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Mathematics

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Multi-layer Random Interval Packing Problem

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Abstract

We extend the study of random interval packing problem to the case of multiple copies of resources, which we call multi-layer interval packing problem. Exact characterization of the key performance quantities in integral tranformation form are obtained. This establishes foundations for further investigation of this important problem.

1 Motivations

Random interval packing problem is an important probability model to many branches of physics, chemistry and engineer. Following A. Rényi's fundamental study, many research have been conducted in extending the understanding of the problem from both the theoretical and application aspects. This paper intend to serve the same purpose. We consider a variation of the problem with multiple copies of resources, we called it *multi-layer interval packing problem*. This model enriched the problem with multidimensional features. We demonstrate that how the conventional procedures, which includes dynamic programming and Laplace transform, can be employed recursively to obtained the performance analysis for the new model.

Apart from being a natural extension of the classical model mathematically, problems with multiple copied of resources arise in many important applications. Kelly [2] studied circuit switched network, a typical loss network in telecommunication, pointed out implicitly that some special cases of the problem can be reduced to an interval packing problem; Lu [8] demonstrated the connections between multi-item inventory system and the multi-layer model.

2 Analysis

The set up for the multiple layers online interval packing problem is the following, there are n different resources, each can be represented as a straight line; intervals arrive randomly in $\mathcal{R}_+^2$ according to a Poisson process representing arrival time epoch t and the left end of the interval, to be more specific, for any pair (t, y), the probability of an arrival in time interval $[t, t + dt]$ with left end in $[y, dy]$ is $dtdy + o(dtdy)$. The length of each interval follow i.i.d random variable L with density function $m(x)$, we assume that the probability function has a support with $[0, d]$ for certain finite d. An arrival interval will be rejected if it can not fit into any of the resource line, i.e. it will overlap with existing interval packed, otherwise it will be packed at the line with lowest index.

Suppose that $L_n(t,x)$ is the cumulative length of the cuccessfully packed intervals whose right end does not exceed x by time t when the number of layer is n, and $K_n(t,x) := \mathsf{E}[L^{(n)}(t,x)]$. We will derive expression of $K_n(t,x)$ recursively for the cases of x being large.

For any time t, conditioning upon the arrival in the time period of $[t, t+\Delta t]$, we have the following observations, which will be key components of our analysis.

- with probability $\Delta t(x - m_1)$, m_1 is the mean interval length, assumed to be finite, there is no interval contained in $[0, x]$ arrives within $[t, t+\Delta t]$; the reason that this is true is that,

$$\mathsf{P}[\text{no interval contained in } [0,x]] = \int_0^x du \int_{x-u}^{\infty} m(dy) = x - m_1$$

- the interval length is $[u, u+\Delta u]$, the left end position is $[y, y+\Delta y]$, with $0 \le y \le x-u$, then $K_n(t+\Delta t, x)$ should be $K_{n-1}(t,x)$ plus, one $K_n(t,y) - K_{n-1}(t,y)$, one $K_n(t, x-y-u) - K_{n-1}(t, x-y-u)$, and u. the probability of this happen is $\Delta t \Delta m(u) \Delta y$,

In summary, we have,

$$\begin{aligned} K_n(t+\Delta t, x) &= [1 - \Delta t(x - m_1)]K_n(t,x) + \Delta t \int_0^{\infty} dm(u) \\ &\int_0^{x-u} \mathsf{E}\left\{L_{n-1}(t,x) + [L_n(t,y) - L_{n-1}(t,y)] + [L_n(t, x-y-u) - L_{n-1}(t, x-y-u)] + u\right\} dy \\ &+ o(\Delta t) \end{aligned} \tag{1}$$

Hence,

$$\begin{aligned} K_n(t+\Delta t, x) &= [1 - \Delta t(x - m_1)]K_n(t,x) + \Delta t \int_0^{d} dm(u) \\ &\int_0^{x-u} \left\{K_{n-1}(t,x) + 2\left[K_n(t,y) - K_{n-1}(t,y)\right] + u\right\} dy + o(\Delta t) \end{aligned}$$

Let $\Delta t \to 0$, we have,

$$\begin{aligned} \frac{\partial}{\partial t} K_n(t,x) &= -(x - m_1)K_n(t,x) + m_1 - m_2 + K + n - 1(t,x)(x - m_1) + \\ &2\int_0^{\infty} dm(u) \int_0^{x-u} \left[K_n(t,y) - K_{n-1}(t,y)\right] dy \end{aligned}$$

where m_2 is the second moment of the interval length distribution, i.e. $m_2 := \int_0^{\infty} z^2 dm(z)$. Since, it is obvious that, $K_n(t,y) - K_{n-1}(t,y) \ge 0$, we can exchange the order of the integrations in the second term, we have,

$$\begin{aligned} \frac{\partial}{\partial t} K_n(t,x) &= -(x - m_1)K_u(t,x) + m_1 x - m_2 + K_{n-1}(t,x)(x - m_1) + \\ &2\int_0^{x} \left[K_n(t,y) - K_{n-1}(t,y)\right] \int_0^{x-y} dm(u) dy \end{aligned}$$

Or, equivalently,

$$\frac{\partial}{\partial t}K_n(t,x) = -(x-m_1)K_n(t,x) + m_1x - m_2 + K_{n-1}(t,x)(x-m_1) + \tag{2}$$

$$2\int_0^x \left[K_n(t,y) - K_{n-1}(t,y)\right] m(x-y)dy \tag{3}$$

When $n = 1$, it is reduced to the case that is considered in [3], and the differential-integral equation of $K_1(t,x)$, which we simplified as $K(t,x)$, then become,

$$\frac{\partial}{\partial t}K(t,x) = -(x-m_1)K(t,x) + m_1x - m_2 + 2\int_0^x K(t,y)m(x-y)dy$$

and we know that $K_0(t,x) = 0$.

Employ the techniques developed in [3], we can obtain $K_n(t,x)$, more accurately its integral transformations, recursively. However, it will be more convenient to study the its z-transform. Let $z \in \mathcal{C}$ be a complex number, and $|z| < 1$, denote

$$\mathcal{K}(t,x,z) = \sum_{n=0}^{\infty} K_n(t,x)z^n,$$

then, from (2), we know that $\mathcal{K}$ satisfies,

$$\frac{\partial}{\partial t}\mathcal{K}(t,x,z) = -(x-m_1)(1-z)\mathcal{K}(t,x,z) + \frac{m_1x+m_2}{1-z} + 2\int_0^x \left[(1-z)\mathcal{K}(t,y,z)\right] m(x-y)dy \tag{4}$$

Apply the following integral transformation,

$$\hat{\mathcal{K}}(t,w,z) := \int_d^{\infty} e^{-wx}\mathcal{K}(t,x,z)dx$$

we have,

$$\frac{\partial}{\partial t}\hat{\mathcal{K}}(t,w,z) = (1-z)\left[\frac{\partial}{\partial w}\hat{\mathcal{K}}(t,w,z) + [m_1 + 2\mathcal{G}(t,w)]\hat{\mathcal{K}}(t,w,z)\right] + g(w) + \mathcal{H}(t,w,z) \tag{5}$$

where,

$$\begin{aligned} g(w) &:= \frac{1}{1-z}\int_d^{\infty} e^{-wx}(m_1x - m_2)dx \\ \mathcal{G}(w) &:= \int_d^{\infty} e^{-wx}m(dx) \\ \mathcal{H}(t,w,z) &:= \int_0^d e^{-wy}(1-z)\hat{\mathcal{K}}(t,y,z)\int_{d-y}^{\infty} e^{-wx}m(dx)dy \end{aligned}$$

Remark It can be easily seen that $\mathcal{H}(t,w,z)$ only depends upon the value of $K_n(t,x)$ when $x \leq d$. When we study the asymptotic behaviour of $K_n(t,x)$, it can be treated as close to constant.

Given the value of $K_n(t,x)$ for $x \leq d$, we can see that $\hat{\mathcal{K}}(t,w,z)$ is the solution of the following differential equation,

$$\frac{\partial}{\partial t}\hat{\mathcal{K}}(t,w,z) = (1-z)\left[\frac{\partial}{\partial u}\hat{\mathcal{K}}(t,w,z) + [m_1 + 2\mathcal{G}(u)]\hat{\mathcal{K}}(t,w,z)\right] + \mathcal{C}(t,w,z) \tag{6}$$

where,

$$\mathcal{C}(t,w,z) := g(w) + \mathcal{H}(t,w,z)$$

Let $\mathcal{M}(v,s,z) := \hat{\mathcal{K}}(v, s-(1-z)v, z)$, then, we have,

$$\frac{\partial \mathcal{M}(v,s,z)}{\partial v} = \frac{\partial \hat{\mathcal{K}}(t,w,z)}{\partial t} - (1-z)\frac{\partial \hat{\mathcal{K}}(t,w,z)}{\partial w}\Bigg|_{t=v,w=s-(1-z)v}$$

and then (6) can be transformed into,

$$\frac{\partial \mathcal{M}(v,s)}{\partial v} = (1-z)[m_1 + 2\mathcal{G}(v,s-v)]\mathcal{M}(v,s,z) + \mathcal{C}(v,s-v,z) \tag{7}$$

and the boundary condition is $\mathcal{M}(0,s,z) = \mathcal{K}(0,s,z) = 0$, the solution will take the following form,

$$\mathcal{M}(v,s,z) = \int_0^v C(\xi, s-\xi, z) \exp\left(\int_\xi^v (1-z)[m_1 + 2\mathcal{G}(x,s-x)]dx\right) d\xi \tag{8}$$

Knowing that $\hat{\mathcal{K}}(t,w,z) = \mathcal{M}(t,(1-z)t+w,z)$, we then have,

$$\hat{\mathcal{K}}(t,w,z) = \int_0^t C(\xi, w+(1-z)t-\xi, z) \exp\left(\int_\xi^w (1-z)[m_1 + \mathcal{G}(x, w+(1-z)t-x)]dx\right) d\xi \tag{9}$$

Hence, given the the value of $K_n(t,x)$ for $x \le d$, we can completely characterize the $K_n(t,x)$. For the value of $K_n(t,x)$, we can employ the recursive procedure developed in [3] to obtain.

With the whole characterization of the integral transform of $K_n(t,x)$, we can conduct analysis upon their asymptotic behavior. In particular, let $w \to 0$ and $z \to 1$, we can easily see that the following is true,

Proposition 1

$$\hat{\mathcal{K}}(t,w,z) \approx \frac{1}{w^2(1-z)^2}[m_1 t - \frac{1}{2}\xi(m_1 d - m_2)t^2 \tag{10}$$

Tauerian theorems then can be applied to indicate the order of $K_n(t,x)$.

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