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On the Complexity of a Class of Mixed Integer Linear Programs

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Abstract. We consider a class of mixed integer linear programs like $z = \{\max cx + dy \mid Ax + By \leq b,\ x \text{ integer valued}\}$, with the property that for any fixed vector $\bar{y}$ the integer program $z(\bar{y}) = \{\max cx \mid Ax \leq b - B\bar{y},\ x \text{ integer valued}\}$, is polynomially solvable. This means that if the continuous variables are fixed, then the remaining integer program is polynomially solvable. Because of this one could expect that the original problem could be solved in polynomial time with a Benders like approach, however we show that it is NP-hard.

1. Introduction

Benders decomposition [2] is a technique to deal with linear programs like

$$
\begin{aligned}
& z = \max cx + dy \\
& \text{subject to} \\
& Ax + By \leq b,
\end{aligned}
$$

with the assumption that when the variables y are fixed then the sub-problem below is "easy" to solve.

$$z(\bar{y}) = \max cx \tag{1}$$

$$\text{subject to} \tag{2}$$

$$Ax \leq b - B\bar{y}. \tag{3}$$

The process consists of solving a master problem that gives a vector $\bar{y}$ to the sub-problem, then the sub-problem gives back a cutting plane to the master problem and the process continues until convergence. This is useful for instance when the sub-problem (1)-(3) can be efficiently solved in a combinatorial way. This process combined with the ellipsoid method can be shown to converge in polynomial time, see [4]. In this paper we study the case when the sub-problem is polynomially solvable but it has the additional constraint that x should be integer valued.

We denote by Π the class mixed integer linear programs like

$$z = \max cx + dy \tag{4}$$

$$\text{subject to}$$

$$Ax + By \leq b \tag{5}$$

$$x \text{ integer valued,} \tag{6}$$

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with the additional property that when the continuous variables y are fixed then

$$z(\bar{y}) = \max cx \tag{7}$$

$$\text{subject to}$$

$$Ax \leq b - B\bar{y} \tag{8}$$

$$x \text{ integer valued,} \tag{9}$$

is polynomially solvable.

Because of this last property one could expect that problem (4)-(6) could be solved in polynomial time with a Benders like approach. We prove that Π is NP-hard, namely we show that the max-cut problem admits a MILP formulation with the properties described above. We refer to [3], [6] for a discussion on NP-hardness.

Given a graph $G = (V, E)$ an $U \subseteq V$ the set of edges with exactly one endnode in U is called a *cut*. If each edge e has a weight $c(e)$ the *max-cut problem* consists of finding a cut C such that the sum of the weights of the edges in C is maximum. This problem appears in the initial list of NP-hard problems given by Karp [5].

2. A MILP formulation of max-cut

In this section we show that the max-cut problem admits a MILP formulation with the properties described before. We start with some definitions.

Let $G = (V, E)$ be a weighted graph and assume that there is a node v_0 that is adjacent to all other nodes in V. This can always be achieved by adding edges of weight zero. We denote by G' the subgraph $G \setminus v_0$. For an edge e incident to u and v we also denote e by uv. For an edge $uv \in G'$ we denote by T_{uv} the triangle $T_{uv} = \{v_0u, uv, vv_0\}$. For a set $S \subseteq E$ the vector $x^S \in \Re^E$ defined by $x^S(e) = 1$ if $e \in S$, and $x^S(e) = 0$ if $e \notin S$, is called the *incidence vector* of S.

We need a series of Lemmas below. The following is well known in graph theory.

Lemma 1. *A 0-1 vector x is the incidence vector of a cut if and only if*

$$x \cdot y \equiv 0 (mod\ 2), \tag{10}$$

for every vector y that is the incidence vector of a cycle.

Proof. Clearly the intersection between a cut and a cycle has even cardinality.

Now let us assume that S is an edge-set whose intersection with any cycle has even cardinality. We describe a procedure to deduct a cut from the set S. We first pick a spanning tree. Then we give to a node the label $+$. Then we traverse the tree giving the labels $+$ and $-$ to the nodes based on the following rule. If an edge in the tree is in S then its endnodes should have opposite labels, if the edge is not in S then its endnodes should have the same label.

Now we claim that for the every edge $e \notin T$ its endnodes have opposite labels if and only if $e \in S$. Let u and v be the endnodes of e. If $e \in S$ because of the initial assumption, in the path in T between u and v there is an odd number of edges in S, this means that there is an odd number of changes in the labels, therefore u and v have different labels. Similarly if $e \notin S$ then there is an even number of edges in S in the path between u and v, thus u and v have the same label.

The labels partition V into V_+ and V_-. The cut given by this partition is exactly the set S. ☐

Thus the set of incidence vectors of cuts are the $0-1$ vectors that satisfy

$$Ax \equiv 0 (\text{mod } 2), \tag{11}$$

where the rows of the matrix A are the incidence vectors of cycles. It is enough to include in A a basis of the vector space over $GF(2)$ generated by the rows of A, this is called a *cycle basis.*

Lemma 2. *The set of incidence vectors of triangles T_{uv} for every edge $uv \in G'$, form a cycle basis.*

Proof. Since every edge $uv \in G'$ appears in exactly one triangle, this set of vectors is linearly independent.

Now consider a cycle $\{u_1u_2, \cdots, u_ku_1\}$ in G'. Its incidence vector can be obtained by adding (mod 2) the incidence vectors of $T_{u_1u_2}, \cdots, T_{u_ku_1}$. Finally consider a cycle $\{v_0u_1, u_1u_2, \cdots, u_kv_0\}$, its incidence vector can be obtained by adding (mod 2) the incidence vectors of $T_{u_1u_2}, \cdots, T_{u_{k-1}u_k}$. □

Lemma 3. *Let $uv \in G'$ and let x_1, x_2, x_3 be associated with v_0u, uv, vv_0 respectively. Condition* (10) *for T_{uv} is equivalent to*

$$x_1 + x_2 + x_3 \leq 2 \tag{12}$$
$$x_1 - x_2 - x_3 \leq 0 \tag{13}$$
$$-x_1 + x_2 - x_3 \leq 0 \tag{14}$$
$$-x_1 - x_2 + x_3 \leq 0 \tag{15}$$
$$x_i \in \{0, 1\},\ \textit{for } i = 1, 2, 3. \tag{16}$$

Proof. It is easy to see that a 0-1 vector x satisfies (12)-(16) if and only if $x_1 + x_2 + x_3$ is even. □

Inequalities (12)-(15) are called *triangle inequalities*, cf. [1]. From the lemmatta above we have that an integer programming formulation of max-cut is given by

$$\max cx \tag{17}$$
$$x \text{ satisfies (12)-(16) for all } T_{uv}, uv \in G'. \tag{18}$$

The next lemma shows that we can relax the integrality of the variables associated with edges in G'.

Lemma 4. *The integrality of x_1 and x_3 in* (12)-(15) *implies the integrality of x_2.*

Proof. First notice that (13) and (15) imply $x_2 \geq 0$.

If $x_1 = x_3 = 1$ then (12) implies $x_2 \leq 0$. If $x_1 = x_3 = 0$ then (14) imply $x_2 \leq 0$.

Now notice that (12) and (14) imply $x_2 \leq 1$. If $x_1 = 1$, $x_3 = 0$ then (13) implies $x_2 \geq 1$. If $x_1 = 0$, $x_3 = 1$ then (15) implies $x_2 \geq 1$. □

The lemma below shows that inequalities $0 \leq x_i \leq 1$ are not needed.

Lemma 5. *Inequalities* (12)-(15) *imply $0 \leq x_i \leq 1$.*

Proof. Just add all possible pairs of these inequalities. □

Let us partition the vector x into y and z, where y is associated with edges incident to v_0 and z is associated with edges in G'. Now we can write the following mixed integer linear programming formulation.

$$\max dy + ez \tag{19}$$

$$(y, z) \text{ satisfies (12)-(15) for all } T_{uv}, uv \in G' \tag{20}$$

$$y \text{ integer valued.} \tag{21}$$

Now we shall see that if z is fixed then (19)-(21) is easy to solve.

Lemma 6. *If the vector z is fixed then* (19)-(21) *can be solved in linear time.*

Proof. Consider first the case when z has a fractional component. Lemma 4 implies that there is no integer feasible vector y.

Now assume that z is integer valued. Consider a triangle T_{uv} and inequalities (12)-(15). If $x_3 = 0$ then (13) and (14) imply $x_1 = x_2$. If $x_3 = 1$ then (12) and (15) imply $x_1 = 1 - x_2$.

So we can pick a node $v \in G'$ and let $\lambda = y(v_0 v)$. Then for an edge $vw \in G'$ we set $y(v_0 w)$ equal to either λ or $1 - \lambda$ depending on the value of $z(vw)$. We continue until either a contradiction is found, or all variables y have an assigned value. Then we decide whether λ should be 0 or 1. $\square$

Problem (19)-(21) is a mixed integer linear programming formulation of max-cut with the property that when the continuous variables are fixed, the remaining integer program is polynomially solvable. Therefore we can state our main result.

Theorem 1. *Problem Π is NP-hard.*

3. Final Remarks

We studied the complexity of Π after failing to produce a polynomial time algorithm for this class of mixed integer linear programs. We still find it surprising that this is NP-hard.

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