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GERSGORIN VARIATIONS II: ON THEMES OF FAN AND GUDKOV

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Dedicated to Charles A. Micchelli, in celebration of his 60th birthday

and our 30 years of friendship

Abstract. Assume $F = \{f_1, \cdots, f_n\}$ is a family of nonnegative functions of $n-1$ nonnegative variables such that, for every matrix A of order n, $|a_{ii}| > f_i$ (moduli of off-diagonal entries in row i of A) for all i implies A nonsingular. We show that there is a positive vector x, depending only on F, such that for all $A = (a_{ij})$, and all i, $f_i \geqq \sum_j |a_{ij}| \frac{x_j}{x_i}$. This improves a theorem of Ky Fan [F], and yields a generalization of a non-singularity criterion of Gudkov [Gu].

1. Introduction. If a complex matrix $A = (a_{ij})$ satisfies

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \text{ for all } i, \tag{1.1}$$

then A is nonsingular. This famous Levy-Desplanques sufficient condition for nonsingularity [L] is equivalent to the more famous Gersgorin theorem [Ge]: every eigenvalue of A lies in

$$\bigcup_i \left\{ z \;\middle|\; |a_{ii} - z| \leqq \sum_j |a_{ij}| \right\}. \tag{1.2}$$

There are many many generalizations and adumbrations of (1.1) and (1.2), and we have decided to call them Gersgorin Variations [Ho4]. In this paper, we recall a variation due to Gudkov [Gu], another variation due to Ostrowski [O], and combine them.

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Gudkov defines inductively

$$(1.3)\qquad \begin{cases} R_1(A) &= \displaystyle\sum_{j>1} |a_{ij}|, \\ R_k(A) &= \displaystyle\sum_{j<k} |a_{kj}| \frac{R_j(A)}{a_{jj}} + \sum_{j>k} |a_{kj}|, \quad k = 2, \cdots, n. \end{cases}$$

His theorem states: if

$$(1.4)\qquad |a_{ii}| > R_i(A) \text{ for } i = 2, \cdots, n,$$

then A is nonsingular. Since (1.1) implies (1.4), Gudkov's theorem implies Levy-Desplanques.

Ostrowski proved many generalizations of Levy-Desplanques, among them [O]: if

$$(1.5)\qquad \begin{cases} p > 0, q > 0, \dfrac{1}{p} + \dfrac{1}{q} = 1, \displaystyle\sum \frac{1}{1+\alpha_i^q} \leqq 1, \text{ and} \\ |a_{ii}| > \alpha_i \Big(\displaystyle\sum_{j \neq i} |a_{ij}|^p \Big)^{\frac{1}{p}} \text{ for all } i, \end{cases}$$

then A is nonsingular.

Our aim is to prove a theorem which extends (1.5)-indeed a considerable generalization of (1.5)-in the same way that (1.4) extends (1.1). Before stating this extension, we need a definition.

A family $F = \{f_1, \cdots, f_n\}$ of nonnegative functions of the moduli of the off-diagonal entries of a complex matrix of order n is a "G-function" (G for Gersgorin) if for every matrix A, $|a_{ii}| > f_i(A)$ for all i implies A nonsingular (the concept was introduced in [N] and named in [Ho1]. See also [NT, Ho2, Ho3, CV], and [HV]). F is a "row G-function" if, for all i, f_i depends only on the moduli of the off-diagonal entries in A_i (the ith row of A).

THEOREM 1.1. *Let $F = \{f_i, \cdots, f_n\}$ be a row G-function, A a matrix of order* *n. Define*

$$(1.6)\qquad \begin{cases} F_1(A) &= f_1\big(|a_{12}|,\cdots,|a_{1n}|\big) \\ F_k(A) &= f_k\big(|a_{k1}|\frac{F_1(A)}{|a_{11}|},\cdots,|a_{k,k-1}|\frac{F_{k-1}(A)}{|a_{k-1,k-1}|}; \\ & \qquad |a_{k,k+1}|,\cdots,|a_{kn}|\big), k=2,\cdots,n, \end{cases}$$

If

$$(1.7)\qquad |a_{ii}| > F_i(A) \text{ for } i=2,\cdots,n,$$

then A is nonsingular.

Theorem 1.1 is a consequence of theorem 1.2, which is an improvement of a theorem of Ky Fan [1] [F].

Ky Fan proved (see also [CH]) that if $F=\{f_1,\cdots,f_n\}$ is a G-function, then for every matrix A, there exists a positive vector x such that

$$(1.8)\qquad f_i(A) \geqq \sum_{j\neq i} |a_{ij}|\frac{x_j}{x_i}, \quad i=1,\cdots,n.$$

We shall show that if F is a row G-function (so that we may write $f_i(A_i)$ instead of $f_i(A)$), then the order of the quantifiers preceding (1.8) can be interchanged.

THEOREM 1.2. *If F is a row G-function, then there is positive vector x such that*

$$(1.9)\qquad \text{for every } A, f_i(A_i) \geqq \sum_{j\neq i} |a_{ij}|\frac{x_j}{x_i}, \quad i=1,\cdots,n.$$

2. Proof of Theorem 1.2. We shall call a function f monotone if $f(x) \leqq f(y)$ wherever $x \leqq y$.

LEMMA 2.1. *Let $F=\{f_1,\cdots,f_n\}$ be a row G-function, and let*

$$(2.1)\qquad \overline{f}_i(A_i) = \inf f_i(B_i) : |b_{ij}| \geqq |a_{ij}|,\ i\neq j;\ i,j=1,\cdots,n,$$

[1] I would like this paper, whose appearance will approximate the 90th birthday of Ky Fan, to signify my admiration for this intellectual giant, who is also a very nice man.

then

(2.2a) $$\overline{F} = \{\overline{f}_1, \cdots, \overline{f}_n\} \text{ is a row G-function;}$$

(2.2b) $$\overline{f}_i \text{ is monotone, } i = 1, \cdots, n,$$

(2.2c) $$f_i \geqq \overline{f}_i, i = 1, \cdots, n.$$

Since (2.2b) and (2.2c) are immediate from (2.1), all we need prove is (2.2a). Let $\epsilon > 0$ be given. Then $\overline{f}_i(A_i) > f_i(B_i) - \frac{\epsilon}{2}$ for some B_i with $b_{ij} \geqq |a_{ij}|, i \neq j, i, j = 1, \cdots, n$ from (2.1). So $\overline{f}_i(A_i) + \epsilon > f_i(B_i) + \frac{\epsilon}{2}, i = 1, \cdots, n$. Let B be the matrix with off-diagonal rows $B_1, \cdots, B_n$. By (1.8), there is a positive vector $x(\frac{\epsilon}{2})$ such that, for all i,

(2.3) $$\overline{f}_i(A_i) + \epsilon > f_i(B_i) + \frac{\epsilon}{2} > \sum_{j \neq i} |b_{ij}| \frac{x_j}{x_i} \geqq \sum_{j \neq i} |a_{ij}| \frac{x_j}{x_i}.$$

Rewrite (2.3) as

(2.3a) $$x_i\big(\overline{f}_i(A_i) + \epsilon\big) \geqq \sum_{j \neq i} |a_{ij}| x_j.$$

Since (2.3a) is homogeneous, we may assume $x \in S_n$, the simplex of all nonnegative vectors $x = \{x_1, \cdots, x_n\}$ with $\sum x_j = 1$. In (2.3a) the vector $x = x(\frac{\epsilon}{2})$. Choose a sequence of ϵ's tending to 0 such that the corresponding $x = x(\frac{\epsilon}{2})$ converge, say to $\overline{x}$. Then (2.3a) becomes

(2.3b) $$\overline{x}_i \overline{f}_i(A_i) \geqq \sum_{j \neq i} |a_{ij}| \overline{x}_j$$

Now each $\overline{x}_i$ is different from 0. If, for example $\overline{x}_1 = 0$, then some $\overline{x}_k > 0$, because $\overline{x} \in S_n$. Then (2.3b) would assert(if $|a_{1k}| \neq 0$)

$$0 \geqq \sum_{j \neq i} |a_{ij}| \overline{x}_j \geqq |a_{1k}| \overline{x}_k > 0$$

a contradiction. Hence (2.3b) holds for all i, and each $\overline{x}_i > 0$. This prove

$$\overline{f}_i(A_i) \geqq \sum_{j \neq i} |a_{ij}| \frac{\overline{x}_j}{\overline{x}_i}. \tag{2.3c}$$

Let X be the diagonal matrix whose entries are taken from the vector x. From (2.3c), we see that

$$|a_{ii}| > \overline{f}_i(A_i) \geqq \sum_{j \neq i} |a_{ij}| \frac{\overline{x}_j}{\overline{x}_i} \quad \text{for all } i$$

implies, from (1.1), that $X^{-1}AX$ is nonsingular. Therefore, $\overline{F} = \{\overline{f}_1, \cdots, \overline{f}_n\}$ is a row G-function, since $X^{-1}AX$ nonsingular means A is nonsingular. So lemma 2.1 is true.

We prove Theorem 1.2 by induction on n. The theorem is trivially true if $n = 1$. Further, reasoning as in the proof of Lemma 2.1, all we need prove is that there exists $x \in S_n$ such that, for all i and A_i,

$$\overline{f}_i(A_i)x_i - \sum_{j \neq i} |a_{ij}| x_j \geqq 0, \tag{2.4}$$

since $f_i \geqq \overline{f}_i$ by (2.2c).

Now, for each i, (2.4) asserts that x lies in the intersection of an infinite set of closed half-spaces; and, considering all i, we must show x is in the intersection of n infinite sets of closed half-spaces. These half-spaces are closed convex sets, and we will invoke Helly's theorem([He], see [DGK] for a wonderful discussion). One form of Helly's theorem asserts that if $\{K_\alpha\}$ is a(possibly infinite) family of closed convex sets of a compact region of Euclidean m-space, then $\bigcap_\alpha K_\alpha \neq \emptyset$ if, for every $\alpha_1, \alpha_2, \cdots, \alpha_{m+1}$, $\bigcap_1^{m+1} K_{\alpha_i} \neq \emptyset$. Since S_n is a compact region of Euclidean $n-1$ space, all we need prove is that, given any n inequalities of (2.4), they are satisfied by some $x \in S_n$.

Suppose that the row indices of the n inequalities are all distinct. Then (2.4) holds because of Ky Fan's theorem, (1.8).

Suppose that the row indices are a proper subset $T = \{1, 2, \cdots, n\}$. For ease of notation, assume $T = \{1, 2, \cdots, m\}$, $m < n$. Let B be a matrix of order n which is the leading principal submatrix of a matrix A of order n, and where every diagonal entry of A is nonzero and every off-diagonal entry of A not in B is 0. Define, for $i = 1, 2, \cdots, m$, $\overline{f}_{i,m}(B_i)$ to be $\overline{f}_i(A_i)$ of the aforementioned A. Since A is nonsingular if and only if B is nonsingular, if follows that $\{\overline{f}_{1,m}, \cdots, \overline{f}_{m,m}\}$ is a row G-family of order m. Using the induction hypotheses and (2.2b), the n inequalities in this case of (2.4) are also consistent. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.1. Let x be the positive vector (from Theorem 1.2) satisfying (1.9) and (2.4), and let X be the diagonal matrix whose diagonal entries come from X. We shall show that, for each k,

$$F_k(A) \geqq R_k(X^{-1}AX), \tag{3.1}$$

from which Theorem 1.1 follows at once. We prove (3.1) by induction on k.

If $k = 1$, (3.1) is (1.9) for $i = 1$. Assume (3.1) has been shown for $1, 2, \cdots, k-1$. Then

$$\begin{aligned}
\overline{F}_k(A) &= f_k\left(|a_{k1}|\frac{F_1(A)}{|a_{11}|}, \cdots, |a_{k,k-1}|\frac{F_{k-1}(A)}{|a_{k-1,k-1}|}; |a_{k,k+1}|, \cdots, |a_{k,n}|\right)\\
&\geqq \overline{f}_k\left(|a_{k1}|\frac{F_1(A)}{|a_{11}|}, \cdots, |a_{k,k-1}|\frac{F_{k-1}(A)}{|a_{k-1,k-1}|}; |a_{k,k+1}|, \cdots, |a_{k,n}|\right), \quad \text{by (2.2c)}\\
&\geqq \overline{f}_k\left(|a_{k1}|\frac{R_1(X^{-1}AX)}{|a_{11}|}, \cdots, |a_{k,k-1}|\frac{R_{k-1}(X^{-1}AX)}{|a_{k-1,k-1}|}; |a_{k,k+1}|, \cdots, |a_{k,n}|\right),\\
&\qquad\qquad \text{by induction and (2.2b)}\\
&\geqq R_k(X^{-1}AX), \quad \text{by (2.4)}\ .
\end{aligned}$$

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