delta\mathbf{L}_i.$$

and

$$\Delta\mathbf{y}_i = (d_{ii} + d_i \mathbf{w}_i)\Delta\mathbf{L}_i.$$

Assume that $\mathbf{w}_i$ and $\Delta\mathbf{L}_i$ are independent, then

$$\sigma^2\{\Delta\mathbf{y}_i\} = d_{ii}^2\sigma^2\{\Delta\mathbf{L}_i\} + d_i^2\sigma^2\{\mathbf{w}_i\}\left[\,\sigma^2\{\Delta\mathbf{L}_i\} + E^2\{\Delta\mathbf{L}_i\,\}\right]. \tag{39}$$

From equations (3) and (6-8) we have

$$\sigma^2\{\Delta\mathbf{L}_i\} = \frac{1}{V_i^2}\left[\,\delta^2(x_p-1)^2\sigma^2\{\mathbf{L}_T - \mathbf{L}_D\} + (x_D-1)^2\sigma^2\{\mathbf{L}_D\}\right]. \tag{40}$$

where we have made the plausible assumption that the measurement uncertainties in the tributary dissolved load $\mathbf{L}_D$ and undissolved particulate load $\mathbf{L}_T - \mathbf{L}_D$ are independent. This would lead to an expression for $\sigma\{\Delta\mathbf{y}_i\}$ that would be nonlinear in the x variables. To avoid the nonlinearities we specify $\sigma_a\{\Delta\mathbf{y}_i\}$ and $\sigma_a\{\Delta\mathbf{L}_i\}$ as the linear combination of the additive terms in (39) and (40)

$$\sigma_a\{\Delta\mathbf{y}_i\} := d_{ii}\sigma_a\{\Delta\mathbf{L}_i\} + d_i\sigma\{\mathbf{w}_i\}\left[\sigma_a\{\Delta\mathbf{L}_i\} + E\{\Delta\mathbf{L}_i\,\}\right] \tag{41}$$

and

$$\sigma_a\{\Delta\mathbf{L}_i\} := \frac{1}{V_i}\,\left[\delta(x_P-1)\sigma\{\mathbf{L}_T - \mathbf{L}_D\} + (x_D-1)\sigma\{\mathbf{L}_D\}\right]. \tag{42}$$

In equations (39) and (40) all the coefficients of x are positive and the behavior of the 'new' σ_a is similar to the standard deviations as defined through (39) and (40). Substituting the terms involving x in (42) into (41) yields

$$\sigma_{ai} = V_i^{-1}\,\Big(x_P\left[(d_{ii} + d_i\sigma\{\mathbf{w}_i\})\,\delta\,(\sigma\{\mathbf{L}_T\} - \sigma\{\mathbf{L}_D\})\right.$$

$$+d_i\sigma\{\mathbf{w}_i\}\,\delta\,(E\{\mathbf{L}_T\} - E\{\mathbf{L}_D\})]$$
$$+x_D\,[(d_{ii} + d_i\sigma\{\mathbf{w}_i\}\,\sigma\{\mathbf{L}_D\}) + d_i\sigma\{\mathbf{w}_i\}\,(E\{\mathbf{L}_D\} - (1-r_t)x_{SN}L_{SN})]$$
$$+x_{SN}d_i\sigma\{\mathbf{w}_i\}\,L_{SN} + x_s d_i\sigma\{\mathbf{w}_i\}\,L_S\Big)$$

Collecting terms we obtain the coefficients σ_{ij} that appear in the objective of the linear program (38). (A more detailed, but similar, derivation also yields the expression for the standard derivation when there is mass exchange between neighboring basins.)

The arguments that we have used to justify the expectation-variance model are mostly of a heuristic nature, in that they rely on a good understanding of the problem at hand and 'engineering' intuition. In the formulation of the models of this section, the objective has usually been formulated in terms of finding control measures such that the observed concentration levels (water quality indicators) are not too far from pre-set goals (given trophic states). If by 'not too far' we mean that

$$E\{\mathbf{y}_i(x)\} + \sigma\{\mathbf{y}_i(x)\} - \gamma_i \tag{43}$$

should be as small as possible, we could also reformulate the problem, in terms of the nominal concentration levels. Instead of minimizing (43), we could maximize

$$E\{\Delta\mathbf{y}_i(x)\} - \sigma\{\Delta\mathbf{y}_i(x)\} \tag{44}$$

and this should give about the same results. This is the motivation behind the formulation of (38), see Figure 3.

There is, however, another approach that does not rely so extensively on heuristic considerations, which leads us to the model (37), i.e., the nonlinear version of the expectation-variance model. The fourth model, described in Section 4.4, which integrates both reliability considerations and penalties for fixing the reliability levels, led us to the nonlinear program

$$\begin{array}{ll}
\text{find } x \in R^n \text{ such that} & \\
\quad r_j^- \le x_j \le r_j^+, & j = 1,\ldots,n, \\
\quad \sum_{j=1}^n \alpha_{ij}x_j \le b_i, & i = 1,\ldots,m, \\
\quad \sum_{j=1}^n c_j(x_j) \le \beta & \\
\quad \text{prob}\left\{\sum_{j=1}^n t_{ij}(\omega)\; x_j - h_i(\omega) - y_i < 0\right\} \ge \alpha_i, & i = 1,\ldots,m, \\
\text{and } z = \sum_{i=1}^m q_i[y_i - \gamma_i]_+ \text{ is minimized.} &
\end{array}$$

Because these probabilistic constraints are very difficult to handle, we may consider finding an approximate solution by replacing the probabilistic constraints by

$$(1-\alpha_1)^{-2}\left(\sum_{j=1}^n\sum_{k=1}^n\; \sigma_{ijk}x_jx_k\right)^{1/2} + \sum_{j=1}^n \mu_{ij}x_j \le y_i \tag{45}$$

where

$$t_{0i}(\cdot) = -h_i(\cdot),$$

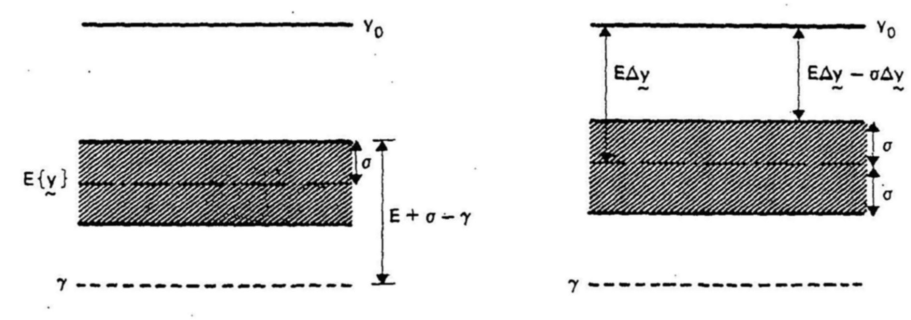

Figure 3: Objective of expectation-variance model.

and for $j = 1, \ldots, n$ and $k = 1, \ldots, n$

$$\mu_{ij} := E\{t_{ij}(\omega)\},$$

$$\sigma_{ijk} := \mathrm{cov}\,(t_{ij}(\cdot), t_{ik}(\cdot)).$$

If the random variables $\{t_{ij}(\cdot)_{j=1}^{j}\}$ are jointly normal, then the restrictions generated by the deterministic constraints (45) are exactly the same as those imposed by the probabilistic constraints, but in general, they are more restrictive, cf. Propositions 1.25 and 1.26 in Wets (1983b) without going into the details, we can see that (45) is obtained by applying Chebyshev's inequality and this, in general, determines an upper bound for the probabilistic event

$$\left\{\omega | \sum_{j=1}^{n} t_{ij}(\omega) x_j \;\; \geq 0\right\}.$$

So, given x, if we can justify a near normal behavior for the random variable

$$\sum_{j=1}^{n} t_{ij}(\omega) x_j - h_i(\omega) =: \;\; y_i(x, \omega),$$

we can use the constraints (45) instead of the probabilistic constraints to obtain an approximate solution of (31). In this setting, 'near normality' of the $\mathbf{y}_i(x)$ is a much more natural, and weaker, assumption than normality of the $t_{ij}(\omega)$. If we proceed in this fashion, we obtain the nonlinear program

find $x \in R^n$ such that
$r_j^- \leq x_j \leq r_j^+, \qquad j = 0, \ldots, n,$
$\sum_{j=1}^{r} \alpha_{ij} x_j \leq b_i, \quad i = 1, \ldots, m,$
$\sum_{j=1}^{n} c_j(x_j) \leq \beta,$
and $z = \sum_{i=1}^{m} q_i \left[\sum_{j=1}^{n} \; \mu_{ij} x_j + (1-\alpha_i)^{-2} (\sum_{j=1}^{n} \sum_{k=1}^{n} \; \sigma_{ijk} x_j x_k)^{1/2} - \gamma_i \;\right]_+$
is minimized.

(46)

We have eliminated the variables $(y_i, i = 1, \ldots, m)$ from the formulation of the problem by using the fact that the optimal y_i^* can always be chosen so that (45) is satisfied with equality. Moreover, if the desired concentration levels γ_i are low enough, then we know that the optimal solution will always have $y_i^* > \gamma_i$ and thus we can rewrite (46) as

$$
\begin{array}{l}
\text{find } x \in R^n \text{ such that} \\
\quad r_j^- \le x_j \le r_j^+, \qquad j = 0, \ldots, n, \\
\quad \sum_{j=1}^r \alpha_{ij} x_j \le b_i, \quad i = 1, \ldots, m, \\
\quad \sum_{j=1}^n c_j(x_j) \le \beta \\
\text{and } z = \sum_{i=1}^m q_i \left[\sum_{j=1}^n \ \mu_{ij} x_j + (1 - \alpha_i)^{-2} (\sum_{j=1}^n \sum_{k=1}^n \ \sigma_{ijk} x_j x_k)^{1/2} - \gamma_i \right] \\
\text{is minimized.}
\end{array}
\tag{47}
$$

The objective of this optimization problem is sublinear, i.e., convex and positively homogeneous. Assuming that the cost functions c_j are linear, or more realistically have been linearized (14), we are thus confronted with a nearly linear program that we could solve by specially designed subroutines (nondifferentiability at 0), or by a linearization scheme that would allow us to use linear programming packages. The nonlinear program (47) is exactly of the same type as (37) if we make the following adjustments:

(i) in the objective of (47) replace the covariance term $\sum_{j=1}^n \sum_{k=1}^n \ \sigma_{ijk} x_k x_j$ by the sum of the variances $\sum_{j=1}^n \sigma_{ijk} x_j^2$;

and

(ii) if for all $i = 1, \ldots, m$, the α_i are the same set, $\Theta = (1 - \alpha_i)^{-1}$, otherwise we replace Θ by $\Theta_i = (1 - \alpha_i)^{-2}$ in (37).

To justify (i), we appeal to (35).

In the derivation that led us from (31) to (47), we stressed the fact that the solution of (47) and thus equivalently of (37), would be feasible for the original program (31), and that, in fact, it would more than meet the probabilistic constraints specified in (31). The further linearization of the objective bringing us from (37) to (38) overstresses (only slightly, one hopes) the role that the variance will play in meeting the prescribed reliability levels. In terms of model (31), we can thus view the solution of (38) as a 'conservative' solution that overestimates the importance to be given to the stochastic aspects of the problem. In that sense, the solution of (38), especially in comparison to that of the deterministic problem (26), always indicates how we should adjust the decisions so as to take into account the stochastic features of the problem.

In our analysis (see Section 6), we have used the linear programming version (38) of this expectation-variance model; the wide availability of reliable linear programming packages makes it easy to implement, and thus an attractive approach, provided one keeps in mind the reservations expressed earlier.

5 Solving the Stochastic Model

We briefly outline here the method used to solve the full stochastic version of the eutrophication management model (25). We recognize it as a stochastic program with quadratic simple recourse, with stochastic technology matrix T and stochastic right-hand side h. When only h is stochastic and the objective function is piecewise linear, efficient procedures are available; Wets (1983a). But in order to deal with this problem new techniques were required.

A procedure called the Lagrange finite generation method was developed Rockafellar and Wets (1985) that exploits the properties of the dual associated to problem (25). Each iteration ν of the algorithm solves the dual over a convex hull spanned by a given basis of dual solutions. This is a finite-dimensional quadratic program that produces a feasible primal solution x^ν. A new dual solution is calculated from x^ν, added to the collection of dual solutions, and the cycle repeats. (The procedure is started by solving the deterministic problem (26) with expected values for the stochastic parameters.) The Lagrange finite generation algorithm produces a sequence $\{x^\nu\}$ that converges at a known rate to an optimal solution of the original stochastic program. An experimental version of this algorithm was implemented at IIASA by A. King; King, Rockafellar, Somlyódy, and Wets (1988).

The distribution used to calculate the dual solutions is derived by sampling from the stochastic model (10). For a number of reasons (including numerical stability considerations) it is recommended to start with a relatively small sample, increasing its size only for verification purposes. We found that a relatively small sample (about 50) will gave surprisingly accurate results, as was confirmed by studying the performance of the obtained solutions using independent simulations. (The early literature on the statistical convergence of solutions for sampled stochastic programs was inspired by this sampling approach to solving the Lake Balaton problem; see King (1986).) Asymptotic error bounds for the solutions of piecewise linear-quadratic problems were derived in King and Rockafellar (1993). See Shapiro (2000) for a survey.

6 APPLICATION TO LAKE BALATON

Lake Balaton (Figure 4), one of the largest shallow lakes of the world, which is also the center of the most important recreational areas in Hungary, has recently exhibited the unfavorable signs of artificial eutrophication. An impression of the major features of the lake-region system, the main processes and activities, the underlying research, data availability and control alternatives can be gained from Figure 4 and Table 1; for details see Somlyódy, Herodek, and Fischer (1983) and Somlyódy and van Straten (1985). Four basins of different water quality can be distinguished in the lake determined by the increasing volumetric nutrient load from east to west. The absolute loads are roughly equal for the four basins, but the biologically available load, BAP, is about ten times higher in Basin I than in Basin IV (see Table 1, line 6). This is due to the asymetric geometry of the system: the smallest western basin drains half of the total watershed, while only 5% of the catchment area belongs to the larger basin (Table 1, lines 2 and 3).

Based on observations for the period 1971-1982 the average deterioration of water quality of the entire lake is about 20% (in terms of Chl-a). According to the OECD classification, the western part of the lake is in a hypertrophic, while the eastern portion of it is in an eutrophic stage (Table 1, line 7).

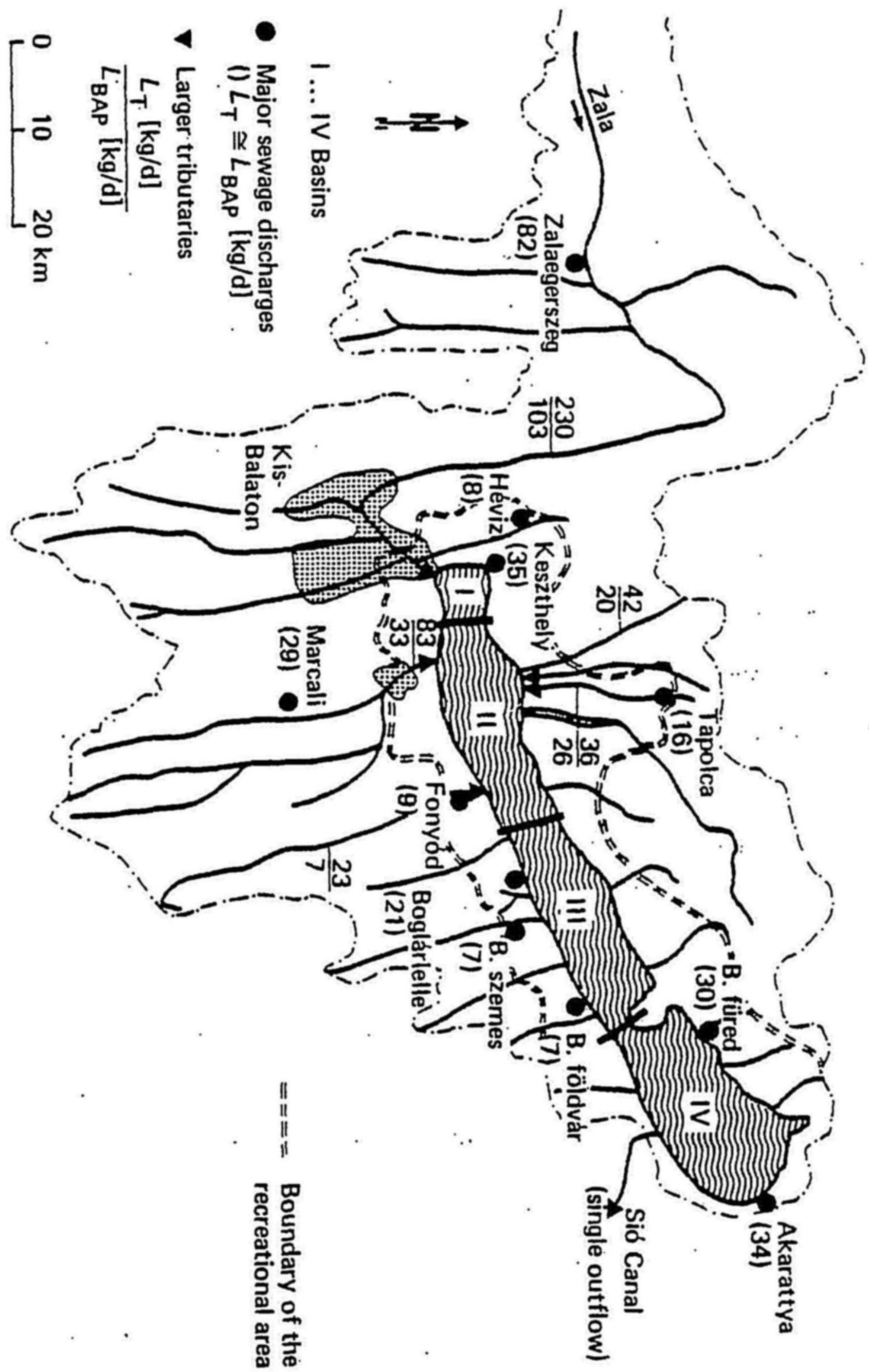

Figure 4: Lake Balaton: Major nutrient sources and control options.

1.	**Basin**	**I**	**II**	**III**	**IV**	**Lake**
2.	**Watershed area** [km^2]	2750	1647	534	249	5180
3.	**Lake surface area** [km^2]	38	144	186	228	536
4.	**Volume** [$10^6 m^3$]	82	413	600	802	1907
5.	**Depth** [m]	2.3	2.9	3.2	3.7	3.2
6.	**BAP load** [$mg/m^3 d$]	1.70	0.30	0.15	0.14	0.25
7a.	$(\mathrm{Chl}-\mathrm{a})_{\max}[\mathrm{mg/m}^3]$	75	38	28	20	(late 70's)
7b.		150	90	60	35	(1982)

8. Use of the watershed. Agriculture and intensive tourism (main season: July and August)

9. Climatic influences. No stratification; large fluctuation in temperature (up to $25 - 28^o C$); 2-month ice cover; strong wind action

10. Eutrophic Status. Hyper-eutrophic state: P limitation till the end of the seventies; large year-to-year fluctuation in Chl-a depending on meteorology and hydrology; 20% per year increase in Chl-a during 1971-1982; marked longitudinal gradient

11. Sediment. Internal load nowadays is roughly equal to the external BAP load

12. Data. Long hydrological and weather records; regular water quality and load survey since 1971 and 1975, respectively

13. Research. Increasing activity in Hungary in various institutes during the past 30 years; joint study of IIASA, the Hungarian Academy of Sciences and the Hungarian National Water Authority, 1978-1982, see Somlyódy et al. (1983).

14. Models developed. Various alternative models, see Somlyódy et al. (1983)

15. Methodologies. ODE and PDE models, regression analysis, Kalman filtering, time series analysis, Monde Carlo simulations, uncertainty analyses, optimization techniques

16. Measures of short-term control. P precipitation on existing treatment plants; pre-reservoirs

17. Policy making. Government decision in 1983: P control is under realization (as of 1985).

Table 1: Major features of Lake Balaton and its watershed.

The lakes' total P load $\mathbf{L}_T$ is on average $315t/yr$ (the BAP load is $170t/yr$) but depending on the hydrologic regime it can reach $550t/yr$. 53% of total load is carried by tributaries 30% of which is of sewage origin (see e.g., the largest city of the region, Zalaegerszeg in Figure 4). 17% of total load is associated to direct sewage discharges. Atmospheric pollution is responsible for 8% of the lake's P load and the rest is from direct runoff (urban and agricultural). Tributary load increases from east to west, while the change in the direct sewage load goes in the opoosite direction. The sewage contribution (direct and indirect loads) to $\mathbf{L}_T$ is 30%, but it contributes about 52% of the biologically available load $\mathbf{L}_D + \delta(\mathbf{L}_T - \mathbf{L}_D)$. The load of agricultural origin can be estimated as 47 and 33%, respectively. This suggests the importance of sewage load from the viewpoint of the short-term eutrophication control. Figure 4 indicates also the loads of sewage discharges and tributaries which were involved in the management optimization model. These cover about 85% of the nutrient load[1] which we consider controllable on the short term.

Control alternatives are sewage treatment (upgraded biological treatment and P precipitation) and the establishment of pre-reservoirs as indicated in Figure 4. The Kis-Balaton reservoir system is planned for a surface area of about $75km^2$. Besides Hungarian research activities, the problem of Lake Balaton was studied in the framework of a four-year cooperative research project on Lake Balaton involving the International Institute for Applied Systems Analysis, IIASA (Laxenburg, Austria), the Hungarian Academy of Sciences, and the Hungarian National Water Authority; Somlyódy (1982 and 1983b); Somlyódy and van Straten (1985). The development of the management model to be discussed here formed a part of the Case Study. The results achieved were then utilized in 1982[2] in the policy making procedure associated with the Lake Balaton water quality problem which was completed by a governmental decision in 1983 (Láng, 1985).

6.1 Specification of elements of EMOM for Lake Balaton

(a) Nutrient Load model
The nutrient load model for Lake Balaton can be derived on the basis of Figure 4 from relation (10). The tributary loads $\mathbf{L}_T$ and $\mathbf{L}_D$ are computed from regression models; Somlyódy and van Straten (1985).

$$\mathbf{L} = (L_0 + \alpha_1 \ \mathbf{Q} + \mathbf{L}_\rho)(\xi^- + \hat{\xi}) \tag{48}$$

where $\mathbf{Q}$ is the stream flow rate, $\mathbf{L}_\rho$ is the residual, and the variable $\hat{\xi}$ accounts for the influence of infrequent sampling (ξ^- is the lower bound). The most detailed data set, consisting of 25 years of continuous records for $\mathbf{Q}$ and 5 years of daily observations for the loads, was available for the Zala River[3] (see Figure 4)

[1]The rest represented by several small creeks and sewage outlets were neglected for the sake of simplicity.

[2]At that time only the expectation-variance model was available.

[3]Its annual load estimated from daily data can be considered accurate.

draining half of the watershed and representing practically the total load of Basin I. For the Zala River $\mathbf{L}_\rho$ was found to have a normal distribution, while $\mathbf{Q}$ was approached by a lognormal distribution. The loads of other tributaries were established on the basis of much more scarce observations. For modeling the uncertainty component of $\hat{\xi}$, first a Monte Carlo analysis was performed on the Zala River data by assuming various sampling strategies. Subsequently, the conclusions were extended to the other rivers and the parameters of the (assumed) gamma distributions of $\hat{\xi}$ were estimated.

(b) Control variables and cost functions
All the optimization models implemented use real-valued control variables. Integer $\{0, 1\}$ variables for the two reservoir systems (see Figure 4)were also used by simply fixing the variable values of 0 and 1 as part of the input. The elaboration of cost functions was based on analyzing a variety of technological process combinations (leading to different removal efficiencies) for treatment plants included in the analysis (Figure 4). As an example, the cost function for the largest treatment plant, Zalaegerszeg (see Figure 4) the capacity of which is $Q_l = 15000m^3/d$, is given in Figure 8[4]. Three groups of expenses are illustrated in the figure:

(i) Investment cost required for upgrading biological treatment;

(ii) Investment cost of P precipitation which increases rapidly with increasing requirements. The use of piecewise linear cost functions required the introduction of three dummy variables for each treatment plants.

(iii) Running cost.

6.2 Results of the Expectation-Variance Model

In order to gain an impression of the character of the problem and the behavior of the solution, first we specify a 'basic situation' (which is close to the real case) having the following features and with the following assumptions (Somlyódy, 1983b):

(i) control variables are continuous;

(ii) no effluent standard prescription is given;

(iii) no P retention takes place in rivers ($r_t = 0$ in equation (10));

(iv) the capital recovery factor is equal for all the projects, $\alpha_j = \alpha = 0.1$ and

(v) equal weighting is adopted (see q_i and Θ in Section 4.5).

[4]Roughly US $1 is equivalent to 50 Forints (Ft).

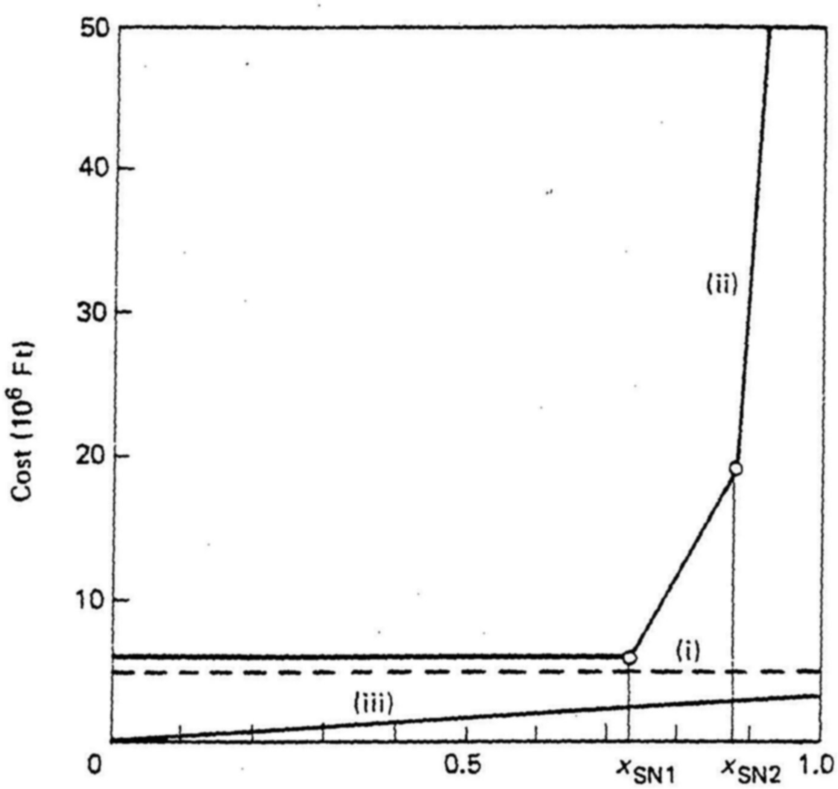

Figure 5: Costs of sewage treatment (Zalaegerszeg).

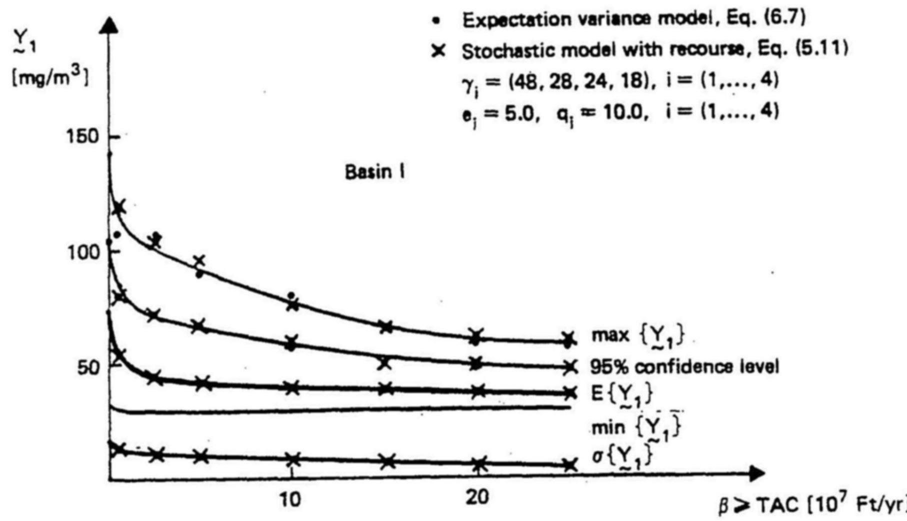

Figure 6: Water quality indicator $(\mathrm{Chl}-\mathrm{a})_{\max}$ as a function of the total annual cost.

With these assumptions optimization was performed under different budgetary conditions ($TAC \leq \beta = 0.5 to 25 \times 10^7\ Ft/yr$). Statistical parameters (expectation, standard deviation and extremes) of the water quality indicators gained from Monte Carlo procedure[5] are illustrated in Figure 6 for the Keszthely basin as a function of the total annual cost, TAC[6].

In Figure 7, we record the changes in the two major control variables (x_{SN1} and x_{D1}) associated to the treatment plant of Zalaegerszeg and the reed lake segment of the Kis-Balaton system (see Figure 4). There is a significant trade-off between these two variables. For decision-making purposes, it is important to observe that there are four ranges of possible values of β (the budget), in which the solution has different characteristics.

(i) In the range of $\beta = 0.5 to 5 \times 10^7\ Ft/yr$, it appears that sewage treatment can be intensified and tertiary treatment introduced. Expectation of the concentration levels will decrease considerably, but not the fluctuations. Under very small costs ($\sim 0.3 \times 10^7\ Ft$ investment costs, IC) it turns out that only the sewage of Zalaegerszeg (Figure 4) should be treated. Under increasing budget, potential treatment plants are built, going from west to east.

(ii) If β is between 5×10^7 and $10 \times 10^7\ Ft/yr$, the effectiveness of sewage treatment cannot be increased further but reservoir systems are still too

[5]1000 simulations were performed in each case.
[6]Running cost is about ten times larger than TAC.

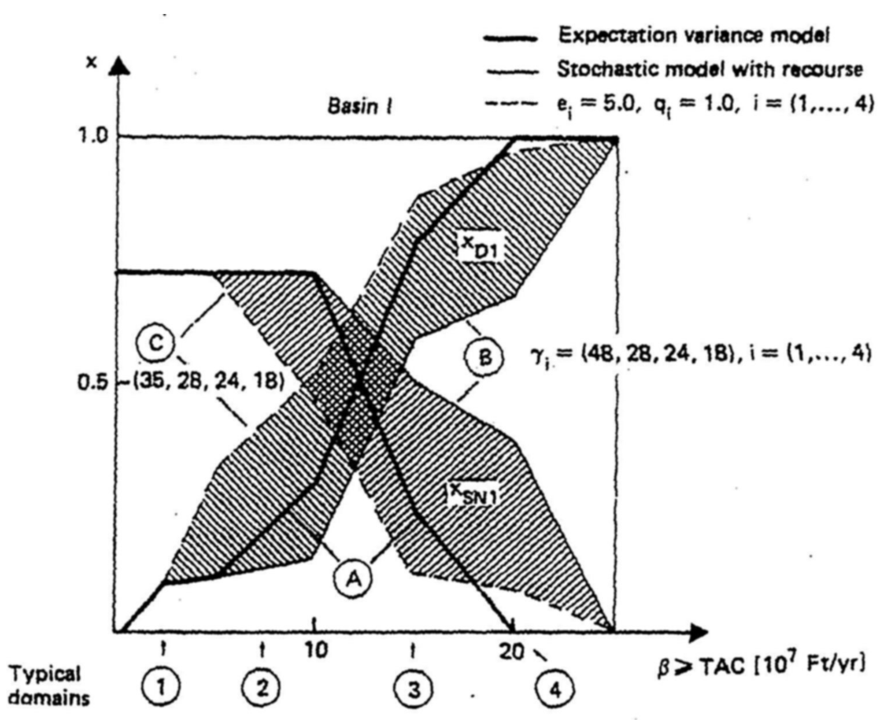

Figure 7: Change of major decision variables.

expensive.

(iii) At about $\beta = 15 \times 10^7$ Ft/yr the solution is a combination of tertiary treatment and reservoirs. Fluctuations in water quality are reduced by the latter control alternatives.

(iv) Finally, around $\beta = 20 \times 10^7$ Ft/yr, tertiary treatment is dropped in regions where reservoirs can be built. After constructing all the reservoirs, no further water quality improvement can be achieved.

Concerning the model sensitivity on major parameters, the following conclusions can be drawn (for details see Somlyódy (1983a), and Somlyódy and van Straten (1985)):

(i) Fixed water quality standard not reflecting the properties of the system (spatial non-uniformities) can result in a strategy far from the optimal one, since the distribution of a portion of the budget is *a priori* determined by the pre-set standard.

(ii) Under increasing P retention in rivers the improvement in water quality is less remarkable in the budget range $0 - 10 \times 10^7$ Ft/yr than in the basic case. The worst – nevertheless, nearly unrealistic – situation is if all the phosphorus were removed along the river and still treatment has to be performed: the budget should be partially allotted for investments having no influence on the lake's load.

(iii) If only deterministic effects are considered ($\Theta = 0$), reservoir projects enter the solution under much larger budget values.

(iv) If the capital recovery factor is smaller for reservoir projects than for sewage treatment plants (12) reservoir projects start to be feasible at smaller budgets. Errors in the efficiency or in costs of reservoirs cause similar shifts in the solution.

(v) When selecting properly the model parameters, the combination of the absolute load reductions for the four basins is maximized by the model (as it is suggested most frequently in the literature, see the Introduction). Since, however, the absolute loads alone do not reflect the spatial changes in water quality, the policy drastically differs from the optimal one.

Subsequently we give the 'realistic' solution for the Lake Balaton management problem by using

- actual retention coefficients (ranging between $0.3 - 0.5$)
- upper limits 0.9 for the P removal rate of reservoirs; and
- fixed variables $\{0, 0.9\}$ for the Kis-Balaton reservoir system.

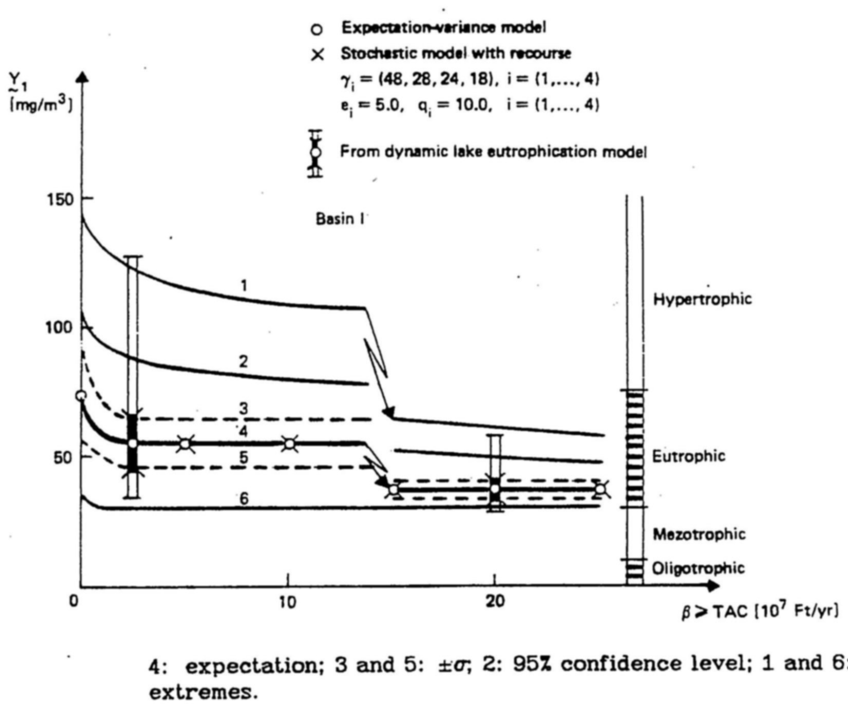

Figure 8: Solutions of EMOM for Lake Balaton, Basin I.

Figure[7] 8, which refers again to the Keszthely bay, shows remarkable differences as compared to Figure 6. First of all, the drastic effect of reservoirs upon expectation but even stronger upon fluctuation of water quality is stressed. Reservoirs enter the solution between 15×10^7 and 17.5×10^7 Ft/yr total annual cost resulting in a reduction in the mean $(\mathrm{Chl-a})_{\max}$ concentration from about 55 to 35 mg/m^3 and in the extreme values from more than 100 to about 60 mg/m^3.

While Figure 6 offers several solutions for a decision maker depending on the budget available, on the basis of Figure 8, only two feasible alternatives come to mind:

(i) If total annual cost of about 2.5×10^7 Ft/yr is available, all the sewage projects can and should be realized (going from west to east). Through this alternative the expectation of $\mathbf{Y}_1 = (\mathrm{Chl-a})_{\max}$ is reduced to about 55 mg/m^3 (tertiary treatment affects the water quality at a slightly smaller extent than in the basic case due to P retention of tributaries) but still extremes larger than 110 mg/m^3 can occur (hypertrophic domain according

[7]In the Figure $\pm$ standard deviation and the upper 95% confidence level are also illustrated (the distributions are bound towards small $\mathbf{Y}_1$ concentrations and the lower 95% confidence level values are close to the minimum).

to the classification of OECD, 1982). Further increase in the budget (up to 10×10^7 Ft/yr) has no impact on water quality (under the alternatives included in the analysis).

(ii) If budget around 20×10^7 Ft/yr is given not only the Kis-Balaton, but all the reservoirs can be established and tertiary treatment can be realized for direct sewage sources. The mean $(\mathrm{Chl}-\mathrm{a})_{\mathrm{max}}$ concentration is about 35 mg/m^3 while the maximum about 60 mg/m^3 (eutrophic stage).

In Figure 8 the results of a detailed simulation model for two optimal solutions ($TAC = 2.5 \times 10^7$ Ft and 20×10^7 Ft) are also given. The agreement between the calculated concentration indicators suggests that the aggregated lake eutrophication model is quite appropriate for our present purpose.

Figure 9 compares the typically skewed probability density functions of two considerably different solutions ($\beta = 2.5 \times 10^7$ Ft and 20×10^7 Ft respectively) for four basins, derived from Monte Carlo simulations. (The non-controlled state is also given in this figure.) Also from this figure we can conclude that tertiary treatment is more effective than reservoirs (when both alternatives are available) for controlling the mean concentration, but fluctuation can be controlled by reservoirs only. In the first case ($\beta = 2.5 \times 10^7$ Ft/yr) Basin I remains hypertrophic, Basins II and III eutrophic, whilst Basin IV mesotrophic. In the second situation ($\beta = 20 \times 10^7$ Ft/yr) the spatial differences and stochastic changes are much smaller: Basins I, II and III are eutrophic and Basin IV mesotrophic (the long-term improvement of water quality is certainly larger than the short-term one discussed here).

From all that we learned through this management model, it follows that in order to realize the optimal short-term strategy of eutrophication management

- tertiary treatment of direct sewage discharges should be introduced (from west to east);

- depending on the budget available tertiary treatment of indirect sewage discharges of pre-reservoirs (again from west to east) should be realized.

For further details of the management strategy worked out for Lake Balaton and other management models not discussed in this paper, the reader is referred to Somlyódy, (1983b) and Somlyódy and van Straten (1985).

6.3 Results of the Stochastic Recourse Model

As seen from Table 1 (line 7), the nominal state of water quality is given by the indicator vector $\mathbf{Y}_{01} = (75, 38, 28, 20), (i = (1, \cdots, 4))$. Goals were specified by $\gamma_i = (48, 28, 24, 18)$ expressing the desire that Basin I should be shifted to the eutrophic, and other segments to the mesotrophic state (see Figure 8), but without forcing a completely homogeneous water quality in the entire lake on the short term, which would be unrealistic.

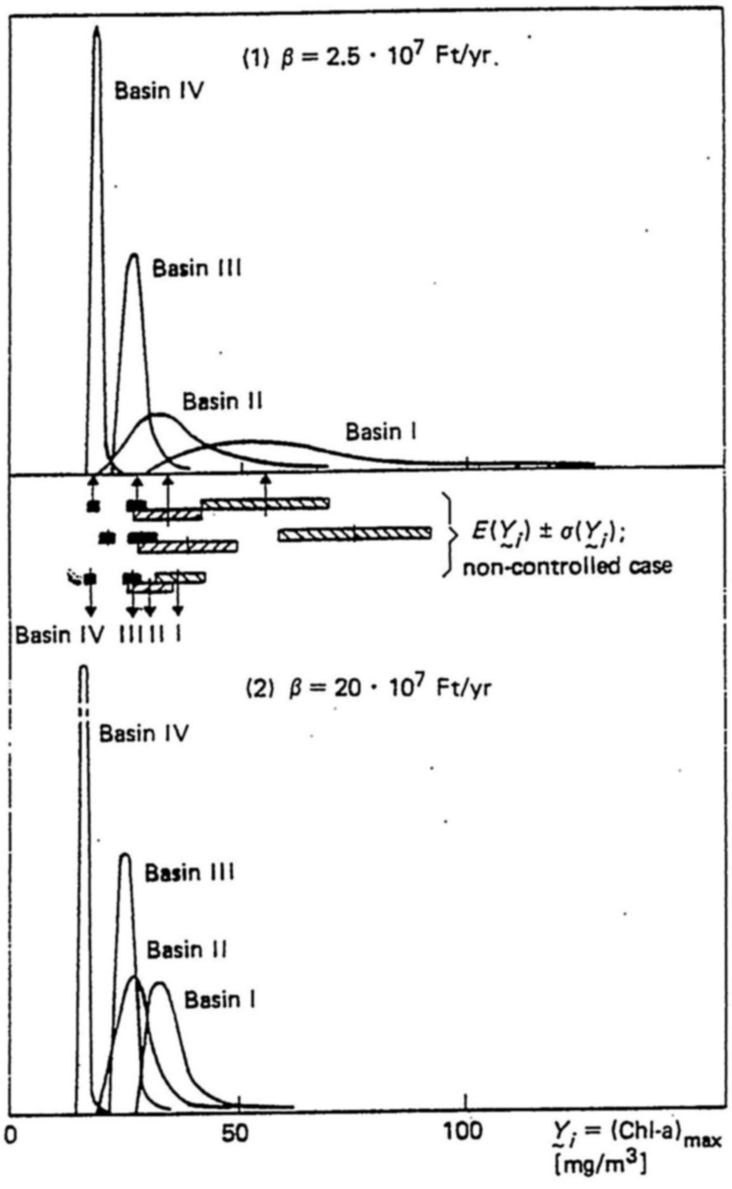

Figure 9: Probability density functions for two different situations (from 1000 Monte Carlo simulations).

The definition of these goals, however, means that the improvement intended to be achieved is quite uniform for the four basins in a relative sense: as compared to the maximal possible reduction in the water quality indicator on the short term (see Figure 6), we plan $50 - 60\%$ improvement for Basins I, II, and III. Basin IV (with 20%) is the only exception as its water quality is presently quite good, but this segment plays a secondary role from the viewpoint of the management problem.

The parameters e_i and q_i of the objective function, see (25) and Figure 2, were selected uniformly for the four basins: $e_i = 5$ and $q_i = 10, i = 1, \cdots, 4$. This corresponds, in the region $z_i \leq 5$, to a 'variance formulation' of the objective function (being similar to (6.1) as $q/2e = 1$ With these parameter values, the quadratic portion of the utility function is predominant in Basins II, II, and IV, while for Basin I the upper linear portion of the utility functional is also of importance.

Results of the stochastic optimization model with recourse are also illustrated in Figures 9-11, in comparison with that of the expectation-variance model. As seen from Figures 9-11, the two models produce practically the same results in terms of the water quality indicator (including also its distribution). With respect to details there are minor deviations. According to Figure 7, the expectation-variance model gives more emphasis to fluctuations in water quality and consequently to reservoir projects, than the stochastic recourse model (with the parameters specified above). This is in accordance with the remarks made in Section 4.5 that the role of the variance is overstressed in the expectation-variance model.

From this quick comparison of the performance of the two models, we may conclude that the more precise stochastic model validates the use of the expectation-variance model in the case of Lake Balaton.

For a more systematic comparison of the two models, the difference in the objective functions should be kept in mind. The stochastic model has more parameters than the expectation-variance model, in particular, the exclusion of the water quality goals γ_i from the expectation-variance model plays an important role. Figure 7 illustrates clearly that the prescription of the goal close to the lowest realizable value for Basin I (see Figure 6) leads to a stronger emphasis on reservoirs as compared to the expectation-variance model. The faster increase in x_{D1} as a function of the budget β is associated with a decrease in x_{SN1} — as expected — in addition to smaller allocations to the other basins. Depending on the value of γ_I the solutions lie in the shaded regions indicated in Figure 7. The solution to the expectation-variance model is located in the "center" of these regions.

As mentioned before, the expectation-variance model gives more weight to variance than the stochastic recourse model. For computational justification we compared curves (A) and (B) in Figure 7. The rationale is that lacking water quality goals, the expectation-variance model follows to some extent the principle of "equal relative" water quality improvements in all basins (other factors — eg. the distribution of costs for basins — also play a role) and in this sense its solution can best be compared to solution (B) of the stochastic method.

We complete this section with the following conclusions:

1. The stochastic optimization model with recourse justified the applicability of the much simpler expectation-variance model for Lake Balaton.

2. Replacing the stochastic objective function with a deterministic version leads to a strikingly different and incorrect management strategy.

3. The most influential parameters in the stochastic model are the prescribed water quality goals for the different basins. The inclusion of the goal in the objective function is the primary advantage in comparison with the expectation-variance model.

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