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Ying Sue Huang
Pace University
Pleasantville, NY 10570

Chai Wah Wu
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

A UNIFYING PROOF OF GLOBAL ASYMPTOTICAL STABILITY OF NEURAL NETWORKS WITH DELAY

Ying Sue Huang[†] *and Chai Wah Wu*[††]

[†]Department of Mathematics, Pace University, Pleasantville, NY 10570, U. S. A.
[††]IBM T. J. Watson Research Center, P. O. Box 218, Yorktown Heights, NY 10598, U. S. A.
e-mail: yhuang@pace.edu, chaiwah@watson.ibm.com

ABSTRACT

We present some new global stability results of neural networks with delay and show that these results generalize recently published stability results. In particular, several different stability conditions in the literature which were proved using different Lyapunov functionals are generalized and unified by proving them using the same Lyapunov functional. We also show that under certain conditions, reversing the directions of the coupling between neurons preserves the global asymptotical stability of the neural network.

1. INTRODUCTION

Recently, there has been much activity to study the global stability of neural networks with delays and many stability criteria are proposed [1–10]. In this paper, we present a new criterion for the global stability of neural networks with delays. We show that this criterion is more general than those presented in the past.

Our criterion is proved using a different strategy than the recent literature. In recent papers, the most general case is proved by incorporating all the generalities into the Lyapunov functional, resulting in a complicated Lyapunov functional. Instead, we first establish the result for a simple canonical case, and deduce the result for the general case by means of simple state transformations. This technique, which was also used in [11] allows for simpler proofs of the stability results with simpler Lyapunov functionals.

2. GLOBAL STABILITY OF NEURAL NETWORKS WITH DELAY

Consider a neural networks with delays described by the state equation:

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + c \qquad (1)$$

where D is a positive definite diagonal matrix and

$$f(x) = (f_1(x_1), \dots f_n(x_n)).$$

The slopes of the continuous scalar functions f_i are bounded:

$$0 \leq \frac{f_i(a) - f_i(b)}{a-b} \leq 1 \text{ when } a \neq b \qquad (2)$$

We assume that Eq. (1) has an equilibrium point x^*. The existence of an equilibrium point can be guaranteed if f_i is bounded. This follows from a simple application of Brouwer's fixed point theorem. By shifting x^* to the origin, we obtain the following canonical form:

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) \qquad (3)$$

where $f(0) = 0$. We will only work with real matrices and vectors. To simplify notation, we will sometimes write $x(t)$ as x, $f(x(t))$ as $f(x)$ and the transpose of A^{-1} as A^{-T}.

The following simple Lemma will be useful in our stability proofs.

Lemma 1 $X^TY + Y^TX \leq X^TX + Y^TY$. *In particular, if X and Y are vectors,* $X^TY \leq \frac{X^TX+Y^TY}{2}$.

Proof: Follows from the fact that $X^TX + Y^TY - X^TY - Y^TX = (X-Y)^T(X-Y) \geq 0$. □

The following fact will also be useful. Given nonsingular matrices P and K where K is symmetric and positive constant β,

$$K > 0 \Leftrightarrow \beta K > 0 \Leftrightarrow K^{-1} > 0 \Leftrightarrow PKP^T > 0$$

Our main stability result is the following which gives conditions under which Eq. (3) has a globally asymptotically stable solution.

Theorem 1 *If there exists a symmetric positive definite matrix K and a factorization of $B = B_1B_2$ such that $R = 2D - A - A^T - B_1KB_1^T - B_2^TK^{-1}B_2$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3).*

Proof: Let $V_1 = \frac{1}{2}x^Tx$ and

$$V_2 = \sum_i \int_0^{x_i(t)} f_i(s)ds + \int_{t-\tau}^{t} f(x(\eta))^T W f(x(\eta))d\eta$$

for some symmetric positive semidefinite matrix W. Consider the Lyapunov functional $V = \epsilon_1 V_1 + V_2$ where the scalar $\epsilon_1 > 0$ and the matrix W are to be determined later. The derivative of V along trajectories of Eq. (3) is:

$$\dot{V} = \epsilon_1 \dot{V}_1(x) + \dot{V}_2(x)$$

where $\dot{V}_1(x) = -x^T Dx + x^T Af(x) + x^T Bf(x(t-\tau))$ and

$$\begin{aligned}\dot{V}_2(x) &= -f(x(t))^T Dx(t) + f(x(t))^T Af(x(t)) \\ &\quad + f(x(t))^T Bf(x(t-\tau)) + f(x(t))^T Wf(x(t)) \\ &\quad - f(x(t-\tau))^T Wf(x(t-\tau))\end{aligned}$$

Since

$$\begin{aligned}\dot{V}_1(t) &= -x^T Dx + \left(x^T \frac{D^{\frac{1}{2}}}{\sqrt{2}}\right)(\sqrt{2}D^{-\frac{1}{2}}Af(x(t))) \\ &\quad + \left(x^T \frac{D^{\frac{1}{2}}}{\sqrt{2}}\right)(\sqrt{2}D^{-\frac{1}{2}}Bf(x(t-\tau)))\end{aligned}$$

Several applications of Lemma 2 results in

$$\begin{aligned}\dot{V}_1(t) &\leq -\frac{1}{2}x^T Dx + f(x)^T A^T D^{-1} Af(x) \\ &\quad + f(x(t-\tau))^T B^T D^{-1} Bf(x(t-\tau)) \\ &\leq -\frac{1}{2}x^T Dx + Mf(x)^T f(x) \\ &\quad + Mf(x(t-\tau))^T f(x(t-\tau))\end{aligned}$$

where $M = \max(\|A^T D^{-1} A\|_2, \|B^T D^{-1} B\|_2) \geq 0$.

Let the Cholesky factorization of K be $K = QQ^T$. Since $x_i f_i(x_i) \geq (f_i(x_i))^2$, we have

$$-f(x)^T Dx \leq -f(x)^T Df(x).$$

By writing $B = B_1 QQ^{-1} B_2$, the term $\dot{V}_2$ can be written as:

$$\begin{aligned}\dot{V}_2(t) &\leq -f(x)^T (D - A - W)f(x) \\ &\quad + (f(x)^T B_1 Q)(Q^{-1} B_2 f(x(t-\tau))) \\ &\quad - f(x(t-\tau))^T Wf(x(t-\tau))\end{aligned}$$

An application of Lemma 2 results in

$$\begin{aligned}\dot{V}_2(t) &\leq -f(x)^T (D - A - W - \frac{1}{2}B_1 QQ^T B_1^T)f(x) \\ &\quad - f(x(t-\tau))^T (W - \frac{1}{2}B_2^T Q^{-T} Q^{-1} B_2)f(x(t-\tau)) \\ &= -f(x)^T (D - A - W - \frac{1}{2}B_1 KB_1^T)f(x) \\ &\quad - f(x(t-\tau))^T (W - \frac{1}{2}B_2^T K^{-1} B_2)f(x(t-\tau))\end{aligned}$$

Let $\epsilon_2 > 0$ be such that $R - 2\epsilon_2 I > 0$. Choose $W = \frac{1}{2}B_2^T K^{-1} B_2 + \frac{\epsilon_2}{2} I$ which is a symmetric positive definite matrix. This implies that $\dot{V}_2(t) \leq -\frac{\epsilon_2}{2} f(x)^T f(x) - \frac{\epsilon_2}{2} f(x(t-\tau))^T f(x(t-\tau))$. If we choose $\epsilon_1 = \frac{\epsilon_2}{2M} > 0$[1], then $\dot{V} = \epsilon_1 \dot{V}_1 + \dot{V}_2 \leq -\frac{\epsilon_1}{2} x^T Dx$. Since $V(x) \geq \frac{\epsilon_1}{2} x^T x$, the origin is the unique equilibrium point and is globally asymptotically stable by applying Lyapunov's direct method [12, Chapter 5, Corollary 3.1]. □

[1] If $M = 0$, we choose $\epsilon_1 = 1$.

Note that depending on the factorization of $B = B_1 B_2$, the matrix K is not necessarily of the same dimension as A and B. Some examples of the factorization $B = B_1 B_2$ are: $(B_1, B_2) = (I, B)$, $(B_1, B_2) = (B, I)$, and when B is singular $(B_1, B_2) = (BB^+, B)$ and $(B_1, B_2) = (B, B^+ B)$ where B^+ is the Moore-Penrose pseudoinverse of B.

When B is symmetric positive (negative) definite, we can choose $(B_1, B_2) = (I, B)$ and $K = B$ ($K = -B$), to obtain:

Corollary 1 *If B is symmetric positive definite and $D - \frac{1}{2}(A + A^T) - B$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3). The origin is also globally asymptotically stable if B is symmetric negative definite and $D - \frac{1}{2}(A + A^T) + B$ is positive definite.*

Corollary 2 *If there exists a symmetric positive definite matrix K such that either $R_1 = 2D - A - A^T - B^T KB - K^{-1}$ or $R_2 = 2D - A - A^T - BKB^T - K^{-1}$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3).*

Proof: R_1 is obtained from R in Theorem 1 by choosing $(B_1, B_2) = (I, B)$ and changing K to K^{-1} as $K > 0$ if and only if $K^{-1} > 0$. R_2 is obtained from R by choosing $(B_1, B_2) = (B, I)$. The conclusion then follows from Theorem 1. □

Note that R_1 becomes R_2 if we change B to B^T. This implies that if $R_1 > 0$, then Eq. (3) remains globally asymptotically stable if we transpose the matrices A and B.

Corollary 3 *If there exists a symmetric positive definite matrix K such that $2D - A - A^T - B^T KB - K^{-1}$ is positive definite, then the origin is globally asymptotically stable for each of the following four state equations:*

$$\begin{aligned}\dot{x}(t) &= -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) \\ \dot{x}(t) &= -Dx(t) + A^T f(x(t)) + Bf(x(t-\tau)) \\ \dot{x}(t) &= -Dx(t) + Af(x(t)) + B^T f(x(t-\tau)) \\ \dot{x}(t) &= -Dx(t) + A^T f(x(t)) + B^T f(x(t-\tau))\end{aligned}$$

Transposing the feedback matrices A and B corresponds to reversing the coupling direction between neurons. The above result gives condition under which such reversals do not change the global asymptotical stability of the neural network.

3. GENERALIZATIONS

We now generalize Theorem 1 through the use of state scaling transformations.

Corollary 4 *If there exists a factorization of $B = B_1 B_2$, a symmetric positive definite matrix K and a positive definite diagonal matrix P such that $R = 2PD - PA - A^T P - PB_1 K B_1^T P - B_2^T K^{-1} B_2$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3).*

Proof: Consider the state transformation $z = P^{\frac{1}{2}} x$. The state equation then becomes

$$\dot{z} = -P^{\frac{1}{2}} D P^{-\frac{1}{2}} z + P^{\frac{1}{2}} A P^{-\frac{1}{2}} \tilde{f}(z(t)) + P^{\frac{1}{2}} B P^{-\frac{1}{2}} \tilde{f}(z(t-\tau) \quad (4)$$

where $\tilde{f}(z) = P^{\frac{1}{2}} f(P^{-\frac{1}{2}} z)$. Note that since f satisfies the bounded slope condition (Eq. (2)) $\tilde{f}$ also satisfies the bounded slope condition. Applying Theorem 1 to Eq. (4) it follows that the origin is a globally asymptotically stable solution of Eq. (4) and thus also of Eq. (3) if

$$\begin{array}{l} 2P^{\frac{1}{2}} D P^{-\frac{1}{2}} - P^{\frac{1}{2}} A P^{-\frac{1}{2}} - P^{-\frac{1}{2}} A^T P^{\frac{1}{2}} \\ -P^{\frac{1}{2}} B_1 K B_1 P^{\frac{1}{2}} - P^{-\frac{1}{2}} B_2^T K^{-1} B_2 P^{-\frac{1}{2}} \end{array} \quad (5)$$

is positive definite. Applying the transformation $X \rightarrow P^{\frac{1}{2}} X P^{\frac{1}{2}}$ to Eq. (5) which preserves positive definiteness completes the proof. □

In [9, 10] a more general slope condition is imposed on f:

$$0 \leq \frac{f_i(a) - f_i(b)}{a - b} \leq \sigma_i \text{ when } a \neq b \quad (6)$$

for some constants $\sigma_i > 0$. Let $S = \text{diag}(\sigma_1, \ldots \sigma_n)$.

This general case can be reduced via a scaling transformation to the condition in Eq. (2) and the main stability result in its most general form can then be stated as:

Theorem 2 *Suppose f satisfies Eq. (6). If there exists a factorization of $B = B_1 B_2$, a symmetric positive definite matrix K and a positive definite diagonal matrix P such that $R = 2S^{-1} PD - PA - A^T P - PB_1 K B_1^T P - B_2^T K^{-1} B_2$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3).*

Proof: Let $z = Sx$. Then state equation (3) can be written as

$$\dot{z} = -\tilde{D} z + \tilde{A} \tilde{f}(z) + \tilde{B} \tilde{f}(z(t - \tau))$$

where $\tilde{D} = SDS^{-1} = D$, $\tilde{A} = SA$, $\tilde{B} = SB$ and $\tilde{f}(z) = f(S^{-1} z)$. Note that $\tilde{f}$ satisfies condition (2) if f satisfies condition (6). By Corollary 4 the origin $z = 0$, and thus also $x = 0$, is globally asymptotically stable if

$$2\tilde{P} D - \tilde{P} S A - A^T S \tilde{P} - \tilde{P} S B_1 K B_1^T S \tilde{P} - B_2^T K^{-1} B_2 > 0$$

for some positive definite diagonal matrix $\tilde{P}$. Using the substitution $P = S\tilde{P} = \tilde{P} S$, the proof is complete. □

Again, by choosing $(B_1, B_2) = (I, B)$ and $(B_1, B_2) = (B, I)$ we get

Corollary 5 *If there exists a symmetric positive definite matrix K and a positive definite diagonal matrix P such that either $R_1 = 2S^{-1} PD - PA - A^T P - B^T K B - PK^{-1} P$ or $R_2 = 2S^{-1} PD - AP - PA^T - BKB^T - PK^{-1} P$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of Eq. (3).*

An analogue to Corollary 3 is the following:

Corollary 6 *If there exists a symmetric positive definite matrix K and a positive definite diagonal matrix P such that $2S^{-1} PD - PA - A^T P - B^T K B - PK^{-1} P$ is positive definite, then the origin is a globally asymptotically stable equilibrium point of each of the following two state equations:*

$$\begin{array}{rcl} \dot{x}(t) & = & -Dx(t) + Af(x(t)) + Bf(x(t - \tau)) \\ \dot{x}(t) & = & -Dx(t) + A^T f(x(t)) + B^T f(x(t - \tau)) \end{array}$$

4. COMPARISON WITH PREVIOUS RESULTS

Several results on the global asymptotical stability of neural networks with delay have appeared in recent years [1–10], each one improving upon and generalizing on previous results. This series of generalizations culminates in two branches of results, one involving a term of the form BB^T [7] and one involving a term of the form $B^T B$ [9, 10]. As the results in [7, 10] are the most general among them, we will compare our results against those in [7, 10]. In particular, we show that the results in [7, 10] are special cases of Theorem 2, thus unifying these two branches. In [7] it was shown that the origin in Eq. (3) is globally asymptotically stable if there exists a positive definite diagonal matrix $\tilde{P}$ and a symmetric positive definite matrix $\tilde{K}$ such that

1. $\tilde{P} A + A^T \tilde{P} + \tilde{K} < 0$
2. $-2\tilde{P} - \tilde{K} + I + \tilde{P} B B^T \tilde{P} \leq 0$

Theorem 2 is more general than this result since the conditions above imply that $2\tilde{P} - \tilde{P} A - A^T \tilde{P} - I - \tilde{P} B B^T \tilde{P} > 0$. Apply the transformation $X \rightarrow P^{-1} X P^{-1}$ and we get $2\tilde{P}^{-1} - A\tilde{P}^{-1} - \tilde{P}^{-1} A - \tilde{P}^{-2} - BB^T > 0$. This is equivalent to the condition $R_2 > 0$ in Corollary 5 by setting $P = \tilde{P}^{-1}$, $D = K = S = I$. In fact Corollary 5 is strictly more general since the condition $\tilde{P} A + A^T \tilde{P} + \tilde{K} < 0$ implies that $\tilde{P} A + A^T \tilde{P} < 0$ and this requirement is not necessary to apply Corollary 5.

In [10] (and in [9] under slightly stronger conditions) it was shown that the origin is global asymptotically stable under condition (6) if there exists a positive definite matrix K, a positive definite diagonal matrix P and a positive constant β such that

$$-2PDS^{-1} + PA + A^T P + \beta B^T K B + \beta^{-1} P K^{-1} P < 0$$

The constant β is not necessary as it can be absorbed into the matrix K by noting that $\beta K > 0$ if and only if $K > 0$. Thus we can assume $\beta = 1$ without loss of generality. The stability condition is then equivalent to the matrix R_1 of Corollary 5 as $S^{-1}PD = PDS^{-1}$.

Thus we have shown that the results in [7, 9, 10] which were proved using different Lyapunov functionals can now be generalized and proved using the same Lyapunov functional.

5. CONCLUSIONS

We have presented new global stability results of neural networks with delays. We show that several results in the literature can be generalized and unified by using a single Lyapunov functional. We prove our stability result by first proving it for the canonical case and then proving it for the general case by means of state transformations. We also show that under these stability conditions, reversing the direction of coupling between neurons does not affect the stability of the network.

6. REFERENCES

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