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Comparing valid inequalities for cyclic group polyhedra

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Abstract

We study the interpolation procedure of Gomory and Johnson (1972), which generates cutting planes for general integer programs from facets of cyclic group polyhedra, and has recently been re-considered by Gomory, Johnson and Evans (2003). We compare inequalities generated by this procedure with MIR inequalities and also with the two-step MIR inequalities discussed in Dash and Gunluk (2003). Our results generalize a result of Cornuéjols, Li and Vandenbussche (2003) on comparing the strength of the Gomory mixed-integer cut with related inequalities. We also show that in some cases, given an inequality generated by the interpolation procedure, one can generate stronger inequalities via the MIR principle, and a similar two-step MIR principle.

1 Introduction

In a sequence of papers, Gomory [9], and later Gomory and Johnson [10, 11], studied the polyhedral structure of the master cyclic group polyhedron

$$P(n,r) = conv\Big\{ w \in Z^{n-1} \,:\, \sum_{i=1}^{n-1} (i/n)\, w_i \equiv r/n \,(\text{mod } 1), \quad w \geq 0 \Big\} \tag{1}$$

where $n, r \in Z$ and $n > r > 0$, and $a \equiv b\,(\text{mod } 1)$ means that $a - b$ is an integer. For a set $S \subseteq R^n$, we use $conv(S)$ to denote the convex hull of vectors in S. As discussed in [9, 10], facets of $P(n,r)$ can easily be translated into sub-additive functions using an *interpolation procedure*. These functions yield cutting planes for general integer programs. In particular, let

$$Y \;=\; \Big\{ x \in Z^{|J|}, y \in Z \,:\, \sum_{j \in J} a_j\, x_j + y = b, \quad x \geq 0 \Big\} \tag{2}$$

($\sum_{j \in J} a_j x_j + y = b$ could be derived from a row of the simplex tableau) and let $h(v)$ be a sub-additive function derived via interpolation from some $P(n,r)$. Let $\hat{c} = c - \lfloor c \rfloor$ for $c \in R$. Then it is possible to show that

$$\sum_{i=1}^{|J|} h(\hat{a}_j) x_j \geq h(\hat{b}) \tag{3}$$

is a valid inequality for Y. Recently, there has been renewed interest in $P(n,r)$, see [13], [2], [12] and [7]. In particular, Evans [8] and Gomory, Johnson and Evans [13] present an empirical approach to identify "important" facets of $P(n,r)$ for small n and propose using sub-additive functions based on these important facets as cutting planes for general integer programs. The most important classes of facets identified in these empirical studies are the facets based on the Gomory mixed-integer cut (GMIC), and the so-called "2slope facets".

In a recent paper [7], we discussed how valid inequalities for Y can be derived from the mixed-integer rounding (MIR) principle, and a related two-step MIR principle. In this paper, we are primarily interested in comparing such inequalities with cuts derived via the interpolation procedure mentioned above. Our analysis of this procedure yields a generalization of a result of Cornuéjols, Li and Vandenbussche [5] on comparing the strength of the GMIC with related inequalities. Our main result is that in a number of important cases, given an inequality based on the interpolation procedure, one can generate a stronger inequality using the MIR principle or the two-step MIR principle.

The structure of the paper is as follows. In the next section we review earlier results and motivate our study. In Section 4 we study properties of the facet defining inequalities for $P(n,r)$and describe the interpolation procedure of Gomory. In Section 5 we compare the strength of the inequalities generated by the interpolation procedure with MIR based inequalities.

2 Valid inequalities based on simple sets

It is well-know that the basic MIR principle, based on a simple mixed-integer set with two variables, can be used to derive the GMIC. In particular, if

$$Q^1 = \Big\{x \in R,\ z \in Z\ :\ x + z\ \geq\ \beta,\ \ x \geq 0\Big\}$$

where $\beta \notin Z$, then the MIR inequality $x\ \geq\ (\beta - \lfloor\beta\rfloor)(\lceil\beta\rceil - z)$ is valid and facet defining for Q^1 (see [14, 15]). The above inequality can be combined with an appropriate relaxation of Y to obtain the GMIC for Y. In [7], we discuss this idea in some detail and also define t-scaled MIR inequalities, which we need here. As before, let $\hat{c}$ denote $c - \lfloor c\rfloor$ for $c \in R$.

Definition 1 *For $b \in R$, the MIR function with parameter b is defined as:*

$$f^b(v) = \begin{cases} \hat{v}/\hat{b} & \text{if } \hat{v} < \hat{b}, \\ (1-\hat{v})/(1-\hat{b}) & \text{if } \hat{v} \geq \hat{b}, \end{cases}$$

For any integer t such that $tb \notin Z$, the t-scaled MIR inequality

$$\sum_{j \in J} f^{tb}(ta_j)x_j \geq 1 \tag{4}$$

is valid for Y. These inequalities are called k-cuts in [5], and can be viewed as the GMIC applied to $\sum_{j\in J} a_j x_j + y = b$ after scaling it by an integer t.

In [7] we also consider the set

$$Q^2 = \Big\{x \in R,\ y, z \in Z\ :\ x + \alpha y + z\ \geq\ \beta,\ \ x, y \geq 0\Big\},$$

where $1 > \alpha > 0$ and $\beta \notin Z$. We show there that the *two-step MIR inequality*, defined as $x\ \geq\ (\beta - \alpha\lfloor\beta/\alpha\rfloor)(\lceil\beta/\alpha\rceil - \lceil\beta/\alpha\rceil z - y)$, is valid and facet defining for Q^2 provided that $1/\alpha \geq \lceil\beta/\alpha\rceil$. We then use the two-step MIR inequalities to derive valid inequalities for Y.

Definition 2 *For $b, \alpha \in R$ satisfying $\hat{b} > \alpha > 0$, and $1/\alpha \geq \lceil\hat{b}/\alpha\rceil > \hat{b}/\alpha$, the* two-step MIR function *with parameters b and α is defined as:*

$$g^{b,\alpha}(v) = \begin{cases} \dfrac{\hat{v}(1-\rho\tau) - k(v)(\alpha-\rho)}{\rho\tau(1-\hat{b})} & \text{if } \hat{v} - k(v)\alpha < \rho \\[2ex] \dfrac{k(v) + 1 - \tau\hat{v}}{\tau(1-\hat{b})} & \text{if } \hat{v} - k(v)\alpha \geq \rho, \end{cases}$$

where $\rho = \hat{b} - \alpha\lfloor\hat{b}/\alpha\rfloor$, $\tau = \lceil\hat{b}/\alpha\rceil$ and $k(v) = \min\{\lceil\hat{v}/\alpha\rceil, \tau\} - 1$.

For any $t \in Z$ and $\alpha \in R$ such that tb and α are valid parameters for the two-step MIR function, the t-scaled two-step MIR inequality

$$\sum_{j \in J} g^{tb,\alpha}(ta_j)x_j \geq 1 \tag{5}$$

is valid for Y.

When applied to $P(n,r)$ (which has the same form as Y) these inequalities take the form

$$\sum_{i=1}^{n} f^{tr/n}(ti/n)\, w_i \geq 1 \tag{6}$$

and

$$\sum_{i=1}^{n} g^{tr/n,\Delta/n}(ti/n)\, w_i \geq 1 \tag{7}$$

for integers t and Δ. They are valid and facet defining under mild conditions (see [7]). In particular, the inequality in (6) defines a facet of $P(n,r)$ if tr is not a multiple of n; we call such a facet a *t-scaled MIR* facet.

3 Valid inequalities based on interpolation

We first present a result of Gomory [9] that gives a complete characterization of the nontrivial facets (i.e., excluding the non-negativity inequalities) of $P(n,r)$.

Theorem 3 *(Gomory [9]) If $r \neq 0$, then $\sum \eta_j w_j \geq 1$ is a non-trivial facet of $P(n,r)$ if and only if $\eta = (\eta_j)$ is an extreme point of the inequality system*

$$\begin{aligned} \eta_i + \eta_j &\geq \eta_{(i+j)\, mod\, n} \quad \forall i,j \in \{1,\ldots,n-1\}, && (8)\\ \eta_i + \eta_j &= \eta_r \quad \forall i,j \text{ such that } r = (i+j)\, mod\, n, && (9)\\ \eta_j &\geq 0 \quad \forall j \in \{1,\ldots,n-1\}, && (10)\\ \eta_r &= 1. && (11) \end{aligned}$$

The property (8) is called *sub-additivity*; we call the property (9) *r-additivity*. We will only be interested in non-trivial facets of $P(n,r)$.

Let $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ be a facet of $P(n,r)$. Let $h : R \rightarrow [0,1]$ be an associated piecewise-linear function defined by:

$$h(v) = \begin{cases} 0, & \text{if } v = \lfloor v \rfloor\,, \\ \eta_i & \text{if } v - \lfloor v \rfloor = \frac{i}{n} \text{ for } i = 1,\ldots,n-1, \\ \delta h(\frac{i}{n}) + (1-\delta)h(\frac{i+1}{n}) & \text{if } v - \lfloor v \rfloor = \frac{i+\delta}{n}, \text{ for } 0 < \delta < 1. \end{cases}$$

Note that $0 \leq h(v) \leq 1$ as $0 \leq \eta_i \leq \eta_r = 1$, and $h(v) = h(\hat{v})$ for all $v \in R$. We will call $h(v)$ a *facet interpolated template function*, abbreviated as a *template function*. In Figure 1 we present a template function based on the 6-scaled MIR facet for $P(13,12)$.

Gomory and Johnson [10] derived the following *sub-additivity* property of template functions from the sub-additivity property of facet coefficients η_i:

$$h \text{ is a template function of } P(n,r) \Rightarrow h(x) + h(y) \geq h(x+y),\ \forall x,y \in R. \tag{12}$$

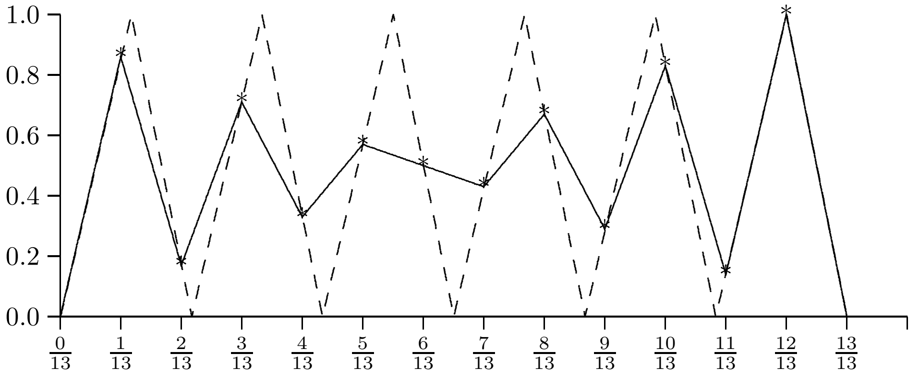

Figure 1: 6-scaled MIR for $P(13, 12)$ and template function

In this paper we will say a function $h(v)$ is *sub-additive over* $[0, 1]$ if $h(v) = h(\hat{v})$ for all $v \in R$, and $h : [0, 1] \to [0, 1]$ and h is sub-additive. The functions f^b and $g^{b,\alpha}$ defined earlier are both sub-additive over $[0, 1]$. The next result implies that template functions can be used to derive cutting planes for Y.

Proposition 4 *(Gomory, Johnson [10]) If $h : R \to [0, 1]$ is sub-additive function over $[0, 1]$, then $\sum_{i\in J} h(a_j)x_j \geq h(b)$ is a valid inequality for Y.*

An important question is: which template function based cutting planes are useful for Y ?

Some template functions for Y can be viewed as being special in the following sense. Let n be the smallest positive integer such that all coefficients of $\sum_{j\in J} a_jx_j + y = b$ become integral when multiplied by n. Define $r = n\hat{b}$, and consider $P(n, r)$ for this choice of n and r; we will call this the *canonical master polyhedron* for Y. Let h be a template function associated with a facet $\eta^T w \geq 1$ of $P(n, r)$. As the values $\hat{a}_j$ are multiples of $1/n$, $h(a_j)$ is equal to the facet coefficient $\eta_{n\hat{a}_j}$. It follows from this, and a result of Gomory [9], that the set of all template functions derived from $P(n, r)$ give the convex hull of Y. That is, every facet of $conv(Y)$ is of the form $\sum_{j\in J} h(\hat{a}_j)x_j \geq h(\hat{b})$ where h is some template function derived from $P(n, r)$.

In most cases, the multiplier n required to make the coefficients of $\sum_{j\in J} a_jx_j+y = b$ integral is too large. In this situation it is impractical to work directly with the canonical $P(n, r)$ and the associated inequality system (8)-(11). However, we do know some classes of facets and template functions for any $P(n, r)$ and thus for the canonical master polyhedron for Y. These are the t-scaled MIR inequalities and the two-step MIR inequalities discussed in Section 2, which can be used to get valid inequalities for Y. We next discuss an alternative approach to using template functions to get valid inequalities for Y.

3.1 Gomory's shooting experiments

In a recent study Gomory, Johnson and Evans [13] describe a computational approach to identify the "important" facets of the small master cyclic group polyhedra, i.e, $P(n, r)$ with $n \leq 20$. They perform a *shooting experiment*, which is a randomized procedure in which they choose a random direction d and identify the facet first encountered along the ray λd, $\lambda > 0$. This facet can be computed by minimizing $d^T \eta$ over the inequalities (8)-(11). By repeating this experiment a number of times (e.g., for 10,000 random directions) they conclude that a relatively small number of facets are important, and that the t-scaled MIR facets (6) are the most important facets for small master polyhedra. Their results are based on a more extensive study by Evans [8] where she also shows that the so-called "2slope facets" constitute the second most important class for small n. These facets form a sub-class of the facets given by inequality (7).

Gomory, Johnson and Evans[13] propose generating cutting planes for Y based on template functions of the important facets of small $P(n, r)$. This approach would then favor using template functions of scaled MIR facets and 2slope facets over template functions of other facets.

4 Comparing cutting planes

In this section we will analyze the relative strengths of different cutting planes for Y. If $\eta^T x \geq 1$ and $\beta^T x \geq 1$ are valid inequalities for Y in (2), and $\eta \leq \beta$, then the first inequality is obviously preferable to the second. How does one compare cutting planes when such obvious criteria do not apply ? As we discussed earlier, one approach to this question was proposed in [13], where facets of low-dimensional master polyhedra are compared via a shooting experiment. Another approach, which we use, is presented in Cornuéjols, Li and Vandenbussche [5], where the authors compare t-scaled MIR cuts, for different scaling factors t, by comparing their coefficients component-wise.

Proposition 4 implies that if $h(v)$ is a sub-additive function over $[0, 1]$, then the inequality h^Y, defined as

$$\sum_{j \in J} \frac{h(a_j)}{h(b)} x_j \geq 1, \tag{13}$$

is valid for Y provided that $h(b) \neq 0$ (we will assume this condition to be true throughout). If $h(v)$ is a template function , we call h^Y a *template inequality*; if $h(v)$ is a template function arising from the canonical master polyhedron for Y, we call h^Y a *canonical template inequality*. The functions f^b and $g^{b,\alpha}$, and template functions for different master polyhedra are piece-wise linear sub-additive functions, and therefore provide recipes for generating valid inequalities for varying Y. To compare the relative strength of the resulting inequalities, we use the following definition.

Definition 5 *Let $h_1(x)$ and $h_2(x)$ be two sub-additive functions over $[0, 1]$. Then h_1* dominates *h_2 if $h_1(v) \leq h_2(v)$ for all $v \in [0, 1]$. If, in addition, there is some interval $[a, b] \subset [0, 1]$ such that $h_1(v) < h_2(v)$ for all $v \in [a, b]$, then h_1* strictly dominates *h_2. Finally, let $\mathcal{X}$*

be a random variable which is uniformly distributed over $[0,1]$*. If* $\mathcal{P}(h_1(\mathcal{X}) < h_2(\mathcal{X})) > \mathcal{P}(h_1(\mathcal{X}) > h_2(\mathcal{X}))$*, then* h_1 *is said to* dominate h_2 *in a* probabilistic sense.

To see the motivation for the definition above, observe that if h_1 dominates h_2 and $h_1(b) = h_2(b)$, then the inequality h_1^Y implies h_2^Y. Further, if the coefficients of Y are chosen randomly from $[n_1, n_2]$, where n_1 and n_2 are integers, then the fractional parts of the coefficients of Y are uniformly distributed over $[0,1]$. If h_1 dominates h_2 in a probabilistic sense and $h_1(b) = h_2(b)$, then it is more likely that $h_1(a_j) < h_2(a_j)$ than $h_2(a_j) < h_1(a_j)$ for an index $j \in J$.

4.1 Properties of facets and template functions

The following property of facets of $P(n,r)$ follows trivially from r-additivity.

Proposition 6 *Let* $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ *and* $\sum_{i=1}^{n-1} \beta_i w_i \geq 1$ *be non-trivial facets of* $P(n,r)$ *with* $r \neq 0$*. (i) The average value of the facet coefficients* η_i *is* $n/(2(n-1))$*, and (ii)* $|\{i : \eta_i > \beta_i\}| = |\{i : \beta_i > \eta_i\}|$*.*

Proof. For any non-zero index i between 1 and $n-1$, if $2i \bmod n = r$ then $\eta_i = 1/2$. On the other hand, if $2i \bmod n \neq r$ then $\eta_i + \eta_{r-i} = 1$. As $\eta_r = 1$, $\sum_{i=1}^{n-1} \eta_i = n/2$ and part (i) follows. Part (ii) follows from the fact that $\eta_i > \beta_i \Leftrightarrow \beta_{r-i} > \eta_{r-i}$. ∎

Proposition 6 says that the coefficients of all non-trivial facets of $P(n,r)$ have the same average value. Also, given two facet-defining inequalities, either inequality has the same number of coefficients which exceed the corresponding coefficients in the other.

To show an analogous statement for template functions , we derive a property of template functions similar to the r-additivity of facet coefficients.

Proposition 7 *If* $h(x)$ *is a template function derived from a facet of* $P(n,r)$*, then* $h(x) + h((r/n) - x) = h(r/n),\ \forall x \in [0,1]$*.*

Proof. By the definition of template functions and because of (9), this is true if $x = i/n$ for any integer i between 0 and n. Let $x = (i+\delta)/n$ for some $\delta \in (0,1)$ and some integer i between 0 and $n-1$. Now,

$$h(x) = \delta h(\frac{i}{n}) + (1-\delta)h(\frac{i+1}{n}), \quad \text{and} \quad h(\frac{r}{n} - x) = \delta h(\frac{r-i}{n}) + (1-\delta)h(\frac{r-i-1}{n}).$$

Adding up the right-hand sides of the equations above we get $\delta(r/n) + (1-\delta)(r/n) = r/n$ and the lemma follows. ∎

Given a sub-additive function over $[0,1]$, say $h(v)$, we will call it b*-additive* if $h(v)+h(b-v) = h(b)$ for all $v \in R$. The previous lemma states that template functions for $P(n,r)$ are (r/n)-additive.

We now show that for any two template functions derived from non-trivial facets of a common master polyhedron, neither dominates the other in a probabilistic sense.

Lemma 8 *Let $h_1(x)$ and $h_2(x)$ be two template functions associated with $P(n,r)$ for some fixed n and non-zero r. Let $\mathcal{X}$ be a random variable which is uniformly distributed over $[0,1]$. (i) The expected value of $h_1(\mathcal{X})$ is $1/2$. (ii) $\mathcal{P}(h_1(\mathcal{X}) > h_2(\mathcal{X})) = \mathcal{P}(h_2(\mathcal{X}) > h_1(\mathcal{X}))$.*

Proof. Define $b = r/n$. Observe that h_1 and h_2 are piece-wise linear functions, with $h_i(x) + h_i(b-x) = h_i(b) = 1$ and $h_i(0) = h_i(1) = 0$ for $i = 1,2$. Hence

$$\int_0^b h_1(x)dx = \int_0^{b/2} (h_1(x) + h_1(b-x))dx = b/2.$$

Similarly $\int_b^1 h_1(x)dx = (1-b)/2$, and part (i) of the proposition follows.

As h_1 and h_2 are piece-wise linear and have n linear segments, there are fewer than n points at which they cross. Therefore one can construct a finite ordered set $S = (s_1, s_2, \ldots, s_m)$, containing the fewer than n crossing points, with the following properties:
(a) $\{0, \frac{b}{2}, b, \frac{1+b}{2}, 1\} \subseteq S \subseteq [0,1]$;
(b) In the interval (s_i, s_{i+1}), for $1 \leq i \leq m-1$, if one of the functions h_1 and h_2 is strictly greater than the other at one point, it is strictly greater throughout the interval;
(c) If $x \in S$, then so does $b - x$.
Now consider an interval $(s_i, s_{i+1}) \subseteq (0, b/2)$. If $h_1(x) > h_2(x)$ for all x in (s_i, s_{i+1}), then because of Lemma 7, $h_2(x) > h_1(x)$ for all x in $(b - s_{i+1}, b - s_i) \subseteq (b/2, b)$. Arguing similarly for intervals in $(b/2, b), (b, \frac{1+b}{2})$ and $(\frac{1+b}{2}, 1)$, it follows that $\mathcal{P}(h_1(\mathcal{X}) > h_2(\mathcal{X})) \leq \mathcal{P}(h_2(\mathcal{X}) > h_1(\mathcal{X}))$. Reversing the roles of h_1 and h_2, the result follows. ∎

Suppose we choose Y to be a row of a simplex tableau associated with the linear relaxation of an integer program. Over many different integer programs, it is reasonable to assume that the fractional parts of the coefficients in Y will be uniformly distributed over $[0,1]$. The above theorem indicates that for any two template functions , neither is likely to yield stronger inequalities than the other, when relative strength is measured by counting how many coefficients of one inequality are less than corresponding coefficients in the other inequality.

We next restrict our attention to template functions arising from t-scaled MIR facets of $P(n,r)$. We define the *t-scaled MIR function*, where t is an integer, in terms of the MIR function described earlier:

$$f^{t,b}(v) = f^{tb}(tv).$$

Then the t-scaled MIR inequality can be written as $\sum_{j \in J} f^{t,b}(a_j) \geq 1$. In Figure 1 we notice that the template function of the 6-scaled MIR cut of $P(13,12)$ does not coincide with the 6-scaled MIR function. This is because the scaling coefficient t does not satisfy the following property.

Proposition 9 *If t is a divisor of n, then the template function $h(v)$ of the t-scaled MIR cut coincides with the t-scaled MIR function $f^{t,b}(v)$.*

Proof. It is sufficient to observe that $h(x) = f^{t,b}(x)$ whenever $f^{t,b}(x) = 0$ *or* 1. ■

Cornuéjols, Li, and Vandenbussche [5][Theorem 1(ii),Theorem 3] compare t-scaled MIR inequalities with the GMIC (1-scaled MIR inequality), for different values of $t \in Z$. They compute $\mathcal{P}(f^{t,b}(\mathcal{X}) > f^{1,b}(\mathcal{X}))$ and $\mathcal{P}(f^{1,b}(\mathcal{X}) > f^{t,b}(\mathcal{X}))$. They also show that these probabilities are equal. We can derive this result from Lemma 8 and Proposition 9 since for any two positive integers t_1,t_2 and a rational number b with $\hat{b} = r/n$, one can use $P(t_1 t_2 n, t_1 t_2 r)$ to compare the t_1-scaled MIR function with the t_2-scaled MIR function. We next state this observation formally.

Corollary 10 *Let b be a rational number and let t_1 and t_2 be positive integers. If $\mathcal{X}$ is uniformly distributed over $[0,1]$, then $\mathcal{P}(f^{t_1,b}(\mathcal{X}) > f^{t_2,b}(\mathcal{X})) = \mathcal{P}(f^{t_2,b}(\mathcal{X}) > f^{t_1,b})(\mathcal{X}))$.*

We know that the expected value of the scaled MIR functions in the interval $[0,1]$ is $1/2$. We close this section with a comment on their variance.

Lemma 11 *Let $t \in Z$ and $1 > b > 0$ be such that $tb \notin Z$. If $\mathcal{X}$ is distributed uniformly in $[n_1, n_2)$, for $n_1, n_2 \in Z$, then $\mathcal{Y} = f^{t,b}(\mathcal{X})$ is distributed uniformly in $[0,1)$.*

Proof. We will show that $\mathcal{P}(\mathcal{Y} \le \delta) = \delta$ for $1 \ge \delta \ge 0$. First note that if $\mathcal{X}$ is distributed uniformly in $[n_1, n_2)$, then $\hat{\mathcal{X}} = t\mathcal{X} - \lfloor t\mathcal{X} \rfloor$ is distributed uniformly in $[0,1)$. Therefore given $\delta \in [0,1)$, we can write

$$\begin{aligned}
\mathcal{P}(\mathcal{Y} \le \delta) &= \mathcal{P}(\mathcal{Y} \le \delta, \hat{\mathcal{X}} < \hat{b}^t) + \mathcal{P}(\mathcal{Y} \le \delta, \hat{\mathcal{X}} \ge \hat{b}^t) \\
&= \mathcal{P}(\hat{\mathcal{X}}/\hat{b}^t \le \delta) \; + \; \mathcal{P}((1-\hat{\mathcal{X}})/(1-\hat{b}^t) \le \delta)) \\
&= \mathcal{P}(\hat{\mathcal{X}} \le \hat{b}^t \delta) \; + \; \mathcal{P}(\hat{\mathcal{X}} \ge 1 - (1-\hat{b}^t)\delta)) \\
&= \delta\, \hat{b}^t + \delta(1-\hat{b}^t) = \delta
\end{aligned}$$
■

It is easy to see that 1-scaled two-step MIR functions, while having an expected value of $1/2$ in $[0,1]$, have a smaller variance than the t-scaled MIR functions. We do not know how this property translates into effectiveness of the corresponding cuts.

5 Comparing template and MIR inequalities

Given Y, we know that template inequalities yield valid inequalities for Y. As discussed in Section 3.1, Gomory, Johnson and Evans [13] propose using template inequalities based on important facets of $P(n,r)$ with small n. Their approach would favour using template inequalities based on t-scaled MIR facets, and two-step MIR facets (which include the "2slope facets" in [2]).

The question we will address in this section is the following: how strong are the above classes of template inequalities when compared to the t-scaled MIR inequalities and two-step MIR inequalities for Y? Recall that the last two classes of inequalities can be viewed as canonical template inequalities for Y in certain cases. We start with a result on the scaled-MIR inequalities. For $c \in R$ and $t \in Z$, define $\hat{c}^t = tc - \lfloor tc \rfloor$.

Theorem 12 *Let $h(x)$ denote the template function of the t-MIR facet of $P(n,r)$, and $f^{t,b}(x)$ be the t-MIR function. If t is a divisor of n, then (i) $f^{t,b}(v) \le h(v)/h(\hat{b})$ for $v \in [0,1]$, (ii) furthermore, if $h(\hat{b}) \ne 1$ and $\mathcal{X}$ is uniformly distributed over $[0,1]$, then $\mathcal{P}(f^{t,b}(\mathcal{X}) < h(\mathcal{X})/h(\hat{b})) \ge \min\{\hat{b}^t, 1-\hat{b}^t\}$.*

Proof. Let ρ stand for r/n, and f for $f^{t,b}$, and h' for $h/h(\hat{b})$. As t is a divisor of n, Lemma 9 implies that $h = f^{t,r/n} = f^{t,\rho}$. Now, both f and h' are periodic functions with period $1/t$ and $f(0) = h'(0) = 0$. Part (i) of the theorem will follow if we show that $f \le h'$ in the interval $[0, 1/t]$. Also, f and h' are piece-wise linear functions, with f being linear in the intervals $[0, \hat{b}^t/t]$ and $[\hat{b}^t/t, 1/t]$, and h' being linear in the intervals $[0, \hat{\rho}^t/t]$ and $[\hat{\rho}^t/t, 1/t]$. As $h(\hat{b}^t/t) = h(\hat{b})$, we know that $f(\hat{b}^t/t) = h'(\hat{b}^t/t) = 1$. Therefore, showing that

$$f(\hat{\rho}^t/t) \le h'(\hat{\rho}^t/t) \tag{14}$$

is enough to prove part (i). Now, $h(\hat{b}) \le 1$ and $h(\hat{\rho}^t/t) = 1$. Therefore $h'(\hat{\rho}^t/t) \ge 1$. As $f(x)$ is never greater than 1, (14) is true and part (i) of the theorem follows. For part (ii), assume $h(\hat{b}) \ne 1$. Then $\hat{b}^t \ne \hat{\rho}^t$.
Case 1: $\hat{b}^t < \hat{\rho}^t$. In this case, $h(\hat{b}) = \hat{b}^t/\hat{\rho}^t$, and thus $h'(\rho^t/t) = \hat{\rho}^t/\hat{b}^t > 1$. Therefore the inequality in (14) is strict, and $f(x) < h'(x)$ if $x \in (\hat{b}^t/t, 1/t)$. Because of the periodicity of f and h', it follows that $f(\mathcal{X}) < h'(\mathcal{X})$ if $\hat{\mathcal{X}}^t > \hat{b}^t$, and the probability of this happening is $1 - \hat{b}^t$.
Case 2: $\hat{b}^t > \hat{\rho}^t$. In this case, $h(\hat{b}) = (1-\hat{b}^t)/(1-\hat{\rho}^t) < 1$. An easy calculation shows that $h'(\rho^t/t) > 1$ and that $f(\mathcal{X}) < h'(\mathcal{X})$ if $\hat{\mathcal{X}}^t < \hat{b}^t$. The probability of the latter being true is $\hat{b}^t$ and the theorem follows. ■

In Figure 5 we give an application of Theorem 12, where $t = 2, \hat{b} = .75, n = 10$ and $r = 7$. The solid line stands for $f^{2,.75}$, and the dashed line for $h(x)/h(.75)$, where h is the template function of the 2-scaled MIR facet of $P(10,7)$. Because of Lemma 9, $h = f^{2,.7}$.

The two most important facets of $P(10,9)$ presented in [13] are t-MIR facets for $t = 5, 2$, both of which are factors of 10. Theorem 12 implies that template functions of these facets should not be used to obtain cutting planes for Y, as they will be dominated by the 5-MIR cut and 2-MIR cut, respectively. In fact, 9 of the 13 important facets presented in [13] are t-MIR facets with have similar divisibility properties, and therefore Theorem 12 implies that template inequalities based on them will be by dominated by t-scaled MIR inequalities.

When t is not a divisor of n, the strict domination of the template function by the scaled-MIR inequality does not necessarily hold. However it is still possible to show that (using the notation of Theorem 12): $\mathcal{P}(f(\mathcal{X}) > h'(\mathcal{X})) > \mathcal{P}(h'(\mathcal{X}) > f(\mathcal{X}))$, assuming some restriction on the choice of h'. These would imply that in many situations it is preferable to directly use scaled MIR cuts rather than the template functions of scaled MIR cuts. As the proofs of such statements are tedious, we omit them here. However we will show that essentially all template inequalities are dominated in a probabilistic sense by the GMIC (or 1-MIR cut).

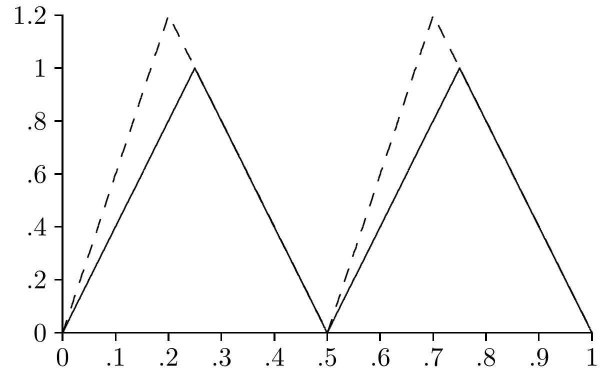

Figure 2: Domination of a template inequality by a scaled-MIR inequality

Theorem 13 *Let $h(x)$ denote the template function of a facet of $P(n, r)$. Consider a point b such that $h(b) \neq 1$ and $|r/n - b| < 1/n$. Let $f^b(x)$ be the 1-MIR function. Then f^b dominates $h(v)/h(b)$ in a probabilistic sense.*

Proof. Define $h'(v) = h(v)/h(b)$. Observe that the sub-additivity of $h(v)$ implies that $h'(v)$ is sub-additive. Therefore $h'(v) + h'(b - v) \geq h'(b) = 1 = f^b(b)$. If $f^b(v) > h'(v)$ for some v, then $f^b(b - v) < h'(b - v)$ as f^b is b-additive. Therefore, if $f^b(v) > h'(v)$ for all v in some interval $[b_1, b_2] \subseteq [0, 1]$, then $f^b(v) < h'(v)$ for all v in $[b - b_2, b - b_1]$. This implies that

$$\mathcal{P}(f(\mathcal{X}) < h'(\mathcal{X})) \geq \mathcal{P}(h'(\mathcal{X}) < f(\mathcal{X})).$$

Now assume $b > r/n$, that is, $b \in (r/n, (r + 1)/n)$. For any template function g of $P(n, r)$, $g(1/n) \geq f^{r/n}(1/n) = 1/r$ (this is because $f^{r/n}$ defines a facet of $P(n, r)$ and because of 8). Also, since $b - r/n < 1/n$, it follows that $g(b - r/n) = (b - r/n)g(1/n)$. From this we can infer that

$$h'(b - r/n) > h(b - r/n) > f^{r/n}(b - r/n) > f^b(b - r/n).$$

The first and third inequalities above are true by definition. Also, it is obvious that $h'(r/n) > 1 > f^b(r/n)$. As both h' and f^b are piecewise-linear, we see that there are two intervals $[b_1, b_2]$ and $[b - b_2, b - b_1]$ contained in $[0, 1]$ such that $f^b(v) < h'(v)$ in both these intervals. This implies the result for the case $b > r/n$.

The case $b < r/n$ is handled in a similar fashion. Finally, since $h(b) \neq 1$, b is not equal to r/n. ■

The condition $|r/n - b| < 1/n$ in Theorem 13 is motivated by the desire to have $h(b)$ as large as possible in (4). As $h(r/n) = 1$ and h is continuous, choosing r/n to be close to b ensures that $h(b)$ is reasonably close to 1. If $|r/n - b| < 1/n$, then the value of $h(b)$ is given by the third case in the definition of template functions and is non-zero.

We will now prove a result analogous to Theorem 12 about template functions arising from 1-scaled two-step MIR facets with $\tau = 2$. Then we will relate it to the template functions based on the 2slope facets in [2]. Given a number b between 0 and 1, let $\alpha \in (b/2, 1/2)$ and $\alpha < b$. Define $\tau = \lceil b/\alpha \rceil$, and let ρ stand for $b - \alpha$. Obviously $1/\alpha > \tau = 2$. Recall the two-step MIR function $g^{b,\alpha}$ in Definition 2, and observe that with the current values of b and α,

$$g^{b,\alpha}(v) = \begin{cases} \dfrac{v(1-2\rho)}{2\rho(1-b)} & \text{if } 0 \le v \le b-\alpha, \\[2ex] \dfrac{1/2 - v}{1-b} & \text{if } b-\alpha < v \le \alpha, \\[2ex] \dfrac{v(1-2\rho)-(b-2\rho)}{2\rho(1-b)} & \text{if } \alpha < v \le b, \\[2ex] \dfrac{1-v}{1-b} & \text{if } b < v \le 1. \end{cases} \tag{15}$$

Thus $g^{b,\alpha}$ is piece-wise linear, and has four linear segments. The next theorem will essentially show that template functions of two-step MIR facets with $\tau = 2$ are dominated by appropriately chosen two-step MIR inequalities with $\tau = 2$.

Theorem 14 *Let b be contained in $(0,1)$. Let $h(x)$ be the template function of a 1-scaled two-step MIR facet of $P(n,r)$, with parameter α' such that $\tau = 2$. Let b' stand for r/n. If $\alpha' \le b$ and $b/2 < b'$, then there exists some $\alpha \in (b/2, \min\{b, 1/2\})$ such that $g^{b,\alpha}(v) \le h(v)/h(\hat{b})$ for all $v \in [0,1]$.*

Proof. Notice that $h(x)$ coincides with the 1-scaled two-step MIR function $g^{b',\alpha'}$. Therefore, we want to show that

$$g^{b,\alpha}(v) \le g^{b',\alpha'}(v)/g^{b',\alpha'}(b) \text{ for all } v \in [0,1]. \tag{16}$$

For convenience, let g stand for $g^{b,\alpha}$. Also let $\rho = b - \alpha$, and $\rho' = b' - \alpha'$. Once α is chosen, (16) will follow if we show that inequality in (16) holds at the points in

$$S = \{b'-\alpha',\ b-\alpha,\ \alpha',\ \alpha,\ b',\ b\}.$$

This is because g and h are piece-wise linear functions in the interval $[0,1]$, with value 0 at $v = 0$ and $v = 1$. Now, $h(b') = 1$ and $h(b) \le 1$. Therefore, $h(v)/h(b)$ equals 1 at $v = b$, and is at least 1 at $v = b'$. At both these points, $g(v) \le 1$ for any choice of α. We will therefore focus on the first four points in S. If $b = b'$, we can set $\alpha = \alpha'$ and satisfy (16) trivially.

Case 1: Assume $b' < b$. Define $\alpha = \max\{\alpha', b/2\}$. It trivially follows that $\alpha \in (b/2, 1/2)$, and $\alpha < b$. Therefore $1/\alpha > \lceil b/\alpha \rceil = 2$, and $g^{b,\alpha}$ is well-defined and has the form in (15). By assumption $b/2 < b'$ and $\alpha' < b'$, and therefore $\alpha' \le \alpha < b'$. Also, $b' - \alpha' < b - \alpha$; if $\alpha = b/2$, then $b' - \alpha' < b'/2 < b/2 = b - \alpha$, and the other case is trivial.

As $b' < b$, $h(b) = (1-b)/(1-b')$. Therefore in the interval $[b'-\alpha', \alpha']$, $h(v)/h(b)$ becomes $(1/2-v)/(1-v)$, which is precisely the form of $g(v)$ in the interval $[b-\alpha, \alpha]$. Also, $h(v)/h(b)$ increases linearly as v increases from α' to b', and $g(v)$ decreases linearly as v decreases from $b - \alpha$ to 0. As $b' - \alpha' < b - \alpha$, and $\alpha' \le \alpha < b'$, we can conclude that (16) holds at the first four points in S, which proves the theorem.

Case 2: Let $b < b'$. Define δ' by

$$\delta' = \frac{b' - 2\rho'}{1 - 2\rho'}.$$

Because $\alpha' < 1/2$, and from the definition of δ' it follows that

$$0 < \delta' < 1/2 \text{ and } \frac{b' - \delta'}{2(1-\delta')} = \rho' = b' - \alpha'.$$

We now define α by

$$\rho = b - \alpha = \frac{b - \delta'}{2(1-\delta')}.$$

We need to verify that α chosen in this manner is valid. From the definitions of α' and α above, it follows that

$$\alpha' - \alpha = (b' - b) - \frac{b' - b}{2(1-\delta')}.$$

As $b' - b > 0$ and $\delta' < 1/2 \Rightarrow 2(1-\delta') > 1$, the right-hand side of the above equation is positive. Therefore $\alpha < \alpha' < 1/2$. By assumption, $\alpha' \leq b$, and therefore $b - \alpha$ is positive. As $(b - \delta')/(1 - \delta') < b$, we see from the above that

$$b - \alpha < b/2 \Rightarrow \alpha > b/2 \Rightarrow \tau = \lceil b/\alpha \rceil = 2.$$

Therefore $g^{b,\alpha}$ is well-defined and has the form in (15). Because $b \in [\alpha', b']$, therefore

$$h(b) = \frac{b(1-2\rho') - (b' - 2\rho')}{2\rho'(1-b')}.$$

One can verify that the slope of $h(v)/h(b)$ in the intervals $[0, b' - \alpha']$ and $[\alpha', b']$ is equal to the slope of $g(v)$ in the interval $[0, b - \alpha]$ and the interval $[\alpha, b]$. In fact, we chose α to ensure this condition. Comparing the expressions for ρ and ρ' above, we see that $b - \alpha < b' - \alpha'$. Observe that $g(v)$ decreases linearly as v increases from $b - \alpha$ to α, and decreases linearly as v decreases from b to α. The above observations, combined with the fact that $h(b)/h(b) = g(b) = 1$, and $h(b-\alpha)/h(b) = g(b - \alpha)$, imply (16). ∎

A simple transformation can be used to show that Theorem 14 is also true for (-1)-scaled two-step MIR functions. It is shown in [7] that the “2slope facets” in [2] are just (-1)-scaled two-step MIR facets with $\tau = 2$. We therefore have the following corollary to the previous theorem.

Corollary 15 *Template inequalities associated with the 2slope facets in [2] are dominated by appropriately chosen (-1)-scaled two-step MIR inequalities.*

6 Concluding Remarks

In summary, we studied the interpolation procedure of Gomory of Johnson [10] and the template functions generated by this procedure. Viewing scaled MIR inequalities as template functions of master cyclic group polyhedra, we showed that some results of Cornuéjols, Li, and Vandenbussche [5] on scaled MIR inequalities follow from general properties of template functions .

We also examined the strength of template inequalities based on small master cyclic group polyhedra. If $cw \geq 1$ is valid for $P(n, r)$, the canonical master polyhedron for Y, then there exists a set of facet defining inequalities $c_j w \geq 1, j \in S$ of $P(n, r)$, and positive multipliers $\lambda_j, j \in S$ such that:

$$c \geq \sum_{j \in S} \lambda_j c_j \text{ and } \sum_{j \in S} \lambda_j = 1. \tag{17}$$

For an arbitrary valid inequality, it is hard to find a decomposition satisfying (17). It may also happen that the inequality is not dominated by the known classes of facets for $P(n, r)$. However, Theorems 12 and 14 imply that if $cw \geq 1$ is a template inequality associated with a scaled MIR or a two-step MIR facet of a master polyhedron, then we do know the decomposition. This suggests that the scaled MIR facets or the two-step MIR facets of small cyclic group polyhedra, identified in [13] as their most important facets, should not be used for template inequalities. It is the current practice to generate at most a few cuts per inequality of an integer program. As the known facets of the canonical master polyhedron associated with Y already yield a plethora of cuts, it is not clear that facets from other master polyhedra should be used.

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