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Synchronization in Arrays of Coupled Systems with Delay and Time-Varying Coupling

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Abstract

We study synchronization in an array of coupled systems with delay and time-varying coupling and present synchronization criteria which generalize previous synchronization results. We show that the array synchronizes when the non-delay coupling term is cooperative and large enough. In particular, as in the nondelay case, the synchronization criteria is related to the second smallest eigenvalue of the matrix describing the coupling topology.

I. INTRODUCTION

Recently, there has been much activity studying the behavior of arrays of coupled systems. In this paper, we study synchronization phenomena in an array of coupled systems with delay where there is a time-varying coupling term between state variables and between delayed state variables. The time-varying coupling and the delay describe the constantly changing nature of the coupling topology and the decentralized nature of real-world coupled systems respectively. Analytical synchronization conditions were presented for the case of constant coupling [1], [2], time-varying coupling [3], [4] and constant coupling with delay [5]. We present synchronization criteria for the case of time-varying coupling with delay that generalize and unify these conditions.

We next list some mathematical notations useful in this brief. We will only work with real matrices and vectors in this paper. A (not necessarily symmetric) matrix A is positive (semi)-definite if $x^TAx > 0$ ($x^TAx \geq 0$) for all $x \neq 0$. We denote the positive (semi) definiteness of A by $A > 0$ ($A \geq 0$). We write $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$). For a symmetric real matrix X, its eigenvalues are written as $\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X)$. We denote the transpose of X as X^T and the transpose of X^{-1} as X^{-T}. The Moore-Penrose generalized inverse or pseudo-inverse of a matrix X is written as $X^\dagger$. The matrix norm used will be the norm induced by $\|\cdot\|_2$ (also called the spectral norm).

II. ARRAYS OF COUPLED SYSTEMS WITH DELAY AND TIME-VARYING COUPLING

We consider an array of coupled systems with delay coupling and time-varying coupling where the state equations are given by:

$$\dot{x}(t) = I \otimes f(x_i, t) + (G(t) \otimes D(t))x(t) + (G_\tau(t) \otimes D_\tau(t))x(t - \tau) \tag{1}$$

where $x = (x_1, \ldots, x_n)$ and $I \otimes f(x_i, t) = (f(x_1, t), \ldots, f(x_n, t))^T$. $G(t)$ and $G_\tau(t)$ describe the time-varying coupling topology of the array while $D(t)$ and $D_\tau(t)$ describe the individual coupling between two systems in the array.

Lemma 1: For matrices X and Y and a symmetric positive semidefinite matrix K of suitable dimensions,

$$X^TKK^\dagger Y + Y^TKK^\dagger X \leq X^TKX + Y^TK^\dagger Y.$$

In particular, if x and y are vectors and K is symmetric positive definite, then $x^Ty \leq \frac{1}{2}x^TKx + \frac{1}{2}y^TK^{-1}y$.
Proof: Let the real Schur decomposition of K be $K = C^T\Gamma C$ where $C = C^{-T}$ is orthogonal and $\Gamma = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues. The Lemma then follows from

$$\begin{aligned} 0 &\leq (\sqrt{\Gamma}CX - \sqrt{\Gamma^\dagger}C^{-T}Y)^T(\sqrt{\Gamma}CX - \sqrt{\Gamma^\dagger}C^{-T}Y) \\ &= X^TC^T\Gamma CX + Y^TC^{-1}\Gamma^\dagger CY - Y^TC^{-1}\sqrt{\Gamma^\dagger}\sqrt{\Gamma}CX - X^TC^T\sqrt{\Gamma}\sqrt{\Gamma^\dagger}C^{-T}Y \end{aligned}$$

as $C^{-1}\sqrt{\Gamma^\dagger}\sqrt{\Gamma}C = C^T\sqrt{\Gamma}\sqrt{\Gamma^\dagger}C^{-T} = KK^\dagger$ and $K^\dagger = C^{-1}\Gamma^\dagger C$. □

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Lemma 1 can be further generalized as follows:

Lemma 2: For matrices X and Y and a symmetric matrix K,

$$X^T\sqrt{K^2}K^\dagger Y + Y^T\sqrt{K^2}K^\dagger X \leq X^T\sqrt{K^2}X + Y^T\sqrt{K^2}^\dagger Y.$$

Proof: As in Lemma 1 let the real Schur decomposition of K be $C^T\Gamma C$. Let $L = \mathrm{diag}(\mathrm{sign}(\lambda_1),\ldots,\mathrm{sign}(\lambda_n))$ where $\mathrm{sign}(x) = 1$ for $x \geq 0$ and -1 otherwise. Note that L is orthogonal and $\sqrt{K^2} = C^T\Gamma LC \geq 0$. By Lemma 1 $X^T\sqrt{K^2}\sqrt{K^2}^\dagger Y + Y^T\sqrt{K^2}\sqrt{K^2}^\dagger X \leq X^T\sqrt{K^2}X + Y^T\sqrt{K^2}^\dagger Y$. Using the invertible transformation $Y \rightarrow C^{-1}LCY$ and the observation that $\sqrt{K^2}^\dagger C^{-1}LC = K^\dagger$ and $C^{-1}LC\sqrt{K^2}^\dagger C^{-1}LC = \sqrt{K^2}^\dagger$, we get the required inequality. □

Definition 1: W_i is the class of irreducible matrices with zero row sums and nonpositive off-diagonal elements.

Definition 2: M is the synchronization manifold defined as the linear subspace $\{x : x_i = x_j, \forall i,j\}$. If $x \rightarrow M$ as $t \rightarrow \infty$, the coupled array is said to *synchronize*.

An element of M can be written as $(1,\ldots,1)^T \otimes z$. The main result in this paper is the following theorem which gives conditions under which the array in Eq. (1) synchronizes.

Theorem 1: Let V be some symmetric positive definite matrix such that $(y-z)^T V(f(y,t) + P(t)y - f(z,t) - P(t)z) \leq -c\|y-z\|^2$ for some $c > 0$. Let U be a symmetric matrix in W_i, $(B_1(t), B_2(t))$ a factorization of $UG_\tau(t) \otimes VD_\tau(t) = B_1(t)B_2(t)$, and $K(t)$ a positive definite symmetric matrix for all t. The array synchronizes if

$$R \stackrel{\triangle}{=} (U \otimes V)(G(t) \otimes D(t) - I \otimes P(t)) + \frac{1}{2}B_1(t)K(t)B_1^T(t) + \frac{1}{2}B_2^T(t)K^{-1}(t)B_2(t) \leq 0 \tag{2}$$

for all t.

Proof: Construct the Lyapunov functional $E = \frac{1}{2}x^T(U \otimes V)x + \int_{t-\tau}^t x^T(s)U_\tau(s)x(s)ds$ where U_τ is a symmetric positive semidefinite matrix to be determined later. Note that $(U \otimes V) \geq 0$. The derivative of E along trajectories of Eq. (1) is:

$$\begin{aligned}\dot{E} \;=\;& x^T(U \otimes V)(I \otimes f(x_i,t) + (I \otimes P)x)\\ &+x^T(U \otimes V)(G \otimes D - I \otimes P)x + x^T(U \otimes V)(G_\tau \otimes D_\tau)x(t-\tau)\\ &+x^T U_\tau x - x(t-\tau)^T U_\tau x(t-\tau)\end{aligned}$$

Using the same argument as [1], [2], we obtain

$$x^T(U \otimes V)(I \otimes f(x_i,t) + (I \otimes P)x) \leq -\mu x^T(U \otimes V)x \tag{3}$$

for some $\mu > 0$. Next, we use Lemma 1 to obtain:

$$\begin{aligned}x^T(U \otimes V)(G_\tau \otimes D_\tau)x(t-\tau) \;&=\; (x^T B_1)(B_2 x(t-\tau))\\ &\leq\; \tfrac{1}{2}x^T B_1 K B_1^T x^T + \tfrac{1}{2}x(t-\tau)^T B_2^T K^{-1} B_2 x(t-\tau)\end{aligned}$$

If we choose $U_\tau = \frac{1}{2}B_2^T K^{-1} B_2$ which is a symmetric positive semidefinite matrix for all t, then

$$\dot{E} \leq -\mu x^T(U \otimes V)x + x^T R x$$

If $R \leq 0$, then by Lyapunov's method [6], [7] the trajectories approach the set $\{x : \dot{E} = 0\}$. If $\dot{E} = 0$, the above equation implies that $x^T(U \otimes V)x = 0$ which in turn implies that $x \in M$ since $U \in W_i$ and $V > 0$ (see [1], [2]). Therefore the set $\{x : \dot{E} = 0\}$ is a subset of the synchronization manifold M and thus the array synchronizes. □

This result has several degrees of freedom: the choice of (B_1, B_2), the choice of K and the choice of U. Next we study several of these choices that simplify the condition in Eq. (2).

A. Choosing the factorization $B_1B_2 = UG_\tau \otimes VD_\tau$

There are several ways to choose the factorization (B_1, B_2). Depending on the factorization, the matrix K can have different dimensions than $G \otimes D$ and $G_\tau \otimes D_\tau$. When the delay coupling term is absent ($G_\tau \otimes D_\tau = 0$), we can pick $B_1 = B_2 = 0$ and the synchronization criterion reverts back to the nondelay criterion in [4]. The factorization should be chosen such that the synchronization manifold M is in the kernel of both B_1^T and B_2. Otherwise, as M is in the the kernel of $(U \otimes V)$, this would mean that the matrix R in Eq. (2) is never negative semidefinite. Therefore if Eq. (2) is satisfied, then G_τ has constant row sums. This can be seen as follows. If G_τ does not have constant row sums, then $UG_\tau(1,\ldots,1)^T \neq 0$ and M is not in the kernel of $UG_\tau \otimes VD_\tau$ and thus also not in the kernel of B_2.

Let J be the matrix of all 1's and $Q = I - \frac{1}{n}J \in W_i$. The eigenvalues of Q are 0 and 1. If X is a matrix with zero column sums, then $JX = 0$ and thus $QX = X$. In particular, $Q^2 = Q$, and $QU = UQ = U$ for $U \in W_i$. By choosing the factorizations $(B_1, B_2) = (U \otimes V, G_\tau \otimes D_\tau)$ and $(B_1, B_2) = (Q \otimes I, UG_\tau \otimes VD_\tau)$ we get the following Corollary:

Corollary 1: Let V be some symmetric positive definite matrix such that $(y-z)^T V(f(y,t) + P(t)y - f(z,t) - P(t)z) \leq -c\|y-z\|^2$ for some $c > 0$. Let U be a symmetric matrix in W_i and $K(t)$ a positive definite symmetric matrix for all t. The array synchronizes if one of the following conditions is satisfied for all t:

$$\begin{aligned}&(U \otimes V)(G(t) \otimes D(t) - I \otimes P(t)) + \frac{1}{2}(U \otimes V)K(t)(U \otimes V)\\&+\frac{1}{2}(G_\tau(t) \otimes D_\tau(t))^T K^{-1}(t)(G_\tau(t) \otimes D_\tau(t)) \leq 0\end{aligned} \tag{4}$$

$$\begin{aligned}&(U \otimes V)(G(t) \otimes D(t) - I \otimes P(t)) + \frac{1}{2}(Q \otimes I)K(t)(Q \otimes I)\\&+\frac{1}{2}(UG_\tau(t) \otimes VD_\tau(t))^T K^{-1}(t)(UG_\tau(t) \otimes VD_\tau(t)) \leq 0\end{aligned} \tag{5}$$

The condition in Eq. (4) was obtained in [5] for the case of constant coupling. If G_τ has zero row sums, we can choose the factorization $(B_1, B_2) = (UG_\tau \otimes VD_\tau, Q \otimes I)$ to get:

Corollary 2: Let V be some symmetric positive definite matrix such that $(y-z)^T V(f(y,t) + P(t)y - f(z,t) - P(t)z) \leq -c\|y-z\|^2$ for some $c > 0$. Let U be a symmetric matrix in W_i and $K(t)$ a positive definite symmetric matrix for all t. If G_τ has zero row sums, then the array synchronizes if the following condition is satisfied for all t:

$$\begin{aligned}&(U \otimes V)(G(t) \otimes D(t) - I \otimes P(t)) + \frac{1}{2}(Q \otimes I)K(t)(Q \otimes I)\\&+\frac{1}{2}(UG_\tau(t) \otimes VD_\tau(t))K^{-1}(t)(UG_\tau(t) \otimes VD_\tau(t))^T \leq 0\end{aligned} \tag{6}$$

B. Choosing the matrix $U \in W_i$

By choosing $U = Q$ as was done in [4] and using the fact that $QX = X$ for X a zero column sum matrix, the synchronization condition can be further simplified:

Corollary 3: Let V be some symmetric positive definite matrix such that $(y-z)^T V(f(y,t) + P(t)y - f(z,t) - P(t)z) \leq -c\|y-z\|^2$ for some $c > 0$. Let $K(t)$ be a positive definite symmetric matrix for all t. Suppose G_τ and G are zero column sums matrices. The array in Eq. (1) synchronizes if one of the following conditions is satisfied for all t:

$$\begin{aligned}&G(t) \otimes VD(t) - Q \otimes VP(t) + \frac{1}{2}(Q \otimes V)K(t)(Q \otimes V)\\&+\frac{1}{2}(G_\tau(t) \otimes D_\tau(t))^T K^{-1}(t)(G_\tau(t) \otimes D_\tau(t)) \leq 0\end{aligned} \tag{7}$$

$$\begin{aligned}&G(t) \otimes VD(t) - Q \otimes VP(t) + \frac{1}{2}(Q \otimes I)K(t)(Q \otimes I)\\&+\frac{1}{2}(G_\tau(t) \otimes VD_\tau(t))^T K^{-1}(t)(G_\tau(t) \otimes VD_\tau(t)) \leq 0\end{aligned} \tag{8}$$

C. Choosing the matrix K

Corollary 4: Let V be some symmetric positive definite matrix such that $(y-z)^T V(f(y,t) + P(t)y - f(z,t) - P(t)z) \leq -c\|y-z\|^2$ for some $c > 0$. Suppose $G_\tau(t)$ and $VD_\tau(t)$ are symmetric for all t, and G and G_τ are zero column sums matrix. Suppose further that G_τ has a simple zero eigenvalue and D_τ is nonsingular for all t. The array in Eq. (1) synchronizes if the following condition is satisfied for all t:

$$G(t) \otimes VD(t) - Q \otimes VP(t) + \sqrt{(G_\tau \otimes VD_\tau)^2} \leq 0 \tag{9}$$

In particular, if in addition $G_\tau(t)\otimes VD_\tau(t)$ is symmetric positive semidefinite for all t, then the array synchronizes if $G(t)\otimes VD(t)+G_\tau(t)\otimes VD_\tau(t)-Q\otimes VP(t)\leq 0$.

Proof: Since Q commutes with G_τ there exists a orthogonal matrix C such that $Q=C^T\Gamma_Q C$ and $G_\tau=C^T\Gamma_\tau C$ with Γ_Q and Γ_τ diagonal matrices of the eigenvalues [8]. Let $\Gamma_\tau=\text{diag}(\lambda_1,\ldots,\lambda_n)$. By hypothesis, $\lambda_1=0$ and $\lambda_i>0$ for $i>1$. Let $H=C^T\text{diag}(1,0,\ldots,0)C$ and $K=\sqrt{(G_\tau\otimes VD_\tau)^2}+H\otimes I$. It is easy to see that K is symmetric positive definite and $K^{-1}=\sqrt{(G_\tau\otimes VD_\tau)^2}^\dagger+H\otimes I$. Since $QH=G_\tau H=0$ and $Q\sqrt{(G_\tau\otimes VD_\tau)^2}=\sqrt{(G_\tau\otimes VD_\tau)^2}$, Eq. (8) reduces to Eq. (9). □

Corollary 5: Let V be some symmetric positive definite matrix such that $(y-z)^TV(f(y,t)+D(t)y-f(z,t)-D(t)z)\leq -c\|y-z\|^2$ for some $c>0$. Suppose G_τ and G are zero column sums matrices. The array in Eq. (1) synchronizes if the following conditions is satisfied for all t and some $\alpha(t)>0$:

$$\begin{aligned}&(G(t)-Q)\otimes VD(t)+\frac{\alpha(t)}{2}(Q\otimes I)\\&+\frac{1}{2\alpha(t)}(G_\tau(t)\otimes VD_\tau(t))^T(G_\tau(t)\otimes VD_\tau(t))\leq 0\end{aligned}\tag{10}$$

If in addition $VD(t)<0$ for all t and G is a zero row sums matrix, then the array synchronizes if

$$\lambda_2\left(\frac{1}{2}(G(t)+G^T(t))\right)\geq 1+\|G_\tau(t)\|\|VD_\tau(t)\|\|(VD(t))^{-1}\|\tag{11}$$

Proof: Eq. (10) follows from Corollary 3 by choosing $P=D$ and $K=\alpha I$. Let us choose $\alpha(t)=\max(\|G_\tau(t)\|\|VD_\tau(t)\|,\epsilon\cdot\nu(VD(t)))$ for some scalar $\epsilon>0$ where $\nu(X)=\|X^{-1}\|^{-1}$ is the co-norm of the matrix X. Eq. (10) is equivalent to the eigenvalues of the symmetric matrix

$$\begin{aligned}F=\quad&\left(\frac{1}{2}(G(t)+G(t)^T)-Q\right)\otimes VD(t)+\frac{\alpha(t)}{2}(Q\otimes I)\\&+\frac{1}{2\alpha(t)}(G_\tau(t)\otimes VD_\tau(t))^T(G_\tau(t)\otimes VD_\tau(t))\end{aligned}\tag{12}$$

begin nonpositive. Let $e=(1,\ldots,1)^T$. For all z, $F(e\otimes z)=0$. For y a unit norm vector orthogonal to e and any unit norm vector z, define $w=y\otimes z$. Since $VD<0$,

$$w^T\left[\left(\frac{1}{2}\left(G(t)+G(t)^T\right)-Q\right)\otimes VD(t)\right]w\leq-\left[\lambda_2\left(\frac{1}{2}\left(G(t)+G(t)^T\right)\right)-1\right]\nu(VD(t))$$

Furthermore, $\frac{\alpha}{2}w^T(Q\otimes I)w\leq\frac{\alpha}{2}$ and

$$\frac{1}{2\alpha}w^T(G_\tau\otimes VD_\tau)^T(G_\tau\otimes VD_\tau)w\leq\frac{\|G_\tau\|^2\|VD_\tau\|^2}{2\alpha}\leq\frac{\alpha}{2}$$

This implies that the eigenvalues of F are nonpositive and that the array synchronizes if

$$\begin{aligned}\lambda_2\left(\frac{1}{2}(G(t)+G^T(t))\right)\ &\geq\ 1+\max(\|G_\tau(t)\|\|VD_\tau(t)\|,\epsilon\cdot\nu(VD(t)))\|(VD(t))^{-1}\|\\&=\ 1+\max(\|G_\tau(t)\|\|VD_\tau(t)\|\|(VD(t))^{-1}\|,\epsilon)\end{aligned}$$

for any $\epsilon>0$. Combine this with the fact that $\mu>0$ in Eq. (3) this implies that the condition in Eq. (11) also synchronizes the array. □

Corollary 5 relates synchronization to the eigenvalues of $G+G^T$. In particular, it says that the array synchronizes if the second smallest eigenvalue of $G+G^T$ is large enough. In [1], [9] it was concluded that sufficiently strong cooperative coupling or an underlying graph that is well connected synchronizes a coupled array. As these conditions correspond to a large $\lambda_2(G+G^T)$ Corollary 5 is an extension of this to the delay case provided the coupling between delayed state variables is small relative to the nondelay coupling.

III. Conclusions

We have presented criteria for synchronization in an array of coupled systems with time-varying coupling and coupling between delayed state variables that generalize and unify previous results in the literature. We show that similar to the nondelay case, the array synchronizes if the nondelay coupling is strong enough, provided the coupling between delayed state variables $G_\tau\otimes D_\tau$ is relatively small. These results are also applicable (after suitable changes to Theorem 1) to the case when $\dot{x}_i=f(x_i,t)$ are retarded functional differential equations.

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